Noname manuscript No.
(will be inserted by the editor)

Contradiction tolerance of Kirchhoff's diffraction theory

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Received: date / Accepted: date

Abstract Complex numbers are basic to exact science. When a flaw exists in complex numbers conceptual difficulties will arise for many subfields concerning wave mechanics. Kirchoff's scalar diffraction theory of optics is already considered inconsistent. Nevertheless it is successfull in experiment. In our study we add the complex number inconsistency to Kirchoff diffraction and see what that does to the experimental value of the Kirchhoff diffraction theory. There are no a priori reasons to include or exclude the obtained inconsistent phase angle. Assuming that the inconsistent phase angle is excluded in nature, we were able to establish the theoretical possibility to observe a substantial diffraction despite a weak intensity point source and small wavelength.

Keywords Basic complex number theory · Euler's identity · Contradiction · Scalar diffraction theory

Acknowledgements The author wishes to acknowledge the support of Ad Popper, director Xilion BV.

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1 Introduction

Here a simple proof of a contradiction in complex numbers is delivered. Then its consequences for scalar Kirchhoff's diffraction theory are researched more deeply. As far as the author knows, there is no existing literature for this kind of exercise. We consulted a few excellent textbooks about Kirchhoff's diffraction. This is sufficient for our purpose.

1.1 Motivation & preliminaries

It is an interesting aspect of the presented mathematics that Kirchhoff's diffraction theory is inconsistent [1, pg. 45-46]. At the same time, Kirchhoff's scalar diffraction is experimentally successful [1, pg 46] and [2]. In [2, pg 482] a partial explanation, concerning geometry & dimensions of the experimental situation, is provided for this success. The presently studied smallness of the wavelength related to the aperture might provide the conditions where the diffraction according to Kirchhoff is correct [3] for a black screen. However, in [3] according to [4], there are boundary conditions that are unequal to Kirchhoff's boundary conditions [2, pg 480]. Note btw that Kirchhoff's theory cannot be seen as a first approximation [3] and [4]. Because of small wavelength, the approximation in equation (3-25) of [1, chap 3] is not made.

It is asked what an additional & *new* contradiction in complex numbers will possibly do to the success of Kirchhoff's diffraction. First the contradiction in complex numbers is demonstrated. Secondly, the ambiguity in phase angle is incorporated in Kirchhoff's diffraction. In the discussion we draw the consequences of the mathematical result.

2 Contradiction & Kirchhoff diffraction

2.1 Contradiction briefly

Here the contradiction in the complex numbers is presente. A paper with more details based on viz. [7], is already under review. Let us look in a $\lambda \to 0^+$ limit at

$$f_\lambda(y) = e^{iy} \left\{ e^{i\lambda} - (1 + \sin(\lambda)) \right\} \tag{1}$$

Here $y = \left(\frac{x+\pi}{2}\right) \in \mathbb{R}$, $x \in \mathbb{R}$. We may rewrite (1) as $f_\lambda(y) = |f_\lambda(y)| \exp(i\varphi_\lambda)$. Here λ is a wave length. With $|f_\lambda(y)|^2 = f_\lambda^*(y) f_\lambda(y)$ and f^* the complex conjugate of f. Applying a number of times the rule of lHopital, the reader can check that

$$\lim_{\lambda \downarrow 0} \frac{\sin^2(\lambda)}{|f_\lambda(y)|^2} = \frac{1}{2} \tag{2}$$

viz. [7]. Similarly, because $\sin(\lambda) \propto \lambda$ for small λ,

$$\lim_{\lambda \downarrow 0} \frac{|f_\lambda(y)|^2}{\lambda^2} = 2 \tag{3}$$

Further, with the use of Euler's identity [6], $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, and $\theta \in \mathbb{R}$, it is possible to rewrite (1)

$$\begin{aligned} \cos(y+\lambda) - (1+\sin(\lambda))\cos(y) &= |f_\lambda(y)|\cos(\varphi_\lambda) \\ \sin(y+\lambda) - (1+\sin(\lambda))\sin(y) &= |f_\lambda(y)|\sin(\varphi_\lambda) \end{aligned} \tag{4}$$

