How the twins each age less than the other

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I show that the asymmetry in a twin paradox experiment resulting in a difference in aging is that the twins move different distances relative to each other, due to length contraction. I show that the twins each age less than the other after accounting for relativity of simultaneity, yet unparadoxically. I give an equation to calculate the constant speed at which a spaceship must travel so that its occupants age a given time during a trip. I give code to numerically integrate a twin paradox experiment involving acceleration.

1 Simplifying the puzzle

Experiment #1: Sue makes a round trip from Earth to the Alpha Centauri star system, 4.4 light years away, while Bob stays on Earth. She launches from Earth, accelerates and decelerates at 1 g to arrive at Alpha Centauri at relative rest, and immediately returns to Earth in the same way.

See the equations of special relativity at The Relativistic Rocket, for a rocket having a constant proper acceleration $a > 0$. The equations that predict the elapsed times for each quarter of the trip are

$$t = \sqrt{(d/c)^2 + 2d/a} \tag{1}$$

$$T = \frac{c}{a}\operatorname{acosh}(ad/c^2 + 1) \tag{2}$$

For one quarter of the trip, the distance $d = 2.2$ ly. Use $c = 1$ ly/yr and $a = 1.03$ ly/yr$^2 \approx g$. Multiply the results of (1) for Bob and (2) for Sue by 4 to see that Bob ages $t \approx 12.1$ yr and Sue ages $T \approx 7.2$ yr. The puzzle of the twin paradox is that each twin should paradoxically age less than the other, as predicted by the gamma factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{t}{T} \tag{3}$$

because each twin sees the other as moving.

Other texts don't resolve the paradox. Just showing that there is asymmetry between the spacetime paths of the twins is insufficient. An actual resolution would show how the twins each age less than the other, yet unparadoxically.

Since (1) and (2) return different agings without a turnaround, the puzzle must be resolvable without considering a turnaround. The clock postulate tells us that acceleration also needn't be considered:

> The clock postulate [says] that even when the moving clock accelerates, the ratio of the rate of our clocks compared to its rate is still [the gamma factor γ]. That is, this ratio depends only on [speed] v, and does not depend on any derivatives of v, such as acceleration. So this says that an accelerating clock will count out its time in such a way that at any one moment, its timing has slowed by a factor (γ) that depends only on its current speed; its acceleration has no effect at all.

2 A clue: missing lines of simultaneity

Missing lines of simultaneity give a clue to resolving the twin paradox:

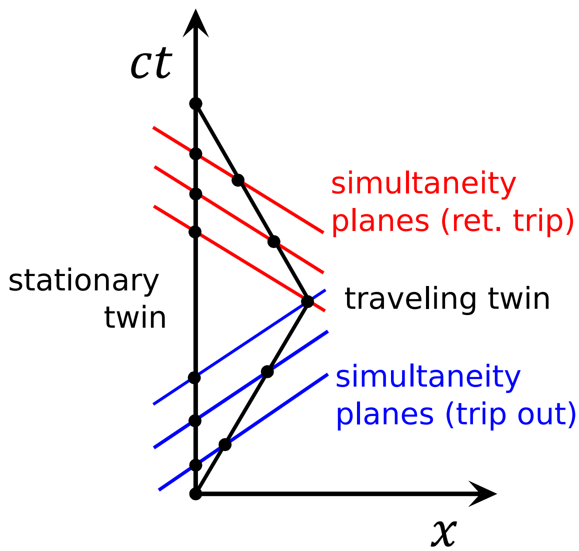

Figure 1: Spacetime diagram of a twin paradox where the traveling twin is actually two travelers moving at constant speed, one outgoing from the starting point and another incoming toward it, passing by each other where the turnaround point would be. At this moment, the clock reading for the first traveler is transferred to the second one. Their trip times are summed at the end of their journey. By Acdx, CC BY-SA 3.0, via Wikimedia Commons.

