Concerning Functions Zeta(s) and Zeta(s,w) and Ramanujan-type Series for 1/Pi

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Abstract

In this paper, in the **Section 1**, we have described some equations concerning the functions $\zeta(s)$ and $\zeta(s,w)$. In this Section, we have described also some equations concerning a transformation formula involving the gamma and Riemann zeta functions of Ramanujan. Furthermore, we have described also some mathematical connections with various theorems concerning the incomplete elliptic integrals described in the "Ramanujan's lost notebook". In the **Section 2**, we have described some Ramanujan-type series for $1/\pi$ and some equations concerning the p-adic open string for the scalar tachyon field. In this Section, we have described also some possible and interesting mathematical connections with some Ramanujan's Theorems, contained in the first letter of Ramanujan to G. H. Hardy. In the **Section 3**, we have described some equations concerning the zeta strings and the zeta nonlocal scalar fields. In conclusion, in the **Section 4**, we have showed some possible mathematical connections between the arguments above mentioned, the Palumbo-Nardelli model and the Ramanujan's modular equations that are related to the physical vibrations of the bosonic strings and of the superstrings.

1. On some equations concerning some observations concerning the functions $\zeta(s)$ and $\zeta(s,w)$ [1]

In the Mathematical Analysis there exist the proof of the following formula:

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + 2\int_0^\infty (1+t^2)^{-\frac{s}{2}} Sin[sArcTan(t)] \frac{dt}{e^{2\pi t}-1} \quad (1.1)$$

that can be rewritten also as follows:

$$\zeta(s)=2\int_0^{\infty}\frac{\sin\left(s\tan^{-1}(t)\right)}{\left(t^2+1\right)^{s/2}\left(e^{2\pi t}-1\right)}dt+\frac{1}{2}+\frac{1}{s-1} \quad (1.1b)$$

where $\zeta(s)$ represent the Riemann zeta function.

Now we analyze in greater detail the formula (1.1)
The function $\zeta(s)$ is represented from the series

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots+\frac{1}{n^s}+\ldots, \quad (1.2)$$

as the real part of the complex variable

$$s\equiv\xi+i\eta$$

is greater than the unity. Under the same condition, we have still that

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}dx}{e^x-1}, \quad (1.3)$$

what is easily proved by using the following equality

$$\frac{1}{v^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}e^{-vx}x^{s-1}dx. \quad (1.4)$$

It is from the expression (1.3) that Riemann reached by an ingenious application of the residue calculus, extending $\zeta(s)$ across all the plan and discover such interesting properties of this function. We set $f(z)=z^{-s}$, where

$$p(\tau,t)=\left(\tau^2+t^2\right)^{-\frac{s}{2}}\cos\left(s\arctan\frac{t}{\tau}\right), \qquad q(\tau,t)=-\left(\tau^2+t^2\right)^{-\frac{s}{2}}\sin\left(s\arctan\frac{t}{\tau}\right). \quad (1.5)$$

The following three conditions:

1° The function $f(z)$ is holomorphic for $\tau\geq\alpha$, for each t;
2° The condition $\lim_{t=\pm\infty}e^{-2\pi|t|}f(\tau+it)=0$ is verified uniformly for $\alpha\leq\tau\leq\beta$, however great β;
3° The function $f(z)$ is subject to the following condition: $\lim_{\tau=\infty}\int_{-\infty}^{+\infty}e^{-2\pi|t|}\left|f(\tau+it)\right|=0$;
are verified in the half-plane $\tau>0$; assuming $\xi>1$, so that the series (1.2) converges, and by $m=1$, we obtain, after the application of the following formula:

$$\sum_m^{\infty}f(v)=\frac{1}{2}f(m)+\int_m^{\infty}f(\tau)d\tau-2\int_0^{\infty}\frac{q(m,t)}{e^{2\pi t}-1}dt, \quad (1.6)$$

$$\zeta(s)=\frac{1}{2}+\frac{1}{s-1}+2\int_0^\infty\left(1+t^2\right)^{-\frac{s}{2}}\sin(s\arctan t)\frac{dt}{e^{2\pi t}-1}, \quad (1.7)$$

that is the eq. (1.1), and, after the application of the following formula,

$$\sum_m^\infty f(v)=\int_\alpha^\infty f(\tau)d\tau-2\int_0^\infty Q(\alpha,t)dt, \quad (1.8)$$

for $\alpha=\frac{1}{2}$,

$$\zeta(s)=\frac{2^{s-1}}{s-1}-2\int_0^\infty\left(\frac{1}{4}+t^2\right)^{-\frac{s}{2}}\sin(s\arctan 2t)\frac{dt}{e^{2\pi t}+1}. \quad (1.9)$$

Another expression for $\zeta(s)$ is derived from the following formula:

$$\sum_m^\infty f(v)=-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left(\frac{\pi}{\sin\pi z}\right)^2 F(z)dz. \quad (1.10)$$

By $\alpha=\frac{1}{2}, z=\frac{1}{2}+it$, we find

$$\zeta(s)=\frac{4\pi}{s-1}\int_0^\infty\left(\frac{1}{4}+t^2\right)^{\frac{1-s}{2}}\frac{\cos[(s-1)\arctan 2t]}{\left(e^{\pi t}+e^{-\pi t}\right)^2}dt. \quad (1.11)$$

Starting from the relationship

$$\left(1-\frac{1}{2^{s-1}}\right)\zeta(s)=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\ldots$$

with the application of the following formula

$$\sum_m^n(-1)^v f(v)=-\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{f(z)dz}{e^{\pi iz}-e^{-\pi iz}}=-2\int_0^\infty Q_1(\alpha,t)dt, \quad (1.12)$$

with $m=1,\alpha=\frac{1}{2}$, we also find

$$\zeta(s)=\frac{2^s}{2^{s-1}-1}\int_0^\infty\left(\frac{1}{4}+t^2\right)^{-\frac{s}{2}}\frac{\cos(s\arctan 2t)}{e^{\pi t}+e^{-\pi t}}dt. \quad (1.13)$$

It is easy to see that the integrals definite appearing in the above expressions are analytic functions, holomorphic for any finite value of s. On the other hand, we deduce from the eq. (1.7), for $s=0$,

$$\zeta(0)=-\frac{1}{2},$$

then, subtracting the two members $\frac{1}{s-1}$, and making tend s to the unity, taking into account the following equality:

$$C=\frac{1}{2}+2\int_0^\infty \frac{t}{1+t^2}\frac{1}{e^{2\pi t}-1}dt\,, \quad (1.14)$$

we obtain

$$\lim_{s=1}\left[\zeta(s)-\frac{1}{s-1}\right]=\frac{1}{2}+2\int_0^\infty \frac{t}{1+t^2}\frac{1}{e^{2\pi t}-1}dt=C\,, \quad (1.15)$$

and finally, by differentiating with respect to s, putting $s=0$ and using the following equality

$$1-2\int_0^\infty \arctan t\frac{1}{e^{2\pi t}-1}dt=\log\sqrt{2\pi}\,, \quad (1.16)$$

we obtain

$$\zeta'(0)=2\int_0^\infty \arctan t\frac{1}{e^{2\pi t}-1}dt-1=-\log\sqrt{2\pi}\,. \quad (1.17)$$

We note that there exist a mathematical connection between π and $\phi=\frac{\sqrt{5}-1}{2}$, i.e. the aurea section, by the simple formula

$$\arccos\phi=0{,}2879\pi\,, \quad (1.18)$$

thence we have that

$$\pi=\arccos\phi\cdot\frac{1}{0{,}2879}\,. \quad (1.19)$$

We can rewrite the eq. (1.17) also as follows:

$$\zeta'(0)=2\int_0^\infty \arctan t\frac{1}{e^{2\pi t}-1}dt-1=-\log\sqrt{2\left(\arccos\phi\cdot\frac{1}{0{,}2879}\right)}\,. \quad (1.19b)$$

Now we consider the function

$$\zeta(s,w)=\frac{1}{w^s}+\frac{1}{(w+1)^s}+\frac{1}{(w+2)^s}+\ldots, \quad (1.20)$$

which reduces to $\zeta(s)$ for $w=1$. We must replace our general formulas

$$f(z)=(z+w)^{-s}, \quad (1.21)$$

where

$$p(\tau,t)=\left[(\tau+w)^2+t^2\right]^{-\frac{s}{2}}\cos\left(s\arctan\frac{t}{\tau+w}\right),\quad q(\tau,t)=-\left[(\tau+w)^2+t^2\right]^{-\frac{s}{2}}\sin\left(s\arctan\frac{t}{\tau+w}\right). \quad (1.22)$$