Then the first equation is

$$\cos(\varphi_\lambda) = -\sin\left(\frac{x}{2}\right) \frac{\cos(\lambda)\ -1}{\sin(\lambda)} \left(\frac{\sin(\lambda)}{|f_\lambda(y)|}\right) + \frac{\sin(\lambda)}{|f_\lambda(y)|} \left(\ \sin\left(\frac{x}{2}\right)\ -\ \cos\left(\frac{x}{2}\right)\ \right) \tag{5}$$

Now, with $\cos\left(\frac{x+\pi}{2}\right) = -\sin(\frac{x}{2})$, the first term on the right hand side of (5) vanishes because,

$$\lim_{\lambda \downarrow 0} \frac{\cos(\lambda)\ -1}{\sin(\lambda)} = 0 \tag{6}$$

From (2) and the existence of $\varphi = \lim_{\lambda \downarrow 0} \varphi_\lambda$ it follows

$$\cos(\varphi) = \frac{1}{\sqrt{2}} \left(\ \sin\left(\frac{x}{2}\right)\ -\ \cos\left(\frac{x}{2}\right)\ \right) \tag{7}$$

This is our first equation of concern. The second equation is derived similarly. We have

$$\sin(\varphi_\lambda) = \sin\left(\frac{x+\pi}{2}\right)\frac{\cos(\lambda)}{|f_\lambda(y)|} + \cos\left(\frac{x+\pi}{2}\right)\frac{\sin(\lambda)}{|f_\lambda(y)|} - \frac{(1+\sin(\lambda))}{|f_\lambda(y)|}\sin\left(\frac{x+\pi}{2}\right) \tag{8}$$

And so, with (6), (2), $\sin\left(\frac{x+\pi}{2}\right) = \cos(\frac{x}{2})$ and along a similar path as previously we may derive the second equation of concern

$$\sin(\varphi) = -\frac{1}{\sqrt{2}} \left(\ \sin\left(\frac{x}{2}\right)\ +\ \cos\left(\frac{x}{2}\right)\ \right) \tag{9}$$

The example case $x/2 = \pi/3$: Let us assume that $x = 2\pi/3$. Moreover, let us restrict the interval of the limit phase angle φ, with, $-\pi \leq \varphi \leq \pi$. Then, $\sin(x/2) = \frac{\sqrt{3}}{2} \approx 0.866$ and $\cos(x/2) = 1/2 = 0.500$. From equations (7) and (9) we then obtain $\cos(\varphi) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \approx 0.259$ and $\sin(\varphi) = -\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) \approx -0.966$. Following the path of the angular analysis this gives $\cos(\varphi) + \sin(\varphi) = -\frac{1}{\sqrt{2}}$ and $\cos(\varphi) - \sin(\varphi) = \frac{\sqrt{3}}{\sqrt{2}}$. In addition, we also have $\cos(\varphi)\sin(\varphi) == \left(-\frac{1}{2}\right) \times \frac{1}{2} = -\frac{1}{4}$ And so $\sin(2\varphi) = 2\cos(\varphi)\sin(\varphi) = -\frac{1}{2}$ and $\cos(2\varphi) = \cos^2(\varphi) - \sin^2(\varphi) = -\frac{\sqrt{3}}{2}$. Therefore, with $-2\pi \leq 2\varphi \leq 2\pi$ and both cos and sin negative, we are allowed to set $2\varphi = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$, with *both* $\sin(2\varphi) = -\frac{1}{2}$ *and* $\cos(2\varphi) = -\frac{\sqrt{3}}{2}$. Hence, $\varphi = \frac{7\pi}{12}$ and the φ is in the interval $-\pi \leq \varphi \leq \pi$. But $\varphi = \frac{7\pi}{12}$ gives $\cos(\varphi) = \cos\left(\frac{7\pi}{12}\right) \approx -0.259$ and $\sin(\varphi) = \sin\left(\frac{7\pi}{12}\right) \approx 0.966$. This is *a contradiction*. Moreover, when we select $2\varphi = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$ it is $-2\pi \leq 2\varphi \leq 2\pi$. Then, for $\varphi = -\frac{5\pi}{12}$ in the required interval of φ, there is no contradiction. There are two different φ in the studied range $[-\pi, \pi]$.

