Continued fractions and the Riemann zeta: connections with string theory

Rosario Turco, Maria Colonnese, Michele Nardelli[1,2]

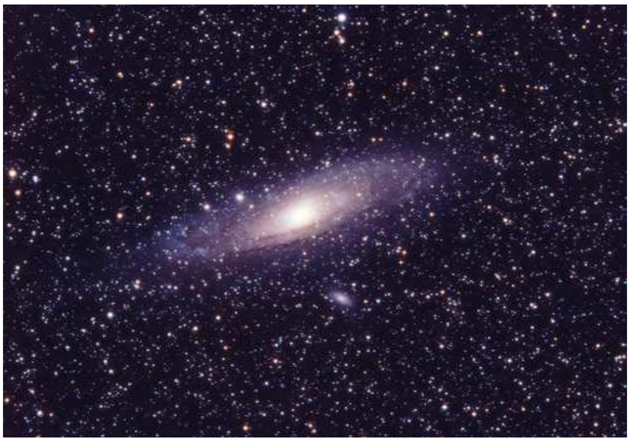
[1]Dipartimento di Scienze della Terra
Università degli Studi di Napoli Federico II, Largo S. Marcellino, 10
80138 Napoli, Italy

[2]Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"
Università degli Studi di Napoli "Federico II" – Polo delle Scienze e delle Tecnologie
Monte S. Angelo, Via Cintia (Fuorigrotta), 80126 Napoli, Italy

Abstract

Physics and astrophysics owe much to Mathematics: knowledge of the universe today would be impossible without it.

What is surprising every day? The simplicity of the physical and mathematical models that Nature uses.



Also behind elementary mathematics topics as continued fractions are hidden problems with greater complexity.

The design of Nature is as if it were conceived in "bottom up" from the small elementary brick, until completion of the "cathedrals of the universe."

It was discovered relatively recently that other scientific fields such as medicine, bioengineering, music, economics, etc., can draw on mathematical models of number theory.

The authors in this article show how, starting from the simple continued fractions, one can reach the most advanced theories of physics, as the connections between the prime numbers and the strings adic, adelic and zeta-strings, furthermore the connections

between the mathematics of the fractals and the golden number.

In particular the areas examined in the following are: "zeta non-local scalar fields", "Lagrangians with Riemann zeta functions" and "Lagrangians for adelic strings.

1. Continued Fractions

Continued fractions are of the type:

$$\alpha = \frac{a}{b} = ao + \cfrac{1}{a1 + \cfrac{1}{a2 + \cfrac{1}{a3 + \cfrac{1}{a4 + \ldots}}}}$$

Each number $\alpha \in P$ can be expressed in this form, with generation finite or infinite depending on whether it is rational or irrational.

For example:
α=116/43; with the Euclidean algorithm we obtain

116 = 2 × 43 + 30

Dividend for 43:

116/43 = 2 + 30/43 = 2 + 1/43/30

so:

43 = 1 × 30 + 13

Dividend for 30:

43/30 = 1 + 13/30 = 1 + 1/30/13

If we continue until the Euclidean algorithm gives no rest, we obtain that:

$$\frac{116}{43} = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}}} = [2,1,2,3,4]$$

Euclid's algorithm allows to find the gcd (a, b) without knowing either the divisors of a, nor those of b.

Semi-prime numbers: factorization and periodic expansions

Among the most curious and funny factorizations of a Semiprime N = p*q, there is the technique of "periodic expansion in base 10 of the fraction 1/N".

For example: N = 1517. It takes three steps to the factorization of semi-prime.

Step 1: find the length of the period T(1/N) of the fraction 1/N

1/N=0.**000659195781147**000659195781147

where for simplicity we have marked in bold red only the "regular part" of the fraction 1/N. If we count the digits of this period (bold red) we obtain that T(1/N) = 15.

Step 2: factorization of the length of the period of the fraction 1/N

We factor a number that is lower than that of departure, especially now we make the factorization of the length of the period T(1/N) = 15 = 3*5. Same result is obtained by PARI/GP with factor (15)), where k1 = 3 and k2 = 5.

Step 3*: find the gcd of N with 10^k1-1 and N with 10^k2-1*

GCD(1517,10^3-1)=37=p
GCD(1517,10^5-1)=41=q

So N=1517=37*41.

Let the counter-test method above. We know that N = 1517 = 37*41.

Now try for p = 37 and q = 41 the smallest values k1 and k2 such that are entire the quantity:

(10^k1-1)/p=3
(10^k2-1)/q=5

What are K1 and K2? These are the length of the periods of expansions 1/p and 1/q.

As it is shown that all right? Meanwhile remember that lcm is the least common multiple (lcm) and LCM is the product of least common multiple in the game; also denote by T the period of a fraction.

If N = p*q is then: T(N) = LCM (T(p), T(q))

If we let g = GCD [T(p), T(q)] then T(N) = T(p)T(q)/g.

So for some factorization T(N) = abg we obtain:

T(p) = ag e T(q) = bg

and in conclusion:

p = GCD [N, 10^ag - 1]
q = GCD [N, 10^bg - 1]

In all this we have used the base B = 10 but in the proof instead of 10 we could put a general B:

p = GCD [N, B^ag - 1]
q = GCD [N, B^bg - 1]

It’s nice! It is a method that is good and easily to apply to a semi-prime number; while things get more complicated if the number is not semi-prime or if the period is unable to locate.

If N is semi-prime for example in PARI/GP we can know it with the function bigomega (N): If N is semi-prime in fact it returns 2. The bigomega gives the number of prime factors even if repeated.

There is always a period? No, not always. If N is prime, 1/N is periodic, except in the case N = 2 and N = 5. If N is semi-prime N = p*q, then 1/N is not periodic if p = q or p and q are prime factors equal to 2 or powers of 2 or equal to 5 or powers of 5 or products of powers of 2 and 5. Obviously, the result could also give an irrational or no period.

Difficulties:

- In PARI/GP there isn’t a default function (built-in) that identifies the period: we need to write an algorithm for T (1 / N);
- Not always the fraction is periodic;
- For N large, PARI/GP returns the value in exponential notation (example of exponential notation: 23 E-21)

For point 1 to search for the period of a fraction, as seen from above, is to research "the smallest value k such that B^k = 1 mod N"; but an exhaustive search of this is not always faster than a "Trial Division". However, in PARI/GP there exist an easy way to place anyway under an algorithm, for example, we know that sqrt (21) = 4,1,1,2,1,8, where the underlined part is the period T (sqrt (21)).