Assuming the real part of w positive, applying the following formula

$$\sum_m^\infty f(v)=\frac{1}{2}f(m)+\int_m^\infty f(\tau)d\tau-2\int_0^\infty\frac{1}{e^{2\pi t}-1}q(m,t)dt, \quad (1.23)$$

we obtain

$$\zeta(s,w)=\frac{w^{1-s}}{s-1}+\frac{w^{-s}}{2}+2\int_0^\infty\left(w^2+t^2\right)^{-\frac{s}{2}}\sin\left(s\arctan\frac{t}{w}\right)\frac{1}{e^{2\pi t}-1}dt, \quad (1.24)$$

valid expression in the whole plan and shows that $\zeta(s,w)$ is a uniform function admitting for singularity at finite distance, only the pole $s=1$ of residue 1. We conclude, on the other hand, for $s=0$,

$$\zeta(0,w)=\frac{1}{2}-w, \quad (1.25)$$

then, subtracting $\frac{1}{s-1}$ and for s tending to the unity,

$$\lim_{s=1}\left[\zeta(s,w)-\frac{1}{s-1}\right]=-\log w+\frac{1}{2w}+2\int_0^\infty\frac{t}{w^2+t^2}\frac{1}{e^{2\pi t}-1}dt, \quad (1.26)$$

and finally, differentiating with respect to s and then for $s=0$, we have that

$$\zeta_s'(0,w)=\left(w-\frac{1}{2}\right)\log w-w+2\int_0^\infty\arctan\frac{t}{w}\frac{1}{e^{2\pi t}-1}dt. \quad (1.27)$$

For the following equalities,

$$\log\Gamma(x)=\log\sqrt{2\pi}+\left(x-\frac{1}{2}\right)\log x-x+J(x), \quad (1.28)$$

$$D_x\log\Gamma(x)=\log x-\frac{1}{2x}+J'(x), \quad (1.29)$$

these last two expressions on reduced respectively to $-\frac{\Gamma'(w)}{\Gamma(w)}$ and $\log\Gamma(w)-\log\sqrt{2\pi}$.

Here it is possible to obtain some mathematical connections with various theorems concerning the incomplete elliptic integrals described in the "Ramanujan's lost notebook".

Let $u(q)$ denote the Rogers-Ramanujan continued fraction defined by

$$u:=u(q):=\cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\dots}}}}, \quad |q|<1, \quad (1.30)$$

and set $v = u(q^2)$. Recall that $\psi(q)$ is defined by

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}. \quad (1.31)$$

Then

$$\frac{8}{5} \int \frac{\psi^5(q)}{\psi(q^5)} \frac{dq}{q} = \log(u^2 v^3) + \sqrt{5} \log\left(\frac{1 + (\sqrt{5} - 2)uv^2}{1 - (\sqrt{5} + 2)uv^2} \right). \quad (1.32)$$

We note that $1 + (\sqrt{5} - 2) = 1{,}236067977 = \frac{\sqrt{5} - 1}{2} \times 2$, i.e. the aurea section multiplied by the number 2, and that $1 - (\sqrt{5} + 2) = -3{,}2360679777 = -\left(\frac{\sqrt{5} + 1}{2} \times 2 \right)$, i.e. the aurea ratio multiplied by the number 2 and with the minus sign.

With $f(-q)$, $\psi(q)$, and $u(q)$ defined by

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty =: e^{-2\pi i z/24} \eta(z), \quad q = e^{2\pi i z}, \quad \operatorname{Im} z > 0, \quad (1.33)$$

by (1.31) and (1.30), respectively, and with $\varepsilon = (\sqrt{5} + 1)/2$, we have that

$$5^{3/4} \int_0^q \frac{f^2(-t) f^2(-t^5)}{\sqrt{t}} dt = 2 \int_{\cos^{-1}\left((\varepsilon u)^{5/2}\right)}^{\pi/2} \frac{1}{\sqrt{1 - \varepsilon^{-5} 5^{-3/2} \sin^2 \varphi}} d\varphi =$$

$$= \int_0^{2 \tan^{-1}\left(5^{3/4} \sqrt{q} f^3(-q^5) / f^3(-q)\right)} \frac{1}{\sqrt{1 - \varepsilon^{-5} 5^{-3/2} \sin^2 \varphi}} d\varphi = \sqrt{5} \int_0^{2 \tan^{-1}\left(5^{1/4} \sqrt{q} \psi(q^5) / \psi(q)\right)} \frac{1}{\sqrt{1 - \varepsilon 5^{-1/2} \sin^2 \varphi}} d\varphi. \quad (1.34)$$

Let v be defined by the following expression

$$v := v(q) := q \left(\frac{f(-q) f(-q^{15})}{f(-q^3) f(-q^5)} \right)^3, \quad (1.35)$$

and let $\varepsilon = (\sqrt{5} + 1)/2$. Then

$$\int_0^q f(-t) f(-t^3) f(-t^5) f(-t^{15}) dt = \frac{1}{5} \int_{2 \tan^{-1}\left(\frac{1}{\sqrt{5}} \sqrt{\frac{1 - 11v - v^2}{1 + v - v^2}} \right)}^{2 \tan^{-1}(1/\sqrt{5})} \frac{1}{\sqrt{1 - \frac{9}{25} \sin^2 \varphi}} d\varphi =$$

$$= \frac{1}{9} \int_{2 \tan^{-1}\left(\frac{1 - v\varepsilon^{-3}}{1 + v\varepsilon^3} \sqrt{\frac{(1 + v\varepsilon)(1 - v\varepsilon^5)}{(1 - v\varepsilon^{-1})(1 + v\varepsilon^{-5})}} \right)}^{\pi/2} \frac{1}{\sqrt{1 - \frac{1}{81} \sin^2 \varphi}} d\varphi = \frac{1}{4} \int_{\tan^{-1}\left((3 - \sqrt{5}) \sqrt{\frac{(1 - v\varepsilon^{-1})(1 - v\varepsilon^5)}{(1 + v\varepsilon)(1 + v\varepsilon^{-5})}} \right)}^{\tan^{-1}(3 - \sqrt{5})} \frac{1}{\sqrt{1 - \frac{15}{16} \sin^2 \varphi}} d\varphi. \quad (1.36)$$

If v is defined by the following expression

$$v := v(q) := q\left(\frac{f(-q)f(-q^{14})}{f(-q^2)f(-q^7)}\right)^4, \quad (1.37)$$

and if $c = \frac{\sqrt{13+16\sqrt{2}}}{7}$, then

$$\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt = \frac{1}{\sqrt{8\sqrt{2}}}\int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1}c} \frac{1}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}}d\varphi. \quad (1.38)$$

Let

$$v = q\frac{f^3(-q^2)f^3(-q^{14})}{f^3(-q)f^3(-q^7)} \qquad \text{and} \qquad c = \frac{9-4\sqrt{2}}{7},$$

Then

$$\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt = 2^{-7/4}\int_{\cos^{-1}\left\{c\frac{1-2\sqrt{2}x}{1+2\sqrt{2}x+1}\right\}}^{\cos^{-1}c} \frac{1}{\sqrt{1-\frac{32-13\sqrt{2}}{64}\sin^2\varphi}}d\varphi. \quad (1.39)$$

1.1 On some equations concerning a transformation formula involving the gamma and Riemann zeta functions of Ramanujan.

In the Ramanujan's lost notebook there is a claim that provides a beautiful series transformation involving the logarithmic derivative of the gamma function and the Riemann zeta function. To state Ramanujan's claim, it will be convenient to use the familiar notation

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{k+x} - \frac{1}{k+1}\right), \quad (1.40)$$

where γ denotes Euler's constant. We also need to recall the following functions associated with Riemann's zeta function $\zeta(s)$. Let

$$\xi(s) := (s-1)\pi^{-\frac{1}{2}s}\Gamma\left(1+\frac{1}{2}s\right)\zeta(s).$$

Then Riemann's Ξ-function is defined by

$$\Xi\left(\frac{1}{2}t\right) := \xi\left(\frac{1}{2}+\frac{1}{2}it\right).$$

Theorem 1

Define

$$\phi(x) := \psi(x) + \frac{1}{2x} - \log x. \quad (1.41)$$

If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left\{\frac{\gamma - \log(2\pi\alpha)}{2\alpha} + \sum_{n=1}^{\infty}\phi(n\alpha)\right\} = \sqrt{\beta}\left\{\frac{\gamma - \log(2\pi\beta)}{2\beta} + \sum_{n=1}^{\infty}\phi(n\beta)\right\} =$$

$$= -\frac{1}{\sqrt{\pi^3}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt, \quad (1.42)$$

where γ denotes Euler's constant and $\Xi(x)$ denotes Riemann's Ξ-function.