It must be noted that criticism where $x = -1$ is computed and then $x^2 = 1$ and then have two different solutions is *not* an equivalent of what is presented. The big difference is that with $x = -1$ a correct soltion is primed. In our analysis *no* correct solution is primed before computation.

The function φ_λ: More details can be found in [7]. In the table below, for the interval $0 \leq x \leq 3\pi/2$, a number of points φ_λ has been computed for particular x. The result presented in Table-1 below extends what has been written in the preprints [7].

Table 1 Table representing a sample of the function $\varphi_\lambda(x)$, with $\lambda \downarrow 0$. We have, $y_{j1} = \sin(\varphi_{\lambda j})$ and $y_{j2} = \cos(\varphi_{\lambda j})$ with $j = 1, 2$ and $\varphi_{\lambda 2} = \varphi_{\lambda 1} + \pi$. Only sin of $2\varphi_{\lambda 1}$ and $2\varphi_{\lambda 2}$ are presented. For later purposes: $\varphi_{\lambda 1} = \varphi_\lambda^{\mathrm{ok}}$ & $\varphi_{\lambda 2} = \varphi_\lambda^{\mathrm{an}}$.

x	$\varphi_{\lambda 1}$	$\varphi_{\lambda 2}$	$\sin(2\varphi_{\lambda 1})$	$\sin(2\varphi_{\lambda 2})$	y_{11}	y_{12}	y_{21}	y_{22}
0.063	-2.325	0.817	0.998	0.998	-0.729	-0.685	0.729	0.685
1.068	-1.822	1.319	0.482	0.482	-0.969	-0.249	0.969	0.249
2.094	-1.309	1.833	-0.500	-0.500	-0.966	0.259	0.966	-0.259
2.136	-1.288	1.854	-0.536	-0.536	-0.960	0.279	0.960	-0.279
3.204	-0.754	2.388	-0.998	-0.998	-0.685	0.729	0.685	-0.729
4.021	-0.346	2.796	-0.637	-0.637	-0.339	0.941	0.339	-0.941
4.398	-0.157	2.985	-0.309	-0.309	-0.156	0.988	0.156	-0.988

2.2 Scalar diffraction

In this section we will employ Kirchhoff's scalar diffraction [1] in the case of small wavelength λ. In the first place let us recapitulate the computation of the complex amplitude, $U(P_0)$, of the observed field in point $P_0 = (x_{01}, x_{02}, x_{03}) = \vec{x}_0$. In scalar diffraction theory, the electric and magnetic field vector in P_0 at time t are generally written like $u(P_0, t) = \Re_e\{U(P_0)\exp[-2i\pi t\nu]\}$, viz. [1, pg 38, eq 3-10]. The expression for $U(P_0)$ is in Kirchoff's theory a surface integral over aperture $\mathcal{A}_p$

$$U(P_0) = \frac{1}{4\pi}\iint_{\mathcal{A}_p}\left\{G(P_1)\frac{\partial}{\partial n}U(P_1) - U(P_1)\frac{\partial}{\partial n}G(P_1)\right\}dS \tag{10}$$

The geometry of the screen plus aperture corresponds with the standard situation presented in [1, pg 45, fig 3.7]. The $U(P_1)$ is a single spherical wave that from P_2, illuminates the screen plus aperture [1, pg 45, fig 3.7]. The sperical wave $U(P_1)$ solves the Helmholtz equation $\left(\nabla_1^2 + k^2\right)U(P_1) = 0$, with k the wave number $k = 2\pi/\lambda$ and ∇_1 is the gradient vector operator $\left(\frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{13}}\right)$. The Green function, $G(P_1)$, also solves the Helmholtz equation (in $\vec{x}_1$). We define