In Fig. 1, notice that the middle part of the stationary twin's world line (on the vertical axis) has no possible line of simultaneity that connects to the traveling twins' world lines, whereas any event on the traveling twins' world lines can connect to the stationary twin's world line. These missing lines of simultaneity allow for the possibility that the twins in any twin paradox experiment each age less than the other, but without paradox, because time periods during the stationary twin's experiment (more specifically, the events during those time periods) don't occur during the traveling twin's experiment. Which is to say that the traveling twin's experiment is a subset of the stationary twin's experiment.

3 Speeding to Andromeda

See the sample problem "Speeding to Andromeda" at Exploring Black Holes, in the chapter "Speeding":

> At approximately what constant speed v_{Sun} with respect to our Sun must a spaceship travel so that its occupants age only 1 year during a trip from Earth to the Andromeda galaxy?

The method therein to get the answer ($1 - v_{\text{Sun}} \approx 1.25 \times 10^{-13}$) takes several steps and requires that the speed is close to the speed of light. This equation works in every case:

$$v = \frac{d/T}{\sqrt{1 + [d/(Tc)]^2}} \tag{4}$$

When $d = 2$ million ly and $T = 1$ yr, (4) returns the same answer as above.

Here is the derivation of (4). From basic physics,

$$t = \frac{d}{v} \tag{5}$$

From The Relativistic Rocket,

$$v = \frac{at}{\sqrt{1 + (at/c)^2}} \tag{6}$$

$$\gamma = \sqrt{1 + (at/c)^2} \tag{7}$$

Refer to the gamma factor (3). Substituting and rearranging:

$$v = \frac{at}{\gamma} \tag{8}$$

$$t = T\gamma = \frac{d}{v} \tag{9}$$

$$at = v\gamma = \frac{d}{T} \tag{10}$$

Substituting the two terms at in (6) with the d/T from (10) gives (4). This completes the derivation of (4).

Rearranging (9):

$$v = \frac{d}{T\gamma} \tag{11}$$

Eq. (11) shows that the speed v needed to get to a destination while aging time T is the speed that length contracts the distance d to that which is covered in time T at that speed.

Experiment #2: Joy ages 10 years while traveling from the star Vega to Earth at a constant speed v relative to our Sun. Bob stays on Earth.

Rearranging (11):

$$T = \frac{d}{v\gamma} \tag{12}$$

When the Earth-Vega distance d in Bob's frame =	25 ly,
and the given aging T for Joy =	10 yr,
then, as calculated by (4), the speed v needed by the twins toward each other ≈	$0.928c$,
and, as calculated by (5), Bob's aging, the time t taken by Bob to reach Joy in his frame ≈	26.9 yr,
and, as calculated by the gamma factor (3), the gamma factor γ for the Earth-Vega system in Joy's frame ≈	2.69,
and the length-contracted distance d/γ between Vega and Earth in Joy's frame ≈	9.28 ly,
and, as calculated by (12), the time T taken by Joy to reach Earth in her frame =	10.0 yr,

which matches the given aging T for Joy that was input into (4), as expected.

4 Length contraction explains the difference in aging

The asymmetry in a twin paradox experiment resulting in a difference in aging is that the distance the twins move relative to each other is less in the traveling twin's frame, due to length contraction, so that the experiment completes faster in that frame. (At any moment the twins have the same speed relative to each other. Moving less distance at the same speed takes less time.) This is shown when their relative speed is constant by plugging (5) and (12) into the gamma factor (3) to get

$$\frac{t}{T} = \frac{d/v}{d/(v\gamma)} = \gamma \tag{13}$$

5 How a true paradox is ruled out

The twins each age less than the other after accounting for relativity of simultaneity, yet unparadoxically.

To see this for experiment #2, overlay the barn-pole paradox: Bob, representing the runner, holds the trailing end of the pole that has a proper length d, the Earth-Vega distance in his frame. Joy stays at the far door of the barn. The experiment starts when she passes Vega and ends when the twins pass each other. Let the proper distance between the barn doors be $D \equiv d/\gamma$, so that in her frame (the barn frame) he's at the near door of the barn when the experiment starts. In his frame, she's distance d away from him when the experiment starts, and the barn is length contracted to $d_b = D/\gamma$. During the experiment in his frame, he ages $t = d/v$ and she ages $T = d/(v\gamma) = D/v$. During the experiment in her frame, she ages $T = D/v$ and he ages $t_b = D/(v\gamma) = d_b/v$, his aging while traversing the barn. They each age the same percentage less than the other, as shown by $t/T = T/t_b = \gamma$. The exact part of his aging that's needed to rule out a paradox, $t - t_b =$ his aging while covering the distance to the barn, doesn't occur during the experiment in her frame. See also the ladder paradox.