Although Ramanujan does not provide a proof of (1.42), he does indicate that (1.42) "can be deduced from"

$$\int_0^{\infty}(\psi(1+x) - \log x)\cos(2\pi nx)dx = \frac{1}{2}(\psi(1+n) - \log n). \quad (1.43)$$

We have that, for $t \neq 0$,

$$\sum_{n=1}^{\infty}\frac{1}{t^2 + 4n^2\pi^2} = \frac{1}{2t}\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right). \quad (1.44)$$

We find that, for $\operatorname{Re} z > 0$,

$$\phi(z) = -2\int_0^{\infty}\frac{tdt}{(t^2 + z^2)(e^{2\pi t} - 1)}. \quad (1.45)$$

We require Binet's integral for $\log\Gamma(z)$, i.e., for $\operatorname{Re} z > 0$, thence

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \int_0^{\infty}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)\frac{e^{-zt}}{t}dt. \quad (1.46)$$

We find that

$$\int_0^{\infty}\left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right)e^{-x}dx = \gamma, \quad (1.47)$$

where γ denotes Euler's constant. Furthermore, by Frullani's integral, we have that

$$\int_0^\infty \frac{e^{-\mu x}-e^{-\nu x}}{x}dx=\log\frac{\nu}{\mu}\ , \quad \mu,\nu>0. \quad (1.48)$$

We now describe a proof of Theorem 1. Our first goal is to establish an integral representation for the far left side of (1.42). Replacing z by $n\alpha$ in (1.45) and summing on n, $1\le n<\infty$, we find, by absolute convergence, that

$$\sum_{n=1}^{\infty}\phi(n\alpha)=-2\sum_{n=1}^{\infty}\int_0^\infty\frac{tdt}{\left(t^2+n^2\alpha^2\right)\left(e^{2\pi t}-1\right)}=-\frac{2}{\alpha^2}\int_0^\infty\frac{tdt}{\left(e^{2\pi t}-1\right)}\sum_{n=1}^{\infty}\frac{1}{(t/\alpha)^2+n^2}. \quad (1.49)$$

Invoking (1.44) in (1.49), we see that

$$\sum_{n=1}^{\infty}\phi(n\alpha)=-\frac{2\pi}{\alpha}\int_0^\infty\frac{1}{\left(e^{2\pi t}-1\right)}\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}+\frac{1}{2}\right)dt. \quad (1.50)$$

Next, setting $x=2\pi t$ in (1.47), we readily find that

$$\gamma=\int_0^\infty\left(\frac{2\pi}{e^{2\pi t}-1}-\frac{e^{-2\pi t}}{t}\right)dt. \quad (1.51)$$

By Frullani's integral (1.48),

$$\int_0^\infty\frac{e^{-t/\alpha}-e^{-2\pi t}}{t}dt=\log\left(\frac{2\pi}{1/\alpha}\right)=\log(2\pi\alpha). \quad (1.52)$$

Combining (1.51) and (1.52), we arrive at

$$\gamma-\log(2\pi\alpha)=\int_0^\infty\left(\frac{2\pi}{e^{2\pi t}-1}-\frac{e^{-t/\alpha}}{t}\right)dt. \quad (1.53)$$

Hence, from (1.40) and (1.43), we deduce that

$$\sqrt{\alpha}\left(\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{n=1}^{\infty}\phi(n\alpha)\right)=\frac{1}{2\sqrt{\alpha}}\int_0^\infty\left(\frac{2\pi}{e^{2\pi t}-1}-\frac{e^{-t/\alpha}}{t}\right)dt-\frac{2\pi}{\sqrt{\alpha}}\int_0^\infty\frac{1}{\left(e^{2\pi t}-1\right)}\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}+\frac{1}{2}\right)dt$$

$$=\int_0^\infty\left(\frac{\sqrt{\alpha}}{t\left(e^{2\pi t}-1\right)}-\frac{2\pi}{\sqrt{\alpha}\left(e^{2\pi t/\alpha}-1\right)\left(e^{2\pi t}-1\right)}-\frac{e^{-t/\alpha}}{2t\sqrt{\alpha}}\right)dt. \quad (1.54)$$

Now, for n real, we have that

$$\int_0^\infty\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\left(\Xi\left(\frac{1}{2}t\right)\right)^2\frac{\cos nt}{1+t^2}dt=\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos nt}{1+t^2}dt=$$

$$=\sqrt{\pi^3}\int_0^\infty\left(\frac{1}{e^{xe^{n}}-1}-\frac{1}{xe^{n}}\right)\left(\frac{1}{e^{xe^{-n}}-1}-\frac{1}{xe^{-n}}\right)dx. \quad (1.55)$$

Letting $n=\frac{1}{2}\log\alpha$ and $x=2\pi t/\sqrt{\alpha}$ in (1.55), we deduce that

$$-\frac{1}{\sqrt{\pi^3}}\int_0^\infty\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt=-\frac{2\pi}{\sqrt{\alpha}}\int_0^\infty\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}\right)dt=$$
$$=\int_0^\infty\left(\frac{-2\pi/\sqrt{\alpha}}{\left(e^{2\pi t/\alpha}-1\right)\left(e^{2\pi t}-1\right)}+\frac{\sqrt{\alpha}}{t\left(e^{2\pi t}-1\right)}+\frac{1}{t\sqrt{\alpha}\left(e^{2\pi t/\alpha}-1\right)}-\frac{\sqrt{\alpha}}{2\pi t^2}\right)dt. \quad (1.56)$$

Hence, combining (1.54) and (1.56), in order to prove that the far left side of (1.42) equals the far right side of (1.42), we see that it suffices to show that

$$\int_0^\infty\left(\frac{1}{t\sqrt{\alpha}\left(e^{2\pi t/\alpha}-1\right)}-\frac{\sqrt{\alpha}}{2\pi t^2}+\frac{e^{-t/\alpha}}{2t\sqrt{\alpha}}\right)dt=\frac{1}{\sqrt{\alpha}}\int_0^\infty\left(\frac{1}{u\left(e^u-1\right)}-\frac{1}{u^2}+\frac{e^{-u/(2\pi)}}{2u}\right)du=0, \quad (1.57)$$

where we made the change of variable $u=2\pi t/\alpha$. In fact, more generally, we show that

$$\int_0^\infty\left(\frac{1}{u\left(e^u-1\right)}-\frac{1}{u^2}+\frac{e^{-ua}}{2u}\right)du=-\frac{1}{2}\log(2\pi a), \quad (1.58)$$

so that if we set $a=1/(2\pi)$ in (1.58), we deduce (1.57).
Consider the integral, for $t>0$,

$$F(a,t):=\int_0^\infty\left\{\left(\frac{1}{e^u-1}-\frac{1}{u}+\frac{1}{2}\right)\frac{e^{-tu}}{u}+\frac{e^{-ua}-e^{-tu}}{2u}\right\}du=\log\Gamma(t)-\left(t-\frac{1}{2}\right)\log t+t-\frac{1}{2}\log(2\pi)+\frac{1}{2}\log\frac{t}{a}, \quad (1.59)$$

where we applied (1.46) and (1.48). Upon the integration of (1.30), it is easily gleaned that, as $t\to 0$,

$$\log\Gamma(t)\approx-\log t-\gamma t, \quad (1.60)$$

where γ denotes Euler's constant. Using this in (1.59), we find, upon simplification, that, as $t\to 0$,

$$F(a,t)\approx-\gamma t-t\log t+t-\frac{1}{2}\log(2\pi)-\frac{1}{2}\log a. \quad (1.61)$$

Hence,

$$\lim_{t\to 0}F(a,t)=-\frac{1}{2}\log(2\pi a). \quad (1.62)$$

Letting t approach 0 in (1.59), taking the limit under the integral sign on the right-hand side using Lebesgue's dominated convergence theorem, and employing (1.62), we immediately deduce (1.58). As previously discussed, this is sufficient to prove the equality of the first and third expressions in (1.42), namely,

$$\sqrt{\alpha}\left\{\frac{\gamma-\log(2\pi\alpha)}{2\alpha}+\sum_{n=1}^{\infty}\phi(n\alpha)\right\}=-\frac{1}{\sqrt{\pi^3}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt\,. \quad (1.63)$$

Lastly, using (1.63) with α replaced by β and employing the relation $\alpha\beta=1$, we conclude that

$$\sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta}+\sum_{n=1}^{\infty}\phi(n\beta)\right\}=-\frac{1}{\sqrt{\pi^3}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log\beta\right)}{1+t^2}dt=$$

$$=-\frac{1}{\sqrt{\pi^3}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log(1/\alpha)\right)}{1+t^2}dt=$$

$$=-\frac{1}{\sqrt{\pi^3}}\int_0^{\infty}\left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2\frac{\cos\left(\frac{1}{2}t\log(\alpha)\right)}{1+t^2}dt\,. \quad (1.64)$$

Hence, the equality of the second and third expressions in (1.42) has been demonstrated, and so the proof is complete.

2. On some Ramanujan-type series for $1/\pi$ [2]

Ramanujan's series representations for $1/\pi$ depend upon Clausen's product formulas for hypergeometric series and Ramanujan's Eisenstein series

$$P(q):=1-24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \qquad |q|<1. \quad (2.1)$$

More precisely, but briefly, by combining two different relations between $P(q)$ and $P(q^n)$, for certain positive integers n, along with a Clausen formula, we can obtain series representations for $1/\pi$.