$$U(P_1) = A_\lambda \frac{\exp[ikr_{21}]}{r_{21}} \tag{11}$$

Here, $A = A_\lambda = \left(e^{i\lambda} - (1 + \sin(\lambda))\right)$ is a constant in $\vec{x}_1$ and $\vec{x}_2$, viz. [1, pg 45]. The r_{21} is the Euclidean distance between point P_2 and P_1 in the aperture, or $r_{21} = ||\vec{x}_1 - \vec{x}_2|| > 0$. Let us subsequently define the Green function in (10) as in Kirchhoff's theory [1, pg43]

$$G(P_1) = \frac{\exp[ikr_{01}]}{r_{01}} \tag{12}$$

P_2 and P_0 are at opposite sides of the screen. They are not necessarily "mirror" images. Furthermore, $r_{01} = ||\vec{x}_1 - \vec{x}_0||$ and let us define

$$g_\lambda(P_1) = \exp[ikr_{01}] \left(e^{i\lambda} - (1 + \sin(\lambda)) \right) \tag{13}$$

Noting that $\frac{\partial}{\partial n} = \hat{n}.\nabla_1$ where $\hat{n}$ is the outward directed (towards P_2) normal, $||\ \hat{n}\ || = 1$, of the aperture $\mathcal{A}_p$, the dot represent the inner product and ∇_1 is the gradient defined previously. The y in (1) is here kr_{01}. Then,

$$\begin{aligned} \frac{\partial}{\partial n} U(P_1) = \cos(\ \hat{n}\ , \vec{x}_{21}) \left(ik - \frac{1}{r_{21}} \right) U(P_1) = \\ \cos(\ \hat{n}\ , \vec{x}_{21}) \sqrt{\left(\frac{1}{r_{21}^2} + k^2 \right)} \exp\left[-i \ \arctan\ (kr_{21}) \right] U(P_1) \end{aligned} \tag{14}$$

From the inner product of $\hat{n}$ and $\vec{x}_{21}$ we can obtain the cosine $\cos(\ \hat{n}\ , \vec{x}_{21}) = (\ \hat{n}\ \cdot \vec{x}_{21})/r_{21}$, etc. The $\cos(\ \hat{n}\ , \vec{x}_{21})$ is a shorthand for $\cos[\measuredangle(\ \hat{n}\ , \vec{x}_{21})]$ and $\measuredangle(\ \hat{n}\ , \vec{x}_{21})$ the angle between $\hat{n}$ and $\vec{x}_{21}$. Similar to (14)

$$\frac{\partial}{\partial n} G(P_1) = \cos(\ \hat{n}\ , \vec{x}_{01}) \sqrt{\left(\frac{1}{r_{01}^2} + k^2 \right)} \exp\left[-i \ \arctan\ (kr_{01}) \right] G(P_1) \tag{15}$$

Under the restriction that $\lambda \approx 0^+$ it follows that $\sqrt{\left(\frac{1}{r^2} + k^2 \right)} \approx k$ for both $r = r_{01}$ as well as for $r = r_{21}$, with $1/r$ finite. Then looking at equations (12) and (3) under $\lambda \approx 0^+$ while $U(P_1) \frac{\partial}{\partial n} G(P_1)$ as well as in $G(P_1) \frac{\partial}{\partial n} U(P_1)$ contains, referring to (13), the term $k|g_\lambda(P_1)|$. Therefore

$$k|g_\lambda(P_1)| \approx 2\pi\sqrt{2} \tag{16}$$

in the evaluation of $U(P_0)$ in (10). If we subsequently have nonzero finite r_{21} then $\arctan\ (kr_{21}) \approx \frac{\pi}{2}$, and also $\arctan\ (kr_{01}) \approx \frac{\pi}{2}$ under $\lambda \approx 0^+$. Therefore

$$U(P_0) \approx \frac{\sqrt{2}}{2} \iint_{\mathcal{A}_p} \frac{\cos(\ \hat{n}\ , \vec{x}_{21}) - \cos(\ \hat{n}\ , \vec{x}_{01})}{r_{01} r_{21}} \exp\left[i \left(\varphi - \frac{\pi}{2} + kr_{21} \right) \right] dS \tag{17}$$