If we use contfrac (sqrt (21)) we obtain the vector

[4,1,1,2,1,8,1,1,2,1,8,1,…]

Now, if a1 is the first element of the vector and j is the j-th element of the vector, then the period T (sqrt (21)) = j-1 if aj = 2*a1. Obviously the method is less applicable if a1 = 0, i.e. the fraction is not periodic but is finite or infinite (irrational number).

The example above was a case of factorization trivial.

Let us examine, instead, N = 66167 T(66167) = 1092.

Factorization of 1092 is 1092 = 2*2*3*7*13. Here we must see how to combine the possible partitions of 1092. For example, after a few attempts we note that 1092 = 21*26*2 and we obtain ag = 42 and bg = 52. In this case T(p) and T(q) aren't co-prime numbers.

From here:

GCD[66167,10^(21*2)-1] = 127
GCD[66167,10^(26*2)-1] = 521

Naturally enough to calculate only one factor, the other is obtained by division, until you get a non-trivial factor such that GCD [N, B^k-1].

This type of factorization can work with numbers less than N, and is especially useful for semi-prime numbers. But, as seen, it is not said to be faster than a factorization of "trial" type.

What justifies that the method of continued fractions is adjusted to the factorization of a number RSA or semi-prime?

We saw in [8] that a method for the factorization of a RSA numbers is to use a quadratic equation: in fact it is the product of the solutions of the equation that their sum (sum related to Goldbach conjecture) allow you to the factorization of a RSA number.

Now a quadratic equation like:

$$x^2 + ax - b = 0$$

we can rewrite as:

$$x(x + a) = b$$

$$x = \frac{b}{a+x} \quad x = -a + \frac{b}{x}$$

In both cases we can generate a continued fraction:

$$\cfrac{b}{a+\cfrac{b}{a+\cfrac{b}{a+\dots}}} \qquad -a+\cfrac{b}{-a+\cfrac{b}{-a+\cfrac{b}{-a+\dots}}}$$

2. Continued fractions and the Riemann zeta

The Riemann hypothesis and the factorization are two problems certainly linked, but are not the same problem. It is also not at all certain that the evidence of RH leads to a fast method for the factorization: how would use the non-trivial zeros of the Riemann zeta for the factorization? If is true, this really should be known already as of now, regardless of proof of RH ...

Previously with "factorizations semi-prime periodic and periodic expansions”, perhaps, has come to you some association of ideas: if the fractions are related to the factorization, they are therefore also linked to the Riemann’s zeta?

Have you tried to prove it? It is not easy, unless you know that you can write the Riemann zeta in another way, through the *Mellin transform*:

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 h(x)x^{s-1}dx \qquad (2.1)$$

where:

$$h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \qquad (2.2)$$

is called the "Gauss map", which represents the expansion in continued fractions of x: the symbol 'square bracket below' denotes the largest value less than or equal to 1 / x.

In particolar is:

$$x = [a1, a2, a3, a4, \ldots] = \cfrac{1}{a1 + \cfrac{1}{a2 + \cfrac{1}{a3 + \cfrac{1}{a4 + \ldots}}}}$$

Obviously h(x) is the inverse, i.e. a "reverse shift operator" on the expansion of continued fractions.

The study of continued fractions is very important, more than it seems: they are related to the Riemann zeta, to the solution of the Pell, to fractals (What? Yet Another Simmetry? Then again the Riemann zeta, Beta and Gamma !!!), to the dynamic systems, to the Farey fractions and to the "symmetrical modular groups" SL(2,Z).

We want to exaggerate and say that from here you can even go to string theory? Well, certainly not wrong at all.

In reality, the "transfer operator" of the Gauss map is known in mathematics as "operator of Gauss-Kuzmin-Wirsing" GKW and has many interesting properties.

Moreover, the Gauss map can be thought as a particular element of the group of permutations acting on an infinite dimensional representation of real numbers.

To be precise, the operator of Ruelle-Perron-Frobenius associated with the Gauss map is the operator of Gauss-Kuzmin-Wirsing (GKW) Λh.

This latter is a linear map between spaces of functions in the unit interval closed (Banach spaces), i.e. if is assigned a vector space of functions, from the unit closed interval to the set of real numbers R $\mathcal{F} = \{ f \mid f:[0,1] \to \mathbb{R} \}$, thence also Λh is a linear operator from $\mathcal{F}$ to $\mathcal{F}$ (see [1]).

Λh is an operator of type:

$$[\mathcal{L}_h f](x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} f\left(\frac{1}{n+x}\right) \qquad (2.3)$$

This operator has not been fully resolved, meaning that there are no known closed forms that express all their eigenvalues and eigenfunctions. We know only an eigenvector:

$$f(x) = \frac{1}{1+x} \qquad (2.4)$$

corresponding to its unit eigenvalue, solution provided by Gauss.

Behind this fact there is a broad class of fractals and discontinuous functions that have eigenvalue 1.

Treatment prototype of solutions associated with them can be through the derivative of the Minkowski's function "Question Mark"? (X), namely:

$$[\mathcal{L}_h ?'](x) = ?'(x)$$

Bearing in mind the operator Λh, we can rewrite (2.1) as follows:

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x[\mathcal{L}_h x^{s-1}] \qquad (2.5)$$

The (2.5) together with a better understanding of gkw may be useful to investigate both the Riemann's zeta and RH.

We can in fact replace the Riemann zeta second of binomial coefficients and arrive at the equivalent RH, Dirichlet L-function, totiente series, Liouville series, etc (see [1], [2], [3], [4]). Because on the operator of before there are no simple solutions, a possible study is to use (2.3) and some associated theorems (see [1]), furthermore we can find models for replace (2.2).

The binomial coefficients mentioned previously for the Riemann zeta are implicated in various fractals forms and the Berry's conjecture suggests that "non-trivial zeros of the Riemann zeta corresponds to the spectrum of an unknown chaotic quantization of a mechanical system chaotic."

For example, (2.5) can be rewritten with a series of Newton (see [4]):

$$\zeta(s) = \frac{s}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{b_n}{n!} (s)_n \qquad (2.6)$$

Where (s)n = s(s−1)…(s−n+1) is the "symbol descendant of Pochhammer".