Theorem 1

For $n=5$, if A_k and B_k, $k\geq 0$, are defined by the following formula

$$A_k:=\frac{\left(\frac{1}{2}\right)_k^3}{k!^3}, \quad B_k:=\frac{\left(\frac{1}{4}\right)_k\left(\frac{1}{2}\right)_k\left(\frac{3}{4}\right)_h}{k!^3}, \quad C_k:=\frac{\left(\frac{1}{6}\right)_k\left(\frac{1}{2}\right)_k\left(\frac{5}{6}\right)_k}{k!^3}, \quad (2.1b)$$

then

$$\frac{2}{\pi}=\sum_{k=0}^{\infty}\left\{4\sqrt{5}k+\sqrt{5}-1\right\}A_k\left(\sqrt{5}-2\right)^{2k+1/2}, \quad (2.2)$$

$$\frac{8}{\pi}=\sum_{k=0}^{\infty}(-1)^k(20k+3)B_k\frac{1}{4^k}. \quad (2.3)$$

For $n=13$*, if* A_k *and* B_k*,* $k\geq 0$*, are defined by the formula (2.1b), then*

$$\frac{72}{\pi}=\sum_{k=0}^{\infty}(-1)^k(260k+23)B_k\frac{1}{18^{2k}}. \quad (2.4)$$

The last two identities was recorded by Ramanujan in the fundamental paper "*Modular equations and approximations to π(1914)*".

Proof of (2.2).
From Ramanujan's second notebook (*Notebooks – 2 volumes – 1957*), we see that

$$1+\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-30\sum_{k=1}^{\infty}\frac{kq^{10k}}{1-q^{10k}}=\frac{1}{4\sqrt{2}}\phi^2(q)\phi^2(q^5)\left\{3+\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\right\}\times$$
$$\times\left\{1+\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\right\}^{1/2}. \quad (2.5)$$

With the help of (2.1) we can rewrite (2.4) in the form

$$5P(q^{10})-P(q^2)=\frac{1}{\sqrt{2}}\phi^2(q)\phi^2(q^5)\left\{3+\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\right\}\times$$
$$\times\left\{1+\sqrt{x(q)x(q^5)}+\sqrt{(1-x(q))(1-x(q^5))}\right\}^{1/2}. \quad (2.6)$$

Now we set $q=e^{-\pi/\sqrt{5}}$ and use, following Ramanujan, the following expressions

$$1-x_n=x_{1/n}, \quad z_{1/n}=\sqrt{n}z_n, \quad (2.7)$$
$$z:=z(q):={}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)=\phi^2(q), \quad (2.8)$$

to deduce that $x(q)=x_{1/5}=1-x_5$, $x(q^5)=x_5$, and $\phi^2\left(e^{-\pi/\sqrt{5}}\right)=\sqrt{5}\phi^2\left(e^{-\pi\sqrt{5}}\right)=\sqrt{5}z_5$.
Thus, from (2.6), we find that

$$5P\left(e^{-2\pi\sqrt{5}}\right)-P\left(e^{-2\pi/\sqrt{5}}\right)=\sqrt{5}z_5^2\left\{3+2\sqrt{x_5(1-x_5)}\right\}\sqrt{\frac{1}{2}\left(1+2\sqrt{x_5(1-x_5)}\right)}=$$
$$=\sqrt{5}z_5^2\left\{3+\sqrt{4x_5(1-x_5)}\right\}\sqrt{\frac{1}{2}\left(1+\sqrt{4x_5(1-x_5)}\right)}. \quad (2.9)$$

The eq. (2.9), putting $X_5=4x_5(1-x_5)$, can be rewritten also as follows

$$5P\left(e^{-2\pi\sqrt{5}}\right)-P\left(e^{-2\pi/\sqrt{5}}\right)=\sqrt{5}z_5^2\left\{3+2\sqrt{x_5(1-x_5)}\right\}\sqrt{\frac{1}{2}\left(1+2\sqrt{x_5(1-x_5)}\right)}=$$

$$=\sqrt{5}z_5^2\left\{3+\sqrt{X_5}\right\}\sqrt{\frac{1}{2}\left(1+\sqrt{X_5}\right)}. \quad (2.9b)$$

But, the singular modulus x_5 is given by

$$x_5=\frac{1}{2}-\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)^3}, \quad (2.10)$$

so that

$$X_5=9-4\sqrt{5} \quad \text{and} \quad \sqrt{X_5}=\sqrt{5}-2. \quad (2.11)$$

Thus, from (2.9), we find that

$$5P\left(e^{-2\pi\sqrt{5}}\right)-P\left(e^{-2\pi/\sqrt{5}}\right)=\sqrt{5}\left(\sqrt{5}+1\right)\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)}z_5^2. \quad (2.12)$$

Next, setting $n=5$ in the following equation

$$nP\left(e^{-2\pi\sqrt{n}}\right)+P\left(e^{-2\pi/\sqrt{n}}\right)=\frac{6\sqrt{n}}{\pi}, \quad (2.13)$$

we find that

$$5P\left(e^{-2\pi\sqrt{5}}\right)+P\left(e^{-2\pi/\sqrt{5}}\right)=\frac{6\sqrt{5}}{\pi}. \quad (2.14)$$

Adding (2.12) and (2.14), we deduce that

$$P\left(e^{-2\pi\sqrt{5}}\right)=\frac{3}{\pi\sqrt{5}}+\frac{\sqrt{5}+1}{\sqrt{5}}\cdot\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)}z_5^2. \quad (2.15)$$

Now, employing the following expression

$$z^2={}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;X\right)=\sum_{k=0}^{\infty}A_kX^k, \quad 0\le x\le\frac{1}{2}, \quad (2.16)$$

in (2.15), we deduce the identity

$$P\left(e^{-2\pi\sqrt{5}}\right)=\frac{3}{\pi\sqrt{5}}+\frac{\sqrt{5}+1}{\sqrt{5}}\cdot\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)}\sum_{k=0}^{\infty}A_kX_5^k. \quad (2.17)$$

Next, setting $n=5$ in the following equation

$$P\left(e^{-2\pi\sqrt{n}}\right)=(1-2x_n)\sum_{k=0}^{\infty}(3k+1)A_k X_n^k\ , \quad (2.18)$$

we find that

$$P\left(e^{-2\pi\sqrt{5}}\right)=(1-2x_5)\sum_{k=0}^{\infty}(3k+1)A_k X_5^k=2\sqrt{\left(\sqrt{5}-2\right)}\sum_{k=0}^{\infty}(3k+1)A_k X_5^k\ . \quad (2.19)$$

Using (2.17) and (2.19), we arrive at (2.2).
Proof of (2.3).
Employing the following expression

$$z^2=\frac{1}{1-2x}{}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1;-W^2\right)=\frac{1}{1-2x}\sum_{k=0}^{\infty}(-1)^k B_k W^{2k}\ , \quad 0\le x\le\frac{1}{2}\left(1-2^{1/4}\sqrt{2-\sqrt{2}}\right), \quad (2.20)$$

in (2.15), we find that

$$P\left(e^{-2\pi\sqrt{5}}\right)=\frac{3}{\pi\sqrt{5}}+\frac{\sqrt{5}+1}{\sqrt{5}(1-2x_5)}\cdot\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)}\sum_{k=0}^{\infty}(-1)^k B_k W_5^{2k}=\frac{3}{\pi\sqrt{5}}+\frac{3+\sqrt{5}}{4\sqrt{5}}\sum_{k=0}^{\infty}(-1)^k B_k W_5^{2k}\ , \quad (2.21)$$

where

$$W_5=\frac{2\sqrt{X_5}}{1-X_5}=\frac{1}{2}. \quad (2.22)$$

Next, setting $n=5$ in the following equation

$$P\left(e^{-2\pi\sqrt{n}}\right)=\sum_{k=0}^{\infty}(-1)^k\frac{3k(1+X_n)+1+X_n/2}{1-X_n}B_k W_n^{2k}\ , \quad (2.23)$$

we find that

$$P\left(e^{-2\pi\sqrt{5}}\right)=\sum_{k=0}^{\infty}\frac{3k(1+X_n)+1+X_n/2}{1-X_n}B_k(-1)^k W_n^{2k}=\left(\frac{3\sqrt{5}}{2}k+\frac{3\sqrt{5}+2}{8}\right)(-1)^k B_k W_5^{2k}\ . \quad (2.24)$$

From (2.21) and (2.24), we readily arrive at (2.3). Thus, we complete the proof. The proof of (2.3b) is similar.