The electric or magnetic vector components in scalar diffraction $u(P_0, t) = \Re_e\{U(P_0) \exp[-2i\pi t\nu]\}$, with $2\pi t\nu = k\frac{tc}{n}$ is, via Euler's identity [6],

$$\begin{aligned} u(P_0, t) \approx \frac{\sqrt{2}}{2} \Re_e \left\{ \iint_{\mathcal{A}_p} \frac{\cos(\ \hat{n}\ , \vec{x}_{21}) - \cos(\ \hat{n}\ , \vec{x}_{01})}{r_{01} r_{21}} \exp\left[i \left(\varphi - \frac{\pi}{2} + k \left(r_{21} - \frac{tc}{n} \right) \right) \right] dS \right\} \\ = \frac{\sqrt{2}}{2} \iint_{\mathcal{A}_p} \frac{\cos(\ \hat{n}\ , \vec{x}_{21}) - \cos(\ \hat{n}\ , \vec{x}_{01})}{r_{01} r_{21}} \cos\left[\varphi - \frac{\pi}{2} + k \left(r_{21} - \frac{tc}{n} \right) \right] dS \end{aligned}$$

The author employed Euler's identity. Now, from the previous section we have learned that, in the first place there are *two* different φ possible. One $\varphi = \varphi_\lambda^{\text{an}}$ and one $\varphi = \varphi_\lambda^{\text{ok}}$. In the second place we may deduce from Table-1 that $\varphi_\lambda^{\text{an}} = \varphi_\lambda^{\text{ok}} + \pi$. It then follows $\cos\left[\varphi_\lambda^{\text{an}} - \frac{\pi}{2} + k\left(r_{21} - \frac{tc}{n}\right)\right] = -\cos\left[\varphi_\lambda^{\text{ok}} - \frac{\pi}{2} + k\left(r_{21} - \frac{tc}{n}\right)\right]$. This implies,

$$u(P_0,t) \approx \frac{\sqrt{2}}{4} \iint_{\mathcal{A}_p} \frac{\cos(\hat{n}, \vec{x}_{21}) - \cos(\hat{n}, \vec{x}_{01})}{r_{01} r_{21}} \cos\left[\varphi - \frac{\pi}{2} + k\left(r_{21} - \frac{tc}{n}\right)\right]_{\varphi=\varphi_\lambda^{\text{an}}} dS \quad (18)$$
$$+ \frac{\sqrt{2}}{4} \iint_{\mathcal{A}_p} \frac{\cos(\hat{n}, \vec{x}_{21}) - \cos(\hat{n}, \vec{x}_{01})}{r_{01} r_{21}} \cos\left[\varphi - \frac{\pi}{2} + k\left(r_{21} - \frac{tc}{n}\right)\right]_{\varphi=\varphi_\lambda^{\text{ok}}} dS$$

Hence, when the anomalous value is included, $u(P_0,t) \approx 0$. Note that A_λ is not present in the two integrals of equation (18) because of the result in (3).

3 Result & discussion

In section-2.1 a contradiction of the complex numbers was demonstrated. Two different phase angles are associated; i.e. the $\varphi_\lambda^{\text{an}}$ and $\varphi_\lambda^{\text{ok}}$, viz. Table-1. The $y = (x + \pi)/2$ in section-2.1 is, as can be seen from equation (13), related to kr_{01}. When, $\ell' \in \mathbb{N}$ then, $y - 2\pi\ell'$ is equivalent to y in the analysis. From $0 \leq x \leq 3\pi/2$ as in our Table-1, it is possible to obtain: $\lambda(\ell' + \frac{1}{4}) \leq r_{01} \leq \lambda(\ell' + \frac{5}{8})$. We can find a possible ℓ' for a small λ such that a set of r_{01} embraces a realistic observer position P_0. In the visible range $\lambda = 4 \times 10^{-7}$ meter. If, $\ell' = 10^7$ then: $4 + 10^{-7} \leq r_{01} \leq 4 + (\frac{20}{8} \times 10^{-7})$ determines r_{01} within Table-1. Obviously, the research for Table-1 can be extended[1].