Experiment #3: Eve ages 10 years while traveling from Earth to the star Vega at a constant speed v relative to our Sun. Bob stays on Earth.

For experiment #3 we can run the barn-pole paradox version of experiment #2 in reverse, to reason that the twins must still each age the same percentage less than the other. The experiment starts when the twins pass each other and ends when Eve passes Vega. The exact part of Bob's aging that's needed to rule out a paradox, $t - t_b =$ his aging while covering the distance from the barn, doesn't occur during the experiment in her frame.

In section 4 I said that the experiment completes faster in the traveling twin's frame. This comes with a caveat: we ignore relativity of simultaneity, by ignoring that the traveling twin's experiment is a subset of the stationary twin's experiment. For example, for experiment #2 we pretend that the experiment starts in Joy's frame at $t = T = 0$, as it does in Bob's frame, when in reality in her frame at $T = 0$ (when she passes Vega) the experiment is already underway in his frame; i.e. $t > 0$. When we account for relativity of simultaneity we find that the experiment completes faster for each twin during the experiment in the other's frame.

6 Numerical integration for an experiment involving acceleration

A twin paradox experiment can be segmented into sub-experiments wherein the twins have a constant speed relative to each other, and the agings in the sub-experiments summed to get agings for the whole experiment.

For example, we segment experiment #2 into three equal sub-experiments. Add two observers Ed and Jeb between Earth and Vega. Bob/Earth, Ed, Jeb, and Vega are positioned in that order, at rest relative to one another, and equally spaced in Bob's frame. The first sub-experiment starts when Joy passes Vega and ends when she passes Jeb. The second sub-experiment starts when she passes Jeb and ends when she passes Ed. The third and final sub-experiment starts when she passes Ed and ends when she passes Bob. For each sub-experiment:

$d = 25/3$	8.3	ly
$T = 10/3$	3.3	yr
v	0.9	c
γ	2.7	
$D = d/\gamma$	3.1	ly
$d_b = D/\gamma$	1.1	ly

$t = d/v$	9.0	yr
$t_b = d_b/v$	1.2	yr

These values are further explained in section 5, re the barn-pole paradox. The cumulative agings in years are

		Start			End	
	n	$(n-1)t$	$nt - t_b$	$(n-1)T$	nt	nT
Jeb	1	0.0	7.7	0.0	9.0	3.3
Ed	2	9.0	16.7	3.3	18.0	6.7
Bob	3	18.0	25.7	6.7	26.9	10.0

where

n	is the segment or sub-experiment number,
$(n-1)t$	is the listed twin's aging when the sub-experiment starts in their frame,
$nt - t_b$	is the listed twin's aging when the sub-experiment starts in Joy's frame,
$(n-1)T$	is Joy's aging when the sub-experiment starts in either twin's frame,
nt	is the listed twin's aging when the sub-experiment ends in either twin's frame, and
nT	is Joy's aging when the sub-experiment ends in either twin's frame.

The listed twin ages $t \approx 9.0$ yr during the sub-experiment in their frame, ages $t_b \approx 1.2$ yr during the sub-experiment in Joy's frame, and she ages $T \approx 3.3$ yr during the sub-experiment in either twin's frame. The $t_b < t$ because the sub-experiment is already underway $(t > 0)$ in the listed twin's frame when the sub-experiment starts in her frame at $T = 0$. For example, Jeb has already aged $t - t_b \approx 7.7$ yr when the first sub-experiment starts in Joy's frame, when she passes Vega.