2.1 On some equations concerning the p-adic open string for the scalar tachyon field. [3] [4]

As a free action in p-adic field theory one can take the following functional

$$S(f)=\int_{Q_p} fDfdx \quad (2.25)$$

where $f = f(x)$ is a function $f : Q_p \to R$, dx is the Haar measure and D is the Vladimirov operator or its generalizations. Boundary value problems for homogeneous solutions of nonlinear equations of motion corresponding to the p-adic string,

$$e^{\Box}\Phi = \Phi^p . \quad (2.26)$$

Here $\Box$ is the d'Alembert operator and the field Φ and its argument are real-valued.
The dynamics of the open p-adic string for the scalar tachyon field is described by the non-linear pseudodifferential equation

$$p^{\frac{1}{2}\Box}\Phi = \Phi^p , \quad (2.27)$$

where $\Box = \partial_t^2 - \partial_{x_1}^2 - ... - \partial_{x_{d-1}}^2$, $t = x_0$, is the d'Alembert operator and p is a prime number, $p = 2,3,5,...$. In what follows p is any positive integer. We consider only real solutions of equations (2.27), since only real solutions have physical meaning. In the one-dimensional case (d = 1) we use the change

$$\varphi(t) = \Phi\left(t\sqrt{2\ln p}\right)$$

and write equation (2.27) in the following equivalent form:

$$e^{\frac{1}{2}\partial_t^2}\varphi = \varphi^p . \quad (2.28)$$

Equation (2.28) is a non-linear integral equation of the following form:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(t-\tau)^2}\varphi(\tau)d\tau = \varphi^p(t), \quad t \in R . \quad (2.29)$$

Solutions of equation (2.29) are sought in the class of measurable functions $\varphi(t)$ such that

$$|\varphi(t)| \le C\exp\{(1-\varepsilon)t^2\} \quad \text{for any} \quad \varepsilon > 0,\ t \in R . \quad (2.30)$$

The following boundary-value problems for the solutions φ of equation (2.29) have physical meaning:

$$\lim_{t\to-\infty}\varphi(t) = 0, \qquad \lim_{t\to\infty}\varphi(t) = 1 \quad (2.31)$$

if p is even, and

$$\lim_{t\to-\infty}\varphi(t) = -1, \qquad \lim_{t\to\infty}\varphi(t) = 1 \quad (2.32)$$

if p is odd.

Assertion 1

I*f φ is a solution of equation (2.29) such that*

$$\lim_{t\to\infty}\varphi(t) = a, \quad |a| < \infty, \quad (2.33)$$

then $a=0$ *or* $a=1$ *if* p *is even and* $a=0$ *or* $a=\pm 1$ *if* p *is odd,* $\lim_{t\to\infty}(\varphi^p)'(t)=0$. *If* $a\neq 0$, *then* $\lim_{t\to\infty}\varphi'(t)=0$.

We deduce from equation (2.29) the following chain of equalities:

$$\lim_{t\to\infty}\varphi^p(t)=\left[\lim_{t\to\infty}\varphi(t)\right]^p=a^p=\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi(\tau)e^{-(t-\tau)^2}d\tau=\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi(t-u)e^{-u^2}=$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{t\to\infty}\varphi(t-u)e^{-u^2}du=a, \quad (2.34)$$

whence $a=0$ or $a=1$ if p is even and $a=0,\pm 1$ if p is odd. Further, we have

$$\lim_{t\to\infty}(\varphi^p)'(t)=-2\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi(\tau)(t-\tau)e^{-(t-\tau)^2}d\tau=-2\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi(t-u)ue^{-u^2}du=$$

$$=\frac{-2}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{t\to\infty}\varphi(t-u)ue^{-u^2}du=-\frac{2}{\sqrt{\pi}}a\int_{-\infty}^{\infty}ue^{-u^2}du=-\frac{2}{\sqrt{\pi}}a\cdot 0=0. \quad (2.35)$$

If $a\neq 0$, then $\lim_{t\to\infty}\varphi'(t)=0$, since

$$\lim_{t\to\infty}(\varphi^p)'(t)=p\lim_{t\to\infty}\varphi^{p-1}(t)\varphi'(t)=pa^{p-1}\lim_{t\to\infty}\varphi'(t)=0. \quad (2.36)$$

We passed to the limit under the integral sign, using Lebesgue's theorem and estimate (2.30). We shall write $a\equiv b$ if the integers a and b are both even or both odd, and $a\neq b$ if one of them is even and the other is odd.
Hermite polynomials are defined to be the polynomials

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \qquad n=0,1,..., \quad (2.37)$$

whence $H_0(x)=1$, $H_1(x)=2x$, $H_2(x)=4x^2-2$, $H_3(x)=8x^3-12x$,.... They form a complete orthogonal system in the Hilbert space L_2^1, and

$$\|H_n\|_1^2=\int_{-\infty}^{\infty}H_n^2(x)d\mu_1(x)=2^n n!. \quad (2.38)$$

Any $f\in L_2^1$ can be expanded in Hermite polynomials:

$$f(x)=\sum_{n=0}^{\infty}(f,H_n)_1\frac{H_n(x)}{2^n n!} \quad \text{in} \quad L_2^1, \quad (2.39)$$

and the Parseval-Steklov equality holds:

$$\|f\|_1^2=\sum_{n=0}^{\infty}|(f,H_n)_1|^2\frac{1}{2^n n!}. \quad (2.40)$$

The expansion in powers of x has the form

$$H_n(x) = n!\sum_{\substack{m=0 \\ m\equiv n}}^{n} c_{n,m}x^m , \quad n = 0,1,..., \quad (2.41)$$

In particular, we have

$$c_{n,n} = \frac{2^n}{n!}, \quad c_{2n,0} = \frac{(-1)^n}{n!}, \quad c_{2n+1,1} = 2\frac{(-1)^n}{n!}, \quad c_{2n,2} = -2\frac{(-1)^n}{(n-1)!}, \quad c_{2n+1,3} = -\frac{4(-1)^n}{3(n-1)!}. \quad (2.42)$$

The integral representation for the modified Hermite polynomials has the form:

$$V_n(x) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} H_n(\tau)e^{-2(t/2-\tau)^2}d\tau , \quad n = 0,1,.... \quad (2.43)$$

Let $f \in L_2^{1/2}$. It follows from (2.28) that $f \in L_2^1$,

$$\sum_{n=0}^{\infty} a_n \frac{H_n(x)}{2^n n!} = f(x) = \sum_{n=0}^{\infty} b_n \frac{V_n(x)}{n!} \quad \text{in} \quad L_2^1. \quad (2.44)$$

We denote by K the linear integral operator in equation (2.29):

$$\varphi \to (K\varphi)(t) \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(t-\tau)^2}\varphi(\tau)d\tau . \quad (2.45)$$

Lemma 1

The operator K assigns to every function $f(t)$ satisfying condition (2.30) an entire function $(Kf)(z)$ with the estimate

$$|(Kf)(z)| \le \frac{C}{\sqrt{\varepsilon}}\exp\left\{y^2 + \left(\frac{1}{\varepsilon} - 1\right)t^2\right\}, \quad z = t + iy . \quad (2.46)$$

The proof follows immediately from (2.30):

$$|(Kf)(z)| \le \frac{1}{\sqrt{\pi}} C\int_{-\infty}^{\infty} e^{(1-\varepsilon)\tau^2}\left|e^{-(t-\tau)^2}\right|d\tau = \frac{C}{\sqrt{\pi}}e^{y^2-t^2}\int_{-\infty}^{\infty} e^{-\varepsilon\tau^2+2t\tau}d\tau = \frac{C}{\sqrt{\varepsilon}}\exp\left\{y^2 + \left(\frac{1}{\varepsilon} - 1\right)t^2\right\}. \quad (2.47)$$

Lemma 2

The operator K assigns to $f \in L_2^{\alpha}$, $0 < \alpha < 2$, an entire function $(Kf)(z)$ with the estimate

$$|(Kf)(z)| \le \|f\|_{\alpha}(2-\alpha)^{-1/4}\exp\left\{y^2 + \frac{\alpha}{2-\alpha}t^2\right\}, \quad z = t + iy . \quad (2.48)$$

The proof follows from the Cauchy-Bunyakovskii inequality (applied to $(Kf)(z)$) and the following estimates:

$$|(Kf)(z)| \le \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |f(\tau)| e^{-\alpha\tau^2/2} \left| \exp\left\{ -z^2 + 2z\tau - \left(1 - \frac{\alpha}{2}\right)\tau^2 \right\} \right| d\tau \le$$

$$\le \|f\|_\alpha \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left| \exp\{2z^2 + 4t\tau - (2-\alpha)\tau^2\} \right| d\tau \right]^{1/2} \le \|f\|_\alpha \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\{2y^2 - 2t^2 + 4t\tau - (2-\alpha)\tau^2\} d\tau \right]^{1/2} =$$

$$= \|f\|_\alpha \exp\left\{ y^2 + \frac{\alpha}{2-\alpha} t^2 \right\} \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \exp\left\{ (2-\alpha)\left(\tau - \frac{2t}{2-\alpha} \right)^2 \right\} d\tau \right]^{1/2} . \quad (2.49)$$

We can rewrite the above equation also as follows:

$$|(Kf)(z)| \le \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} |f(\tau)| e^{-\alpha\tau^2/2} \left| \exp\left\{ -z^2 + 2z\tau - \left(1 - \frac{\alpha}{2}\right)\tau^2 \right\} \right| d\tau =$$

$$= \|f\|_\alpha \exp\left\{ y^2 + \frac{\alpha}{2-\alpha} t^2 \right\} \left[\frac{1}{\pi} \int_{-\infty}^{\infty} \exp\left\{ (2-\alpha)\left(\tau - \frac{2t}{2-\alpha} \right)^2 \right\} d\tau \right]^{1/2} . \quad (2.49b)$$

Lemma 3

The operator $K : L_2^\alpha \to L_2^\beta$, $0 < \alpha < 2$, $\beta > \dfrac{2\alpha}{2-\alpha}$ *, is bounded, and*

$$\|Kf\|_\beta \le \left(2\alpha - \frac{2\alpha^2}{\beta} - \alpha^2 \right)^{-1/4} \|f\|_\alpha , \quad f \in L_2^\alpha . \quad (2.50)$$