In section-2.2 the equation (18) was derived. There is no reason to disallow $\varphi = \varphi_\lambda^{\text{an}}$ in the first and $\varphi = \varphi_\lambda^{\text{ok}}$ in the second integral of (18). The $\varphi = \varphi_\lambda^{\text{an}}$ in section-2.1 is in all aspects, except for the anomaly when looking back to the formulae, equivalent to $\varphi = \varphi_\lambda^{\text{ok}}$.

Therefore, will a further inevitable inconsistency such as presented in section-2.1 play a role in experimental result. In other words: given A_λ is small in the point source (11) i.e. λ is small, then

– if nature *excludes* the $\varphi_\lambda^{\text{an}}$ then despite small A_λ in experiment, it is possible to have $|u(P_0,t)| > 0$. The intensity [4, pg 8] equals $I(P_0) = |U(P_0))|^2$. If $\varphi = \varphi_\lambda^{\text{an}}$ is *excluded*

[1] x is not a coordinate like P_0, P_1 or P_2 is.

then from a faint source $A_\lambda \approx 0$ for $\lambda \approx 0^+$, $I(P_0)$ is independent of the order of magnitude of A_λ.

– if nature does "select" $\varphi_\lambda^{\mathrm{an}}$ then $|u(P_0, t)| \approx 0$, predicted via (18), is found in experiment.

Keller [5] argues that diffraction (coefficients) vanish at small wavelength and only geometrical terms remain [5, pg 116].

Further, one can add $2\ell\pi$, with $\ell \in \mathbb{Z}$, in the cos argument. Therefore, we may look at distances $r_{21} = (tc/n) - \ell\lambda$. The diffraction will then, when $\varphi_\lambda^{\mathrm{an}}$ is excluded , in approximation at least contain the term:

$$\Delta_{excl} = \frac{\sqrt{2}}{4}\left(\frac{1}{(tc/n) - \lambda\ell}\right)\iint_{\mathcal{A}_p}\left(\frac{\cos(\ \hat{n}\ , \vec{x}_{01}) - \cos(\ \hat{n}\ , \vec{x}_{21})}{r_{01}}\right)\sin(\varphi_\lambda^{\mathrm{ok}})dS$$

Use is made of $\cos(\varphi_\lambda^{\mathrm{ok}} - \pi/2) = -\sin(\varphi_\lambda^{\mathrm{ok}})$ in (18), when $2\ell\pi + k\left(r_{21} - \frac{tc}{n}\right) = 0$. Moreover, in this particular example, the approximation is that the aperture P_1 variation doesn't influence the r_{21} much. Hence, the $\cos(\ \hat{n}\ , \vec{x}_{21})$ does not change much in P_1.

Because $r_{21} > 0$, it follows that $\frac{ct}{n\lambda} > \ell$. Hence, when $\varphi_\lambda^{\mathrm{an}}$ is *not* selected and $\nu t > \ell$, & given $\Delta\mathcal{E}_m = \hbar\nu_m$, per "photon". Provided $\Delta E = \sum_{m=1}^{\ell}\Delta\mathcal{E}_m$ which means $\nu_m = \nu$, we can have $\Delta E \Delta t > \hbar\ell$. Therefore, the faint point source $U(P_1)$ in (11), will possibly give $|u(P_0, t)| > 0$ when coherence $\nu_m = \nu$ occurs for ℓ photons. This represents a nonzero diffraction from a quantum coherent bundle of photons originating from a faint source of light despite a small wavelength viz. [5]. There is no a priori rule to exclude or include the contradictory phase angle.

In the paper fundamental mathematics is connected to physical optics. The question is, will the contradictory phase angle of section-2.1 be excluded in wave mechanics experiments yes or no.

Declarations

The author has no conflict of interest. The author was not funded. There is no data associated.

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