The (2.6) has a strong resemblance with a develop in Taylor series. These general similarities are the basic idea of a technique called "umbral calculus". In (2.6) b_n has a similar role to the Stieltje's constants in the Taylor expansion and has several properties:

$$b_n = n(1-\gamma - H_{n-1}) - \frac{1}{2} + \sum_{k=2}^{n} (-1)^k \zeta(k) \qquad (2.7)$$

for n>0 and γ is Euler-Mascheroni constant. Furthermore is:

$$H_n = \sum_{m=1}^{n} \frac{1}{m} \qquad (2.8)$$

that is the armonic number.

The initial values of b_n, derivable from (2.7), are:

$$b_o = \frac{1}{2}$$
$$b_1 = -\gamma + \frac{1}{2}$$
$$b_2 = -2\gamma - \frac{1}{2} + \zeta(2)$$
$$b_3 = -3\gamma - 2 - \zeta(3) + 3\zeta(2)$$

Although the intermediate terms become very large, the result tends, instead, to become small:

$$b_n = \mathrm{O}(n^{1/4} e^{-2\sqrt{\pi n}})$$

From (2.7) one can try to generalize b_n to complex values:

$$b(s) = -s\gamma + \frac{1}{2} + \sum_{k=2}^{\infty} (-1)^k \binom{s}{k} \left[\zeta(k) - \frac{1}{k-1} \right] \qquad (2.9)$$

Form the study of generalization of Riemann, through the L-function, we can to obtain a fractal structure of the distribution of zeros, using the so-called "Rescaled range analysis".

Property of fractal self-similarity (see below) of the distribution of zeros of L-function is of great importance and is characterized by a fractal dimension d = 1.9. A such great fractal dimension was found for many zeros of the Riemann zeta function and also for those of L-functions of another type.

Fractions and Farey's series

Some properties of Farey fractions were discovered by the geologist Farey.

We consider the series Fn with n = 3 obtained from all the set of the fractions less than or equal to 1, having for numerator and denominator all the numbers from n = 0 to, for example, n = 3, then delete the equivalent values and resort them from the lowest to the largest:

$$F_3 = \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$$

Excluding the terms which give 0 or 1, one can observe interesting properties.

Example for the series above we can consider only the elements 1/3, 1/2 and 2/3. The two outer elements said “convergent” if add up give the central element, said “means”. This is generally true for every N and for each trio of fractions excluding external ones that give 0 and 1. Moreover, when N is prime we obtain N-1 fractions or φ(N).

The Farey’s series also produce a proof of an important corollary of Euclid's algorithm: for two integers m and n with gcd (m, n) = 1 and m≤n, there exist positive integers a and b such that ma-nb = 1.

The test was given by Cauchy and a geometric interpretation was provided by Lester R. Ford. The Farey’s series is connected to the “Stern-Brocot’s tree” least in terms of demonstration. (see [5]).

Finally, between two adjacent of Farey one can also define an operation known as the Farey “sum”:

$$p/q \oplus r/s = (p+q)/(r+s)$$

the resulting number is the “means” and is that of lower denominator, which is in the range (p/q, r/s). This “means” is useful to describe the hierarchy of the responses of synchronization in a periodically forced oscillator: the response characterized by “means”, (p+q)/(r+s), has the most important region of stability among those who are in the range defined by the numbers p/q and r/s.

From here it turns playing with the non-linear forced oscillators that “any dynamical system forced by two or more mutually incommensurate frequencies, can’t produce periodic responses”.

Golden Section, Fibonacci series and Farey’s series

A special infinite continued fraction is:

$$(0,1,1,1,1,1,1,1,1,\dots) \qquad (2.10)$$

The (2.10), in fact, corresponds to the number ϕ, the golden section:

$$\phi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}$$

In fact is:

$$\phi = \frac{1}{1+\phi} \to \phi^2 + \phi - 1 = 0 \to \phi_{1-2} = (1 \pm \sqrt{5})/2$$

We can consider also (2.10) as several pieces of convergent terms, for example:

(0) = 0
(0, 1) = 1
(0, 1, 1) = 1/2
(0, 1, 1, 1) = 2/3
(0, 1, 1, 1, 1) = 3/5
(0, 1, 1, 1, 1, 1) = 5/8
(0, 1, 1, 1, 1, 1, 1) = 8/13

The various pieces seen before give us two unexpected **links** of the golden section, one with the Fibonacci’s series, the other with the Farey’s series!

We note in the pieces the repetition of the sequence 1, 2, 3, 5, 8, 13, ... as in the Fibonacci’s numbers. Excluding (0), to obtain the third element, we must add the first two, for to obtain the next term we must add up the previous two etc.

Again from the pieces we note that two successive pieces convergent of the golden section satisfy the relation (ps - qr) = 1. For example with 5/8 and 8/13 we have that 5*13-8*8 = 65-64 = 1.

3. The fractals

It was Benoit Mandelbrot in 1975 to speak for the first time of fractal. *Fractal* comes from the Latin *fractus* meaning irregular or *fragmented*.

Mandelbrot is considered the father of the theory of fractals.

He has formalized the properties of these figures, considered, before him, only curiosity.

Several classics fractals indeed has been described by famous mathematicians of the past as Cantor, Hilbert, Peano, von Koch, Sierpinski, but only with "The Fractal Geometry of Nature" (1982) that they have found an unified and geometric theory, which highlighted the links with typical forms of nature (galaxy, coastlines, trees, mountains, butterflies, ...).

Intuitively, a fractal is a figure in which a single reason is repeated on sliding scale. Zoom in on any part of the figure, we can find a copy in the scale of the figure itself. Fractals, therefore, are also a symptom of recursive symmetry.

In general, a fractal is a set that has one or more of following properties:

- **self-similarity**: is the union of copies of itself at different scales;
- **fine structure**: the detail of the image does not change at every enlargement;
- **irregularities**: can not be described as a place of points that satisfy simple geometric or analytical conditions; the function is recursive and irregular locally and globally
- **fractal dimension**: although it can be represented in a conventional space of two or three dimensions, its size is not necessarily an integer; it can be a fraction, but often also an irrational number. It is usually greater than the topological dimension

The properties above are also expressible mathematically.

The fractal dimension is therefore the number that measures the degree of irregularity and interruption of an object, considered in any scale.

Since Mandelbrot introduced the *fractal geometry*, is born a new description language of complex forms of nature: they require algorithms, simple recursive function, that iterates many times provide a picture.