We prove the lemma by writing the following chain of equalities and inequalities for $f \in L_2^\alpha$:

$$\|Kf\|_\beta^2 = \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} e^{-\beta t^2} |(Kf)(t)|^2 dt = \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} e^{-(\beta+2)t^2} \left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-(\alpha/2)\tau^2 - (1-\alpha/2)\tau^2 + 2t\tau} d\tau \right|^2 dt \le$$

$$\le \frac{1}{\pi} \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} e^{-(\beta+2)t^2} \int_{-\infty}^{\infty} |f(\tau)|^2 e^{-\alpha\tau^2} d\tau \int_{-\infty}^{\infty} e^{-(2-\alpha)\tau^2 + 4t\tau} d\tau dt =$$

$$= \frac{1}{\pi} \sqrt{\frac{\beta}{\alpha}} \|f\|_\alpha^2 \int_{-\infty}^{\infty} e^{-\left(\beta + 2 - \frac{4}{2-\alpha}\right)t^2} dt \int_{-\infty}^{\infty} e^{-(2-\alpha)\tau^2} d\tau = (2\beta - 2\alpha - \alpha\beta)^{-1/2} \sqrt{\frac{\beta}{\alpha}} \|f\|_\alpha^2 . \quad (2.51)$$

This equation can be rewritten also as follows:

$$\|Kf\|_\beta^2 = \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} e^{-\beta t^2} |(Kf)(t)|^2 dt = \frac{1}{\pi} \sqrt{\frac{\beta}{\alpha}} \|f\|_\alpha^2 \int_{-\infty}^{\infty} e^{-\left(\beta + 2 - \frac{4}{2-\alpha}\right)t^2} dt \int_{-\infty}^{\infty} e^{-(2-\alpha)\tau^2} d\tau . \quad (2.51b)$$

Lemma 4

If $f \in L_2^1$, then its image $(Kf)(t)$ can be expanded in the Taylor series

$$(Kf)(t)=\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_n=(f,H_n)_1, \quad (2.52)$$

which converges uniformly on every compact set in R. If $f \in L_2^{1/2}$, then

$$(Kf)(t)=\sum_{n=0}^{\infty} b_n \frac{H_n(t)}{2^n n!} \quad in \quad L_2^1, \qquad b_n=(Kf,H_n)_1, \quad (2.53)$$

and

$$(Kf,H_n)_1=(f,V_n)_{1/2}, \quad n=0,1,\ldots. \quad (2.54)$$

By lemma 3, the function $(Kf)(t)$ is the trace of an entire function $(Kf)(z)$ for $y=0$. Hence, it can be expanded in the Taylor series with the coefficients

$$\frac{d^n}{dt^n}(Kf)(t)\big|_{t=0}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(\tau)\frac{d^n}{dt^n}e^{-(t-\tau)^2}d\tau\big|_{t=0}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(\tau)(-1)^n e^{-(t-\tau)^2}H_n(t-\tau)d\tau\big|_{t=0}=$$
$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(\tau)(-1)^n H_n(-\tau)e^{-t^2}d\tau=(f,H_n)_1=a_n. \quad (2.55)$$

Here we used equality (2.37). Further, if $f \in L_2^{1/2}$, then (2.53) holds by (2.44), since $Kf \in L_2^1$ by Lemma 3. Equalities (2.54) can be proved as follows:

$$(Kf,H_n)_1=(f,K^*H_n)_{1/2}=(f,V_n)_{1/2}. \quad (2.56)$$

Here we used formula (2.43), which implies that $V_n=K^*H_n$, where K^* is the operator adjoint to K.

Let φ be a solution of equation (2.29) belonging to L_2^1, whence $\varphi^p=K\varphi$. Putting $a_n=(\varphi,H_n)_1$, we deduce from (2.39) and (2.40) that

$$\varphi(t)=\sum_{n=0}^{\infty} a_n \frac{H_n(t)}{2^n n!} \quad \text{in} \quad L_2^1, \qquad \sum_{n=0}^{\infty}\frac{a_n^2}{2^n n!}=\|\varphi\|^2. \quad (2.57)$$

The function $\varphi^p(t)$ is the trace of the entire function $A(z)=(K\varphi)(z)$, for which (2.48) holds with $\alpha=1$:

$$|A(z)|\le\|\varphi\|_1 e^{|z|^2}, \quad z\in C. \quad (2.58)$$

By Lemma 4, it can be expanded in the Taylor series (2.52):

$$\varphi^p(t)=\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}. \quad (2.59)$$

The integral equation (2.29) is equivalent to the following boundary-value problem for the heat equation:

$$u_x = \frac{1}{4} u_{tt}, \quad 0 < x \le 1, \quad t \in R, \quad (2.60)$$

$$u(0,t) = \varphi(t), \quad u(1,t) = \varphi^p(t), \quad t \in R. \quad (2.61)$$

Let us note that if there is an interpolating function, it can be represented by Poisson's formula for equation (2.60):

$$u(x,t) = \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \varphi(\tau) \exp\left\{ -\frac{(t-\tau)^2}{x} \right\} d\tau, \quad 0 < x \le 1. \quad (2.62)$$

If φ such that

$$|\varphi(t)| \le C \exp\{\varepsilon t^2\} \text{ for any } \varepsilon > 0, \quad t \in R, \quad (2.63)$$

then formula (2.62) gives its analytic continuation to the domain $x > 1$, $t \in R$ and, further, its analytic continuation with respect to (x,t) to the complex domain $T^+ \times C$, where T^+ is the right half-plane $\operatorname{Re}\zeta = x > 0$.
Equation (2.29) takes the form

$$\varphi^{2q}(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(\tau) e^{-(t-\tau)^2} d\tau, \quad q = 1,2,\ldots. \quad (2.64)$$

If $\varphi(t)$ is a solution of equation (2.64), then $\varphi(-t)$ and $\varphi(t+t_0)$ also are solutions of this equation (for all t_0).

Assertion 2

If $\varphi(t)$ is a solution of equation (2.64) such that (2.63) holds, then

$$\frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \varphi^{2q}(\tau) \exp\left\{ -\frac{(t-\tau)^2}{x} \right\} d\tau \le x^{\frac{q}{2q-1}} (1+x)^{-\frac{1}{4q-2}} \sqrt{\frac{2q-1}{2qx - x - 1}} \quad (2.65)$$

for all $x > 1/(2q-1)$.

We remember that if S is a measurable subset of R^n with the Lebesgue measure, and f and g are measurable real- or complex-valued functions on S, then Hölder inequality is

$$\int_S |f(x) g(x)| dx \le \left(\int_S |f(x)|^p dx \right)^{1/p} \left(\int_S |g(x)|^q dx \right)^{1/q}. \quad (2.65b)$$

Denoting the left-hand side of inequality (2.65) by $J(x,t)$, using the boundary conditions (2.61), the properties of solutions of the heat equation and Holder's inequality, we obtain the following chain of relations for all $x > 1/(2q-1)$:

$$J(x,t) \equiv \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \varphi^{2q}(\tau) \exp\left\{-\frac{(t-\tau)^2}{x}\right\} d\tau = \frac{1}{\sqrt{\pi(1+x)}} \int_{-\infty}^{\infty} \varphi(\tau) \exp\left\{-\frac{(t-\tau)^2}{1+x}\right\} d\tau \leq$$

$$\leq \frac{1}{\sqrt{\pi(1+x)}} \int_{-\infty}^{\infty} \varphi(\tau) \exp\left\{-\frac{(t-\tau)^2}{2qx}\right\} \exp\left\{-\frac{(t-\tau)^2(2qx-x-1)}{(1+x)2qx}\right\} d\tau =$$

$$= \frac{1}{\sqrt{\pi(1+x)}} \left[\sqrt{\pi t} J\right]^{\frac{1}{2q}} \left(\int_{-\infty}^{\infty} \exp\left\{-\frac{(t-\tau)^2(2qx-x-1)}{x(1+x)(2q-1)}\right\} d\tau \right)^{1-\frac{1}{2q}}, \quad (2.66)$$

whence

$$J^{1-\frac{1}{2q}} \leq \frac{(\pi x)^{\frac{1}{4q}}}{\sqrt{\pi x(1+x)}} \left(\sqrt{\frac{\pi x(1+x)(2q-1)}{2qx-x-1}} \right)^{1-\frac{1}{2q}}, \quad (2.67)$$

which implies that (2.65) holds.
Now we can rewrite the eq. (2.65) also as follows:

$$J^{1-\frac{1}{2q}} \leq \frac{(\pi x)^{\frac{1}{4q}}}{\sqrt{\pi x(1+x)}} \left(\sqrt{\frac{\pi x(1+x)(2q-1)}{2qx-x-1}} \right)^{1-\frac{1}{2q}} \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \varphi^{2q}(\tau) \exp\left\{-\frac{(t-\tau)^2}{x}\right\} d\tau \leq x^{\frac{q}{2q-1}} (1+x)^{-\frac{1}{4q-2}} \sqrt{\frac{2q-1}{2qx-x-1}}. \quad (2.67b)$$