In the 80's years with this new geometry have been found fractals in every area: from the nature to the medicine and music and has been developed a branch of fractal geometry which studies the so-called fractal *biomorphic* and one on fractals with *condensing*, using the transformations geometric of the plan, the methods IFS and the L-system.
Obviously the fractals appear also in the study of dynamical systems.
Fractals are used by physicists and engineers to build models that describe the motion of fluid turbulence - *but for the authors are important also for the extra dimensions* - and the phenomena of combustion. Furthermore, they have application in the compression of images and for virtual movies. Finally, they are useful for the reproduction of porous media and the study of hydrocarbons and of Nature in general: geographical coastlines, river courses etc.

In [7] it was shown that in the case of extra dimensions compactified, if you make the analogy with a tube pump in which is driven water by strong pressure, it is first projected strongly on the side panels and then continues in the longitudinal direction. Now, at a distance rather small, the lines of strength of gravity behave at the same way: a short distance or small dimension one has the propagation radially in all directions, and after extend into the larger dimension in a linear mode. From this phenomenon, using a “umbral” technique that clone the turbulent flow of the fluid, we can observe a sort of "*fractal gravity*”.

The differential form of Gauss' law for gravity states:

$$\nabla \cdot \mathbf{g} = -4\pi G\rho$$

where

$\nabla\cdot$ denotes divergence,
G is the gravitational constant of the universe,
ρ is the mass density at each point.

The two forms of Gauss' law for gravity are mathematically equivalent. The divergence theorem states:

$$\oint_{\partial V} \mathbf{g} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{g}\, dV$$

where

V is a closed region bounded by a simple closed oriented surface ∂V,
g is a continuously differentiable vector field defined on a neighborhood of *V*,
dV is an infinitesimal piece of the volume *V.*

Given also that

$$M = \int_V \rho\, dV$$

we can apply the divergence theorem to the integral form of Gauss' law for gravity, which becomes:

$$\int_V \nabla \cdot \mathbf{g}\, dV = -4\pi G \int_V \rho\, dV$$

which can be rewritten:

$$\int_V (\nabla \cdot \mathbf{g})\, dV = \int_V (-4\pi G\rho)\, dV$$

This has to hold simultaneously for every possible volume *V*; the only way this can happen is if the integrands are equal. Hence we arrive at

$$\nabla \cdot \mathbf{g} = -4\pi G\rho$$

which is the differential form of Gauss' law for gravity.

The differential form of Gauss' law for gravity can also be derived from Newton's law of universal gravitation. Using the expression from Newton's law, we get the total field at **r** by using an integral to add up the field at **r** due to the mass at each other point in space with respect to an **s** coordinate system, to give

$$\mathbf{g}(\mathbf{r}) = -G\int_V \frac{\rho(\mathbf{s})(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}dV(\mathbf{s})$$

If we take the divergence of both sides of this equation with respect to **r**, and use the known theorem

$$\nabla \cdot \left(\frac{\mathbf{s}}{|\mathbf{s}|^3}\right) = 4\pi\delta(\mathbf{s})$$

where $\delta(\mathbf{s})$ is the Dirac delta function, the result is

$$\nabla \cdot \mathbf{g}(\mathbf{r}) = -4\pi G\int_V \rho(\mathbf{s})\ \delta(\mathbf{r}-\mathbf{s})\ dV(\mathbf{s})$$

Using the "sifting property" of the Dirac delta function, we arrive at

$$\nabla \cdot \mathbf{g} = -4\pi G\rho$$

which is the differential form of Gauss' law for gravity, as desired.

Since the gravitational field has zero curl (equivalently, gravity is a conservative force), it can be written as the gradient of a scalar potential, called the gravitational potential:

$$\mathbf{g} = -\nabla\phi\ ,$$

Then the differential form of Gauss' law for gravity becomes Poisson's equation:

$$\nabla^2\phi = 4\pi G\rho\ ,$$

This provides an alternate means of calculating the gravitational potential and gravitational field. Although computing **g** via Poisson's equation is mathematically equivalent to computing **g** directly from Gauss's law, one or the other approach may be an easier computation in a given situation.

In radially symmetric systems, the gravitational potential is a function of only one variable (namely, $r = |\mathbf{r}|$), and Poisson's equation becomes:

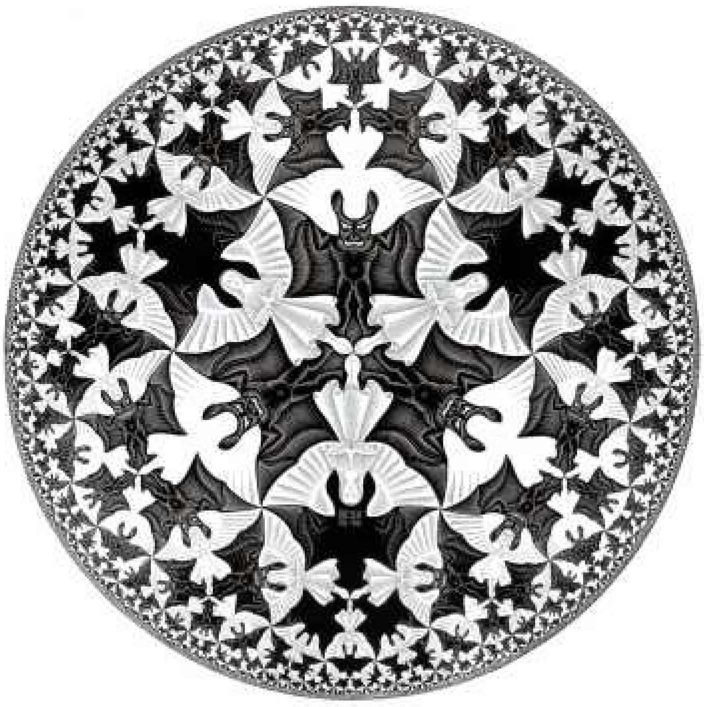
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho(r)$$

while the gravitational field is:

$$\mathbf{g}(\mathbf{r}) = -\mathbf{e_r}\frac{\partial\phi}{\partial r}$$

It is possible that the Poisson’s equation give rise to the “fractals” in the specific case of the extra compactified dimensions? It all depends from the boundary conditions or from the fact that is possible to simplify the equation $\nabla^2\phi = 4\pi G\rho$, in a Laplace’s equation? Yes, it is possible. Indeed, we have said above that at a distance rather small, the lines of strength of gravity behave at the same way: a short distance or small dimension one has the propagation radially in all directions, and after extend into the larger dimension in a linear mode.

The famous incision of Escher *“Limite of the circle IV”* is one "map" of a space to curvature negative that shows exactly as a bidimensional slice of an AdS space it would appear. In it, the figures are alternated without end, falling through in an edge infinite fractal (**also here is present the Aurea ratio Φ**).



We now add the time and we put all together in a figure that represents an anti de Sitter space. We put the time along the vertical axis. Every horizontal section represents the ordinary space to a particular instant. We can think then the AdS as an endless sequence of thin slices of space that, stack one on the other, form a continuous space-time of cylindrical form.
Now we imagine to zoom on a region near to the edge of the figure and to do an enlargement of it such that can to appear the edge almost rectilinear. If we simplify the image replacing the dark figures with squares, the image becomes a kind of network made as soon as more and more of small squares it draws near us to the edge endless fractal. We can imagine the AdS as an infinite "wall" of square bricks: going down along the wall, to every new layer the width of the bricks doubles.
The anti de Sitter space is as one "tin of soup" The horizontal sections of the tin represent the space, while the vertical axis represents the time. The label to the outside of the tin is the edge, while the inside represents the real space-time. The pure AdS space is an empty tin, that can be made more interesting when is full of "soup", thence of matter and energy. Edward Witten has explained that, accumulating enough matter and energy in the tin, it is possible to create a black hole.
The existence of a black hole in the "soup" must have an equivalent on the hologram on the board, but what? In his "theory of edge" Witten sustains that the black hole in the "soup" it is equivalent to a "warm fluid" of elementary particles, essentially gluons. Now, the field theory is a particular case of quantum

mechanics, and in quantum mechanics the information never comes destroyed. The strings theorists immediately understood that Maldacena and Witten had shown without shade of doubt that is not possible to make to disappear information behind the horizon of a black hole.
In the Maldacena paper: “The Large N Limit of Superconformal field theories and supergravity”, we consider a near extremal black D3 brane solution in the following decoupling limit:

$$\alpha' \to 0 , \qquad U \equiv \frac{r}{\alpha'} = \text{fixed}. \quad (3.1)$$

We keep the energy density on the brane worldvolume theory (μ) fixed. We find the following metric:

$$ds^2 = \alpha' \left\{ \frac{U^2}{\sqrt{4\pi gN}} \left[-\left(1 - U_0^4 / U^4\right) dt^2 + dx_i^2 \right] + \sqrt{4\pi gN} \frac{dU^2}{U^2 \left(1 - U_0^4 / U^4\right)} + \sqrt{4\pi gN} d\Omega_5^2 \right\}, \quad (3.2)$$

where

$$U_0^4 = \frac{2^7}{3} \pi^4 g^2 \mu . \quad (3.3)$$

We note that the pure number $\sqrt[4]{\frac{2^7 \pi^4}{3}} = \sqrt[4]{4156,121217} = 8,029195849 \cong 8$ is related to the Aura ratio by the following relation:

$$\left[(\Phi)^{35/7} + (\Phi)^{-7/7} + (\Phi)^{-21/7} + (\Phi)^{-42/7} \right] \cdot \frac{2}{3} = (11,09016994 + 0,61803399 + 0,23606798 + 0,05572809) \cdot \frac{2}{3} = 8$$

Furthermore, we note also that 8 is a Fibonacci’s number.
We could now ask what the low energy effective action for the light $U(1)$ fields is. For large N the action of a D3 brane in $AdS_5 \times S^5$ is $gN >> 1$. More concretely, the bosonic part of the action becomes the Born-Infeld action on the AdS background:

$$S = -\frac{1}{(2\pi)^3 g} \int d^4 x h^{-1} \left[\sqrt{-Det\left(\eta_{\alpha\beta} + h \partial_\alpha U \partial_\beta U + U^2 h g_{ij} \partial_\alpha \theta^i \partial_\beta \theta^j + 2\pi \sqrt{h} F_{\alpha\beta}\right)} - 1 \right] \quad (3.4)$$

$$h = \frac{4\pi gN}{U^4} , \quad (3.5)$$

with $\alpha, \beta = 0,1,2,3, \quad i, j = 1,...,5$; and g_{ij} is the metric of the unit five-sphere. As any low energy action (3.4) is valid when the energies are low compared to the mass of the massive states that we are integrating out. In this case the mass of the massive states is proportional to U. The low energy condition translates into $\partial U / U << U$ and $\partial \theta^i << U$, etc… So the nonlinear terms in the action (3.4) will be important only when gN is large.

Also here we can note that there exists the mathematical connection with the Aurea section. Indeed, we remember that π, that is present in the eqs. (3.4) and (3.5), is related to the Aurea section $\phi = \sqrt{5} - 1/2$ by the following simple relation:

$$\arccos\phi = 0{,}2879\pi \quad (3.6)$$

4. p-adic, adelic and zeta strings [9] [10] [11] [12] [13]

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx = g^2 \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)} \right] = g^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} =$$
$$= g^2 \int DX \exp\left(-\frac{i}{2\pi} \int d^2\sigma \partial^\alpha X_\mu \partial_\alpha X^\mu \right) \prod_{j=1}^{4} \int d^2\sigma_j \exp\left(i k_\mu^{(j)} X^\mu \right), \quad (4.1-4.4)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - \frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s + t + u = -8$, i.e. $a + b + c = 1$.
The p-adic generalization of the above expression