Corollary

For $x=1$ *estimate (2.65) with* $q=2,3,\ldots$ *takes the form*

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi^{2q}(\tau) \exp\left\{-(t-\tau)^2\right\} d\tau \leq 2^{-\frac{1}{4q-2}} \sqrt{\frac{2q-1}{2q-2}}. \quad (2.68)$$

With regard the possible mathematical connections, we note that it is possible to obtain some interesting and new relationships evidencing some Ramanujan's Theorems. The first letter of Ramanujan to G. H. Hardy, contain the bare statements of about 120 theorems, mostly formal identities extracted from his note-books. We take, for the connections regarding this section, the following identities:

$$\int_0^{\infty} \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \ldots dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}, \quad (2.69)$$

$$\int_0^{\infty} \frac{1}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots} dx = \pi \frac{1}{2(1+r+r^3+r^6+r^{10}+\ldots)}, \quad (2.70)$$

If $\alpha\beta = \pi^2$, then

$$\alpha^{-1/4}\left(1+4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}dx\right) = \beta^{-1/4}\left(1+4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}dx\right), \quad (2.71)$$

$$\frac{1}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+\ldots} = \left\{\sqrt{\left(\frac{5+\sqrt{5}}{2}\right)} - \frac{\sqrt{5}+1}{2}\right\}e^{2\pi/5}, \quad (2.72)$$

$$\frac{1}{1+}\frac{e^{-2\pi\sqrt{5}}}{1+}\frac{e^{-4\pi\sqrt{5}}}{1+\ldots} = \left[\frac{\sqrt{5}}{1+\sqrt[5]{\left\{5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1\right\}}} - \frac{\sqrt{5}+1}{2}\right]e^{2\pi/\sqrt{5}}. \quad (2.73)$$

We note that the eqs. (2.49b), (2.51b) and (2.68) can be related with the expression (2.69) and (2.70). Indeed, we have:

$$|(Kf)(z)| \le \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}|f(\tau)|e^{-\alpha\tau^2/2}\left|\exp\left\{-z^2+2z\tau-\left(1-\frac{\alpha}{2}\right)\tau^2\right\}\right|d\tau =$$

$$= \|f\|_\alpha \exp\left\{y^2+\frac{\alpha}{2-\alpha}t^2\right\}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\exp\left\{(2-\alpha)\left(\tau-\frac{2t}{2-\alpha}\right)^2\right\}d\tau\right]^{1/2} \rightarrow$$

$$\rightarrow \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}, \quad (2.74)$$

$$|(Kf)(z)| \le \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}|f(\tau)|e^{-\alpha\tau^2/2}\left|\exp\left\{-z^2+2z\tau-\left(1-\frac{\alpha}{2}\right)\tau^2\right\}\right|d\tau =$$

$$= \|f\|_\alpha \exp\left\{y^2+\frac{\alpha}{2-\alpha}t^2\right\}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\exp\left\{(2-\alpha)\left(\tau-\frac{2t}{2-\alpha}\right)^2\right\}d\tau\right]^{1/2} \rightarrow$$

$$\rightarrow \int_0^\infty \frac{1}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots}dx = \pi\frac{1}{2(1+r+r^3+r^6+r^{10}+\ldots)}, \quad (2.75)$$

$$\|Kf\|_\beta^2 = \sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty}e^{-\beta t^2}|(Kf)(t)|^2 dt = \frac{1}{\pi}\sqrt{\frac{\beta}{\alpha}}\|f\|_\alpha^2\int_{-\infty}^{\infty}e^{-\left(\beta+2-\frac{4}{2-\alpha}\right)t^2}dt\int_{-\infty}^{\infty}e^{-(2-\alpha)\tau^2}d\tau \rightarrow$$

$$\rightarrow \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}, \quad (2.76)$$

$$\|Kf\|_\beta^2 = \sqrt{\frac{\beta}{\pi}}\int_{-\infty}^{\infty} e^{-\beta t^2}\left|(Kf)(t)\right|^2 dt = \frac{1}{\pi}\sqrt{\frac{\beta}{\alpha}}\|f\|_\alpha^2 \int_{-\infty}^{\infty} e^{-\left(\beta+2-\frac{4}{2-\alpha}\right)t^2} dt \int_{-\infty}^{\infty} e^{-(2-\alpha)\tau^2} d\tau \rightarrow$$

$$\rightarrow \int_0^\infty \frac{1}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots} dx = \pi \frac{1}{2(1+r+r^3+r^6+r^{10}+\ldots)}, \quad (2.77)$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \varphi^{2q}(\tau)\exp\left\{-(t-\tau)^2\right\}d\tau \le 2^{-\frac{1}{4q-2}}\sqrt{\frac{2q-1}{2q-2}} \rightarrow$$

$$\rightarrow \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \ldots dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}, \quad (2.78)$$

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \varphi^{2q}(\tau)\exp\left\{-(t-\tau)^2\right\}d\tau \le 2^{-\frac{1}{4q-2}}\sqrt{\frac{2q-1}{2q-2}} \rightarrow$$

$$\rightarrow \int_0^\infty \frac{1}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots} dx = \pi \frac{1}{2(1+r+r^3+r^6+r^{10}+\ldots)}. \quad (2.79)$$

3. On some equations concerning the zeta strings and the zeta nonlocal scalar fields [5]

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (3.1)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq 1}C_n\mathcal{L}_n=\sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (3.2)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1. \quad (3.3)$$

Employing usual expansion for the logarithmic function and definition (3.3) we can rewrite (3.2) in the form

$$L=-\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi+\phi+\ln(1-\phi)\right], \quad (3.4)$$

where $|\phi|<1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2=k_0^2-\vec{k}^2>2+\varepsilon, \quad (3.5)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi} \quad (3.6)$$

which has an evident solution $\phi=0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (3.7)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (3.8)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq 1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (3.9)$$

and one can easily see trivial solution $\phi=\theta=0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (3.10)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (3.10) with p replaced by $n \in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\left[-\frac{1}{2}\phi n^{-\frac{\Box}{2m_n^2}}\phi + \frac{1}{n+1}\phi^{n+1}\right], \quad (3.11)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n = \frac{n-1}{n^{2+h}}, \quad (3.12)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h}\phi + \sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right] \quad (3.13)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h} = \prod_p \frac{1}{1-p^{-\frac{\Box}{2m^2}-h}}. \quad (3.14)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (3.15)$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (3.15) we can rewrite (3.13) in the form

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2} + h\right)\phi + \sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right]. \quad (3.16)$$

Here $\zeta\left(\frac{\Box}{2m^2}+h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk\,, \quad (3.17)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.
We consider Lagrangian (3.16) with analytic continuations of the zeta function and the power series $\sum\frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi+AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right], \quad (3.18)$$

where AC denotes analytic continuation.
Potential of the above zeta scalar field (3.18) is equal to $-L_h$ at $\Box=0$, i.e.

$$V_h(\phi)=\frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h)-AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (3.19)$$

where $h\neq 1$ since $\zeta(1)=\infty$. The term with ζ-function vanishes at $h=-2,-4,-6,\ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi=AC\sum_{n=1}^{+\infty}n^{-h}\varphi^n\,, \quad (3.20)$$

$$\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk=AC\sum_{n=1}^{+\infty}n^{-h}\phi^n\,, \quad (3.21)$$

respectively.
Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (3.18): $h=0$, $h=\pm 1$, and $h=\pm 2$. For $h=-2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2}-2\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-2\right)\tilde{\phi}(k)dk=\frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (3.22)$$

This equation has two trivial solutions: $\phi(x)=0$ and $\phi(x)=-1$. Solution $\phi(x)=-1$ can be also shown taking $\tilde{\phi}(k)=-\delta(k)(2\pi)^D$ and $\zeta(-2)=0$ in (3.22).
For $h=-1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2}-1\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk=\frac{\phi}{(1-\phi)^2}. \quad (3.23)$$

where $\zeta(-1)=-\frac{1}{12}$.
The equation of motion (3.23) has a constant trivial solution only for $\phi(x)=0$.
For $h=0$, the equation of motion is

$$\zeta\left(\frac{\square}{2m^2}\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (3.24)$$

It has two solutions: $\phi = 0$ and $\phi = 3$. The solution $\phi = 3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\square}{2m^2}\right) = \zeta(0) + \sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\square}{2m^2}\right)^n, \quad (3.25)$$

as well as from $\tilde{\phi}(k) = (2\pi)^D 3\delta(k)$.
For $h = 1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+1\right)\tilde{\phi}(k)dk = -\frac{1}{2}\ln(1-\phi)^2, \quad (3.26)$$

where $\zeta(1) = \infty$ gives $V_1(\phi) = \infty$.
In conclusion, for $h = 2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk = -\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw. \quad (3.27)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2)$$

one has trivial solution $\phi = 1$ in (3.27).