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx, \quad (4.5)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty\left(ax^2 + bx\right) d_\infty x \prod_{p\in P} \int_{Q_p} \chi_p\left(ax^2 + bx\right) d_p x = 1, \quad a \in Q^\times, \quad b \in Q, \quad (4.6)$$

what follows from

$$\int_{Q_v} \chi_v\left(ax^2 + bx\right)d_v x = \lambda_v(a)\left|2a\right|_v^{-\frac{1}{2}} \chi_v\left(-\frac{b^2}{4a}\right), \quad v = \infty, 2, \ldots, p \ldots . \quad (4.7)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_v\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q}, q, t)dt\right)D_v q, \quad (4.8)$$

for kernels $K_v(x'', t''; x', t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q}, q) = \frac{1}{2}\left(-\frac{\dot{q}^2}{4} - \lambda q + 1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'', T; x', 0)\prod_{p \in P} K_p(x'', T; x', 0) = 1, \quad x'', x', \lambda \in Q, \ T \in Q^\times, \quad (4.9)$$

where

$$K_v(x'', T; x', 0) = \lambda_v(-8T)\left|4T\right|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + \left[\lambda(x'' + x') - 2\right]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right). \quad (4.10)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 “modes”, i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'', T; x', 0) = \lambda_v(-8T)\left|4T\right|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + \left[\lambda(x'' + x') - 2\right]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right) \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\dfrac{\displaystyle\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \dfrac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\dfrac{10 + 11\sqrt{2}}{4}\right)} + \sqrt{\left(\dfrac{10 + 7\sqrt{2}}{4}\right)}\right]}. \quad (4.10b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x) = \psi_\infty(x)\prod_{p \in P} \Omega\left(\left|x\right|_p\right) = \begin{cases} \psi_\infty(x), x \in Z \\ 0, x \in Q \setminus Z \end{cases}, \quad (4.11)$$

where $\Omega(|x|_p)=1$ if $|x|_p \le 1$ and $\Omega(|x|_p)=0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a)=\int_R |x|_\infty^{a-1}\chi_\infty(x)d_\infty x=\frac{\zeta(1-a)}{\zeta(a)}, \quad \Gamma_p(a)=\int_{Q_p}|x|_p^{a-1}\chi_p(x)d_p x=\frac{1-p^{a-1}}{1-p^{-a}}, \quad (4.12)$$

$$B_\infty(a,b)=\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x=\Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c), \quad (4.13)$$

$$B_p(a,b)=\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}d_p x=\Gamma_p(a)\Gamma_p(b)\Gamma_p(c), \quad (4.14)$$

where $a,b,c\in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in P}\Gamma_p(u)=1, \quad B_\infty(a,b)\prod_{p\in P}B_p(a,b)=1, \quad u\ne 0,1, \quad u=a,b,c, \quad (4.15)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (4.16)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x=\frac{1}{1-p^{-a}}, \quad \mathrm{Re}\,a>1, \quad (4.17)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (4.18)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (4.19)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x. \quad (4.19b)$$

Let us note that $\exp(-\pi x^2)$ and $\Omega(|x|_p)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x) = 2^{\frac{1}{4}} e^{-\pi x_\infty^2} \prod_{p \in P} \Omega\left(\left|x_p\right|_p\right), \quad (4.20)$$

whose the Fourier transform

$$\psi_A(k) = \int \chi_A(kx)\psi_A(x) = 2^{\frac{1}{4}} e^{-\pi k_\infty^2} \prod_{p \in P} \Omega\left(\left|k_p\right|_p\right) \quad (4.21)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a) = \int \psi_A(x)|x|^a d_A^\times x = \int_R \psi_\infty(x)|x|^{a-1} d_\infty x \prod_{p \in P} \frac{1}{1-p^{-1}} \int_{Q_p} \Omega\left(|x|_p\right)|x|^{a-1} d_p x = \sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (4.22)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (4.19).
The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2} \frac{p^2}{p-1} \left[-\frac{1}{2} \varphi p^{-\frac{\square}{2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right], \quad (4.23)$$

where p is any prime number, $\square = -\partial_t^2 + \nabla^2$ is the D-dimensional d’Alambertian and we adopt metric with signature $(-+\ldots+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n \geq 1} C_n \mathcal{L}_n = \sum_{n \geq 1} \frac{n-1}{n^2} \mathcal{L}_n = \frac{1}{g^2} \left[-\frac{1}{2} \phi \sum_{n \geq 1} n^{-\frac{\square}{2}} \phi + \sum_{n \geq 1} \frac{1}{n+1} \phi^{n+1} \right]. \quad (4.24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (4.25)$$

Employing usual expansion for the logarithmic function and definition (4.25) we can rewrite (4.24) in the form

$$L = -\frac{1}{g^2} \left[\frac{1}{2} \phi \zeta\left(\frac{\square}{2}\right) \phi + \phi + \ln(1-\phi) \right], \quad (4.26)$$

where $|\phi|<1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk\,,\quad -k^2=k_0^2-\vec{k}^2>2+\varepsilon\,,\quad (4.27)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.
Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\quad (4.28)$$

which has an evident solution $\phi=0$.
For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}.\quad (4.29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq1}\theta^{\frac{n(n-1)}{2}}\phi^n\,,\quad (4.30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right],\quad (4.31)$$

and one can easily see trivial solution $\phi=\theta=0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p=\frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right],\quad (4.32)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$, as above.

Now, we want to introduce a model which incorporates all the above string Lagrangians (4.32) with p replaced by $n \in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2} \frac{n^2}{n-1} \left[-\frac{1}{2} \phi n^{-\frac{\Box}{2m_n^2}} \phi + \frac{1}{n+1} \phi^{n+1} \right], \quad (4.33)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n = \frac{n-1}{n^{2+h}}, \quad (4.34)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2} - h} \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right] \quad (4.35)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2} - h} = \prod_p \frac{1}{1 - p^{-\frac{\Box}{2m^2} - h}}. \quad (4.36)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (4.37)$$

which has analytic continuation to the entire complex s plane, excluding the point $s=1$, where it has a simple pole with residue 1. Employing definition (4.37) we can rewrite (4.35) in the form

$$L_h = \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \zeta \left(\frac{\Box}{2m^2} + h \right) \phi + \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right]. \quad (4.38)$$

Here $\zeta\left(\frac{\Box}{2m^2} + h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2} + h\right) \phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{2m^2} + h\right) \tilde{\phi}(k) dk, \quad (4.39)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.
We consider Lagrangian (4.38) with analytic continuations of the zeta function and the power series $\sum \frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right], \quad (4.40)$$

where AC denotes analytic continuation.
Potential of the above zeta scalar field (4.40) is equal to $-L_h$ at $\Box = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h) - AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (4.41)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ-function vanishes at $h = -2, -4, -6, \ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi = AC\sum_{n=1}^{+\infty} n^{-h}\varphi^n , \quad (4.42)$$

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n , \quad (4.43)$$

respectively.

Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (4.40): $h = 0,$ $h = \pm 1,$ and $h = \pm 2$. For $h = -2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2}-2\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-2\right)\tilde{\phi}(k)dk = \frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (4.44)$$

This equation has two trivial solutions: $\phi(x) = 0$ and $\phi(x) = -1$. Solution $\phi(x) = -1$ can be also shown taking $\tilde{\phi}(k) = -\delta(k)(2\pi)^D$ and $\zeta(-2) = 0$ in (4.44).
For $h = -1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2}-1\right)\phi = \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk = \frac{\phi}{(1-\phi)^2}. \quad (4.45)$$

where $\zeta(-1) = -\frac{1}{12}$.

The equation of motion (4.45) has a constant trivial solution only for $\phi(x)=0$.
For $h=0$, the equation of motion is

$$\zeta\left(\frac{\square}{2m^2}\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \quad (4.46)$$

It has two solutions: $\phi=0$ and $\phi=3$. The solution $\phi=3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\square}{2m^2}\right)=\zeta(0)+\sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\square}{2m^2}\right)^n, \quad (4.47)$$

as well as from $\tilde{\phi}(k)=(2\pi)^D 3\delta(k)$.
For $h=1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+1\right)\tilde{\phi}(k)dk=-\frac{1}{2}\ln(1-\phi)^2, \quad (4.48)$$

where $\zeta(1)=\infty$ gives $V_1(\phi)=\infty$.
In conclusion, for $h=2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk=-\int_0^\phi\frac{\ln(1-w)^2}{2w}dw. \quad (4.49)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw=\sum_{n=1}^\infty\frac{1}{n^2}=\zeta(2)$$

one has trivial solution $\phi=1$ in (4.49).

Now, we want to analyze the following case: $C_n=\frac{n^2-1}{n^2}$. In this case, from the Lagrangian (4.33), we obtain:

$$L=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)\right\}\phi+\frac{\phi^2}{1-\phi}\right]. \quad (4.50)$$

The corresponding potential is:

$$V(\phi)=-\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2. \quad (4.51)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n\pm 1$ with n=1 and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore, the number 24 is related to the Ramanujan function that has 24 “modes” that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi)=-\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 \Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.51b)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right]\phi=\frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2}. \quad (4.52)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-2\right]\phi=0, \quad (4.53)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=2. \quad (4.54)$$

From (4.54) it follows one solution for $M^2>0$ at $M^2\approx 2.79m^2$ and many tachyon solutions when $M^2<-38m^2$.

We note that the number 2.79 is connected with $\phi=\frac{\sqrt{5}-1}{2}$ and $\Phi=\frac{\sqrt{5}+1}{2}$, i.e. the “aurea” section and the “aurea” ratio. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2^2}\left(\frac{\sqrt{5}-1}{2}\right)=2{,}772542\cong 2{,}78.$$

Furthermore, we have also that:

$$(\Phi)^{14/7}+(\Phi)^{-25/7}=2{,}618033989+0{,}179314566=2{,}79734$$

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n=\frac{n^2-1}{n^2}$, are:

$$L = \frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2} - \zeta\left(\frac{\Box}{2m^2} - 1\right) - \zeta\left(\frac{\Box}{2m^2}\right) - 1\right\}\phi + \frac{\phi^2}{2}\ln\phi^2 + \frac{\phi^2}{1-\phi}\right], \quad (4.55)$$

$$V(\phi) = \frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1) + \zeta(0) + 1 - \ln\phi^2 - \frac{1}{1-\phi}\right], \quad (4.56)$$

$$\left[\zeta\left(\frac{\Box}{2m^2} - 1\right) + \zeta\left(\frac{\Box}{2m^2}\right) - \frac{\Box}{m^2} + 1\right]\phi = \phi\ln\phi^2 + \phi + \frac{2\phi - \phi^2}{(1-\phi)^2}, \quad (4.57)$$

$$\zeta\left(\frac{M^2}{2m^2} - 1\right) + \zeta\left(\frac{M^2}{2m^2}\right) = \frac{M^2}{m^2}. \quad (4.58)$$

In addition to many tachyon solutions, equation (4.58) has two solutions with positive mass: $M^2 \approx 2.67m^2$ and $M^2 \approx 4.66m^2$.
We note also here, that the numbers 2.67 and 4.66 are related to the “aureo” numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 + \frac{1}{2\cdot 5}\left(\frac{\sqrt{5}-1}{2}\right) \cong 2.6798,$$

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 + \left(\frac{\sqrt{5}+1}{2}\right) + \frac{1}{2^2}\left(\frac{\sqrt{5}+1}{2}\right) \cong 4.64057.$$

Furthermore, we have also that:

$$(\Phi)^{14/7} + (\Phi)^{-41/7} = 2{,}618033989 + 0{,}059693843 = 2{,}6777278;$$

$$(\Phi)^{22/7} + (\Phi)^{-30/7} = 4{,}537517342 + 0{,}1271565635 = 4{,}6646738.$$

Furthermore, with regard the value $M^2 < -38m^2$, we have the following connection with the aureo number:

$$\left[(\Phi)^{35/7} + (\Phi)^{-7/7} + (\Phi)^{-21/7} + (\Phi)^{-42/7}\right]\cdot 3 = (11{,}09016994 + 0{,}61803399 + 0{,}23606798 + 0{,}05572809)\cdot 3 = 36$$

and 36 is < of 38.

Now, we describe the case of $C_n = \mu(n)\frac{n-1}{n^2}$. Here $\mu(n)$ is the Mobius function, which is defined for all positive integers and has values 1, 0, – 1 depending on factorization of n into prime numbers p. It is defined as follows:

$$\mu(n) = \begin{cases} 0, & n = p^2 m \\ (-1)^k, & n = p_1 p_2 \ldots p_k,\ p_i \neq p_j \\ 1, & n = 1, (k = 0). \end{cases} \quad (4.59)$$

The corresponding Lagrangian is

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{\frac{\Box}{2m^2}}}\phi + \sum_{n=1}^{+\infty} \frac{\mu(n)}{n+1}\phi^{n+1}\right] \quad (4.60)$$

Recall that the inverse Riemann zeta function can be defined by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1. \quad (4.61)$$

Now (4.60) can be rewritten as

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi + \int_0^\infty \mathcal{M}(\phi) d\phi\right], \quad (4.62)$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n)\phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots$ The corresponding potential, equation of motion and mass spectrum formula, respectively, are:

$$V_\mu(\phi) = -L_\mu(\Box = 0) = \frac{m^D}{g^2}\left[\frac{C_0}{2}\phi^2\left(1 - \ln\phi^2\right) - \phi^2 - \int_0^\phi \mathcal{M}(\phi) d\phi\right], \quad (4.63)$$

$$\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi - \mathcal{M}(\phi) - C_0 \frac{\Box}{m^2}\phi - 2C_0 \phi \ln\phi = 0, \quad (4.64)$$

$$\frac{1}{\zeta\left(\frac{M^2}{2m^2}\right)} - C_0 \frac{M^2}{m^2} + 2C_0 - 1 = 0, \quad |\phi| << 1, \quad (4.65)$$

where usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$ is used.

Now, we take the pure numbers concerning the eqs. (4.54) and (4.58). They are: 2.79, 2.67 and 4.66. We note that all the numbers are related with $\Phi = \frac{\sqrt{5}+1}{2}$, thence with the aurea ratio, by the following expressions:

$$2{,}79 \cong (\Phi)^{15/7}; \qquad 2{,}67 \cong (\Phi)^{13/7} + (\Phi)^{-21/7}; \qquad 4{,}66 \cong (\Phi)^{22/7} + (\Phi)^{-30/7}. \quad (4.66)$$

5. Mathematical connections.

Now, we describe some possible mathematical connections between some equations concerning the **Section 2** and some equations regarding the **Section 4.**
We note that from the eqs. (2.1) and (2.2), we have that:

$$\zeta(s)=\frac{s}{s-1}-s\int_0^1 \frac{1}{x}-\left[\frac{1}{x}\right]x^{s-1}dx\ ; \qquad (5.1)$$

That is related with the eqs. (4.28), (4.43) and (4.49) as follows:

$$\zeta(s)=\frac{s}{s-1}-s\int_0^1 \frac{1}{x}-\left[\frac{1}{x}\right]x^{s-1}dx \Rightarrow \zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}, \quad (5.2)$$

$$\zeta(s)=\frac{s}{s-1}-s\int_0^1 \frac{1}{x}-\left[\frac{1}{x}\right]x^{s-1}dx \Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk=AC\sum_{n=1}^{+\infty}n^{-h}\phi^n\ , \quad (5.3)$$

$$\zeta(s)=\frac{s}{s-1}-s\int_0^1 \frac{1}{x}-\left[\frac{1}{x}\right]x^{s-1}dx \Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk=-\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw\,. \quad (5.4)$$

Now, we note that the eq. (2.5) can be rewritten, for the eqs. (2.6) and (2.7), as follows

$$\zeta(s)=\frac{s}{s-1}-s\int_0^1 dx\ x[\mathfrak{L}_h x^{s-1}]=\frac{s}{s-1}+\sum_{n=0}^{\infty}(-1)^n\frac{(s)_n}{n!}\cdot n(1-\gamma-H_{n-1})-\frac{1}{2}+\sum_{k=2}^{n}(-1)^k\zeta(k)\,. \quad (5.5)$$

Thence, we have the following mathematical connections with the eqs. (4.28), (4.43), (4.49) and (4.62):

$$\zeta(s)=\frac{s}{s-1}-s\int_0^1 dx\ x[\mathfrak{L}_h x^{s-1}]=\frac{s}{s-1}+\sum_{n=0}^{\infty}(-1)^n\frac{(s)_n}{n!}\cdot n(1-\gamma-H_{n-1})-\frac{1}{2}+\sum_{k=2}^{n}(-1)^k\zeta(k)\Rightarrow$$
$$\Rightarrow \zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}, \quad (5.6)$$

$$\zeta(s)=\frac{s}{s-1}-s\int_0^1 dx\ x[\mathfrak{L}_h x^{s-1}]=\frac{s}{s-1}+\sum_{n=0}^{\infty}(-1)^n\frac{(s)_n}{n!}\cdot n(1-\gamma-H_{n-1})-\frac{1}{2}+\sum_{k=2}^{n}(-1)^k\zeta(k)\Rightarrow$$
$$\Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk=AC\sum_{n=1}^{+\infty}n^{-h}\phi^n\ , \quad (5.7)$$

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x[\mathfrak{L}_h x^{s-1}] = \frac{s}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{(s)_n}{n!} \cdot n(1-\gamma-H_{n-1}) - \frac{1}{2} + \sum_{k=2}^{n} (-1)^k \zeta(k) \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2}+2\right) \tilde{\phi}(k)dk = -\int_0^{\phi} \frac{\ln(1-w)^2}{2w} dw\,, \quad (5.8)$$

$$\zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x[\mathfrak{L}_h x^{s-1}] = \frac{s}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{(s)_n}{n!} \cdot n(1-\gamma-H_{n-1}) - \frac{1}{2} + \sum_{k=2}^{n} (-1)^k \zeta(k) \Rightarrow$$

$$\Rightarrow L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)} \phi + \int_0^{\infty} \mathcal{M}(\phi) d\phi \right]. \quad (5.9)$$

Furthermore, also the eq. (3.4) can be related with the eq. (5.5) and we obtain the following interesting mathematical connection:

$$S = -\frac{1}{(2\pi)^3 g} \int d^4 x h^{-1} \left[\sqrt{-Det\left(\eta_{\alpha\beta} + h\partial_\alpha U \partial_\beta U + U^2 h g_{ij} \partial_\alpha \theta^i \partial_\beta \theta^j + 2\pi\sqrt{h} F_{\alpha\beta}\right)} - 1 \right] \Rightarrow$$

$$\Rightarrow \zeta(s) = \frac{s}{s-1} - s\int_0^1 dx\, x[\mathfrak{L}_h x^{s-1}] = \frac{s}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{(s)_n}{n!} \cdot n(1-\gamma-H_{n-1}) - \frac{1}{2} + \sum_{k=2}^{n} (-1)^k \zeta(k)\,. \quad (5.10)$$

Also here we can note that there exists the mathematical connection with the Aurea section. Indeed, we remember that π, that is present in the eqs. (5.2) - (5.4) and (5.6) – (5.8), is related to the Aurea section $\phi = \sqrt{5} - 1/2$ by the following simple but fundamental relation:

$$\arccos\phi = 0{,}2879\pi$$

For further details, click on the following link:

http://150.146.3.132/1163/01/TCN6.pdf

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Eng. Rosario Turco

http://mathbuildingblock.blogspot.com/

Dr. Michele Nardelli (various papers on the string theory)

http://xoomer.virgilio.it/stringtheory/

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