Now, we want to analyze the following case: $C_n = \frac{n^2-1}{n^2}$. In this case, from the Lagrangian (3.11), we obtain:

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)\right\}\phi + \frac{\phi^2}{1-\phi}\right]. \quad (3.28)$$

The corresponding potential is:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2. \quad (3.29)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n\pm 1$ with n=1 and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore, the number 24 is related to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 \Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (3.29b)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right]\phi = \frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2}. \quad (3.30)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-2\right]\phi = 0, \quad (3.31)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=2. \quad (3.32)$$

From (3.32) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79m^2$ and many tachyon solutions when $M^2 < -38m^2$.

We note that the number 2.79 is connected with $\phi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$, i.e. the "aurea" section and the "aurea" ratio. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 + \frac{1}{2^2}\left(\frac{\sqrt{5}-1}{2}\right) = 2,772542 \cong 2,78.$$

Furthermore, we have also that:

$$(\Phi)^{14/7} + (\Phi)^{-25/7} = 2,618033989 + 0,179314566 = 2,79734$$

4. Mathematical connections

We have the following new possible mathematical connections: between eqs. (1.7), (1.17) and (1.19b) with eq. (3.6)

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + 2\int_0^\infty \left(1+t^2\right)^{-\frac{s}{2}} \sin(s\arctan t)\frac{dt}{e^{2\pi t}-1} \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\widetilde{\phi}(k)dk = \frac{\phi}{1-\phi}, \quad (4.1)$$

$$\zeta'(0) = 2\int_0^\infty \arctan t \frac{1}{e^{2\pi t}-1}dt - 1 = -\log\sqrt{2\pi} \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\widetilde{\phi}(k)dk = \frac{\phi}{1-\phi}, \quad (4.2)$$

$$\zeta'(0) = 2\int_0^\infty \arctan t \frac{1}{e^{2\pi t}-1}dt - 1 = -\log\sqrt{2\left(\arccos\phi \cdot \frac{1}{0,2879}\right)} \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\widetilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (4.3)$$

We have further possible mathematical connections between eqs. (1.56) and (1.64) with eq. (3.6):

$$-\frac{1}{\sqrt{\pi^3}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt = -\frac{2\pi}{\sqrt{\alpha}}\int_0^\infty \left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}\right)dt =$$

$$= \int_0^\infty \left(\frac{-2\pi/\sqrt{\alpha}}{\left(e^{2\pi t/\alpha}-1\right)\left(e^{2\pi t}-1\right)} + \frac{\sqrt{\alpha}}{t\left(e^{2\pi t}-1\right)} + \frac{1}{t\sqrt{\alpha}\left(e^{2\pi t/\alpha}-1\right)} - \frac{\sqrt{\alpha}}{2\pi t^2}\right)dt \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\widetilde{\phi}(k)dk = \frac{\phi}{1-\phi}, \quad (4.4)$$

$$\sqrt{\beta}\left\{\frac{\gamma-\log(2\pi\beta)}{2\beta}+\sum_{n=1}^\infty \phi(n\beta)\right\} = -\frac{1}{\sqrt{\pi^3}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\beta\right)}{1+t^2}dt =$$

$$= -\frac{1}{\sqrt{\pi^3}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log(\alpha)\right)}{1+t^2}dt \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\widetilde{\phi}(k)dk = \frac{\phi}{1-\phi}, \quad (4.5)$$

With regard π, we have these possible mathematical connections between eqs (4.4) and (4.5) with some Ramanujan's equations ((2.69) and (2.70)). Indeed:

$$-\frac{1}{\sqrt{\pi^3}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt = -\frac{2\pi}{\sqrt{\alpha}}\int_0^\infty \left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}\right)dt =$$

$$= \int_0^\infty \left(\frac{-2\pi/\sqrt{\alpha}}{\left(e^{2\pi t/\alpha}-1\right)\left(e^{2\pi t}-1\right)} + \frac{\sqrt{\alpha}}{t\left(e^{2\pi t}-1\right)} + \frac{1}{t\sqrt{\alpha}\left(e^{2\pi t/\alpha}-1\right)} - \frac{\sqrt{\alpha}}{2\pi t^2}\right)dt \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \rightarrow$$

$$\rightarrow \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma(b+1)\Gamma\left(b-a+\frac{1}{2}\right)}{\Gamma(a)\Gamma\left(b+\frac{1}{2}\right)\Gamma(b-a+1)}, \quad (4.6)$$

$$-\frac{1}{\sqrt{\pi^3}}\int_0^\infty \left|\Xi\left(\frac{1}{2}t\right)\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1+t^2}dt = -\frac{2\pi}{\sqrt{\alpha}}\int_0^\infty\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)\left(\frac{1}{e^{2\pi t/\alpha}-1}-\frac{\alpha}{2\pi t}\right)dt =$$

$$= \int_0^\infty\left(\frac{-2\pi/\sqrt{\alpha}}{\left(e^{2\pi t/\alpha}-1\right)\left(e^{2\pi t}-1\right)}+\frac{\sqrt{\alpha}}{t\left(e^{2\pi t}-1\right)}+\frac{1}{t\sqrt{\alpha}\left(e^{2\pi t/\alpha}-1\right)}-\frac{\sqrt{\alpha}}{2\pi t^2}\right)dt \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \rightarrow$$

$$\rightarrow \int_0^\infty \frac{1}{\left(1+x^2\right)\left(1+r^2x^2\right)\left(1+r^4x^2\right)\ldots}dx = \pi\frac{1}{2\left(1+r+r^3+r^6+r^{10}+\ldots\right)}. \quad (4.7)$$

We have also some possible mathematical connections with the eqs. (1.19b), (1.34), (1.36) and eq. (3.6). Indeed:

$$\zeta'(0) = 2\int_0^\infty \arctan t\frac{1}{e^{2\pi t}-1}dt - 1 = -\log\sqrt{2\left(\arccos\phi\cdot\frac{1}{0,2879}\right)} \rightarrow$$

$$\rightarrow 5^{3/4}\int_0^q \frac{f^2(-t)f^2\left(-t^5\right)}{\sqrt{t}}dt = 2\int_{\cos^{-1}\left((\varepsilon u)^{5/2}\right)}^{\pi/2}\frac{1}{\sqrt{1-\varepsilon^{-5}5^{-3/2}\sin^2\varphi}}d\varphi =$$

$$= \int_0^{2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3\left(-q^5\right)/f^3(-q)\right)}\frac{1}{\sqrt{1-\varepsilon^{-5}5^{-3/2}\sin^2\varphi}}d\varphi = \sqrt{5}\int_0^{2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi\left(q^5\right)/\psi(q)\right)}\frac{1}{\sqrt{1-\varepsilon 5^{-1/2}\sin^2\varphi}}d\varphi \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}, \quad (4.8)$$

$$\zeta'(0) = 2\int_0^\infty \arctan t\frac{1}{e^{2\pi t}-1}dt - 1 = -\log\sqrt{2\left(\arccos\phi\cdot\frac{1}{0,2879}\right)} \rightarrow$$

$$\rightarrow \int_0^q f(-t)f\left(-t^3\right)f\left(-t^5\right)f\left(-t^{15}\right)dt = \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}\left(1/\sqrt{5}\right)}\frac{1}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}d\varphi =$$

$$= \frac{1}{9}\int_{2\tan^{-1}\left(\frac{1-v\varepsilon^{-3}}{1+v\varepsilon^3}\sqrt{\frac{(1+v\varepsilon)\left(1-v\varepsilon^5\right)}{\left(1-v\varepsilon^{-1}\right)\left(1+v\varepsilon^{-5}\right)}}\right)}^{\pi/2}\frac{1}{\sqrt{1-\frac{1}{81}\sin^2\varphi}}d\varphi = \frac{1}{4}\int_{\tan^{-1}\left(\left(3-\sqrt{5}\right)\sqrt{\frac{\left(1-v\varepsilon^{-1}\right)\left(1-v\varepsilon^5\right)}{(1+v\varepsilon)\left(1+v\varepsilon^{-5}\right)}}\right)}^{\tan^{-1}\left(3-\sqrt{5}\right)}\frac{1}{\sqrt{1-\frac{15}{16}\sin^2\varphi}}d\varphi \rightarrow$$

$$\rightarrow \zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}. \quad (4.9)$$

In conclusion, we have also some possible mathematical connections between the eqs. (1.32) and (1.39) with the following equation $\frac{\pi}{3}=\frac{8}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]$, related to π and with the physical vibrations of the superstrings. Thence, we have:

$$\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q}=\log(u^2v^3)+\sqrt{5}\log\left(\frac{1+(\sqrt{5}-2)uv^2}{1-(\sqrt{5}+2)uv^2}\right)\rightarrow$$

$$\rightarrow\frac{\pi}{3}=\frac{8}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right], \quad (4.10)$$

$$\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt=2^{-7/4}\int_{\cos^{-1}\left\{c\frac{1-2\sqrt{2}x}{1+2\sqrt{2}x+1}\right\}}^{\cos^{-1}c}\frac{1}{\sqrt{1-\frac{32-13\sqrt{2}}{64}\sin^2\varphi}}d\varphi\rightarrow$$

$$\rightarrow\frac{\pi}{3}=\frac{8}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \quad (4.11)$$

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