Revisit experiment #1. The twins' agings can be calculated by numerically integrating one quarter of Sue's round trip, calculating their agings for each segment or sub-experiment that's at constant speed, and then multiplying the sum of those results by 4. Here is code to show this. Click the Run button to get the output:

```
Predicted by the Relativistic Rocket equations:
      Bob ages 12.1 yr
      Sue ages 7.2 yr

Predicted by this code:
      During the experiment in Bob's frame:
            Bob ages 12.1 yr
            Sue ages 7.2 yr
            They move 8.8 ly relative to each other in his frame
            They move 4.6 ly relative to each other in her frame
```

```
During the experiment in Sue's frame:
       Bob ages 4.9 yr
       Sue ages 7.2 yr
       They move 2.7 ly relative to each other in his frame
       They move 4.6 ly relative to each other in her frame
```

The experiment in Sue's frame is a subset of the experiment in Bob's frame, even though the experiment starts and ends when they're together on Earth. That's how they can each age less than the other without paradox. Every event on her world line has a line of simultaneity that connects to his world line, but not vice versa. In each sub-experiment, $t_b < t$, whereas her aging is T in both frames. Her sub-experiment is thus a subset of his sub-experiment. So none of his aging $> t_b$ in any sub-experiment occurs during any part of the whole experiment in her frame.

Appendix A – Helpful equations

These equations can help for calculations or code:

$d = vt$

$T = D/v = t/\gamma$

$v = d/t = D/T = d_b/t_b$

$\gamma = t/T = T/t_b$

$D \equiv d/\gamma = vT$

$d_b = D/\gamma = vt_b$

$t = d/v = \gamma T$

$t_b = d_b/v = T/\gamma$

Appendix B – Code for the program

Below is the Go language code for the numerical integration program that's referenced in section 6, in case the link to the code is broken. You can run the code at the Go Playground after fixing the formatting.

```
package main

import (
       "fmt"
       "math"
)

const (
       // Sue's acceleration in ly/yr^2, = ~1 g
       a = 1.03

       // Half of the distance in ly between Earth and Alpha Centauri as
measured in Bob's frame
       d = 2.2
```

```
	// The number of steps per year in Bob's frame, in the numerical
integration below
	// More steps gives greater accuracy
	segmentCount int = 50

	// The speed of light = 1 ly/yr
	c = 1

	// End of user input

	// Bob's aging in years that is considered in each step
	tSegment = 1 / float64(segmentCount)
)

func main() {
	// For further explanation of these variables, see section 5
	// As predicted by the code:
	// During the experiment in Bob's frame:
	t := 0.0 // His aging
	dCheck := 0.0 // The distance the twins recede from each other in
his frame
	// During the experiment in either twin's frame:
	T := 0.0 // Sue's aging
	D := 0.0 // The distance the twins recede from each other in her
frame
	// During the experiment in her frame:
	tb := 0.0 // His aging
	db := 0.0 // The distance the twins recede from each other in his
frame

	for {
		t += tSegment

		// Get their speed v relative to each other, which could be
measured instead
		at := a * t
		// These equations are from the Relativistic Rocket site
		gamma := math.Sqrt(1 + math.Pow(at / c, 2))
		v := at / gamma

		// Get Sue's aging during tSegment
		// This is the gamma factor equation
		TSegment := tSegment / gamma

		T += TSegment

		// Get the distance that the twins recede from each other in
Sue's frame during
		// TSegment
		DSegment := v * TSegment

		D += DSegment

		tb += TSegment / gamma
```

```
            db += DSegment / gamma

            dCheck += v * tSegment

            if dCheck >= d {
                  // The distance d has been reached
                  break
            }
      }

      // These equations are from the Relativistic Rocket site
      // Bob's aging
      tExpected := math.Sqrt(math.Pow(d / c, 2) + 2 * d / a)
      // Sue's aging
      TExpected := (c / a) * math.Acosh(a * d / math.Pow(c, 2) + 1)

      // Make a round trip
      dRoundTrip := d * 4
      t *= 4
      T *= 4
      D *= 4
      tb *= 4
      db *= 4
      tExpected *= 4
      TExpected *= 4

      fmt.Printf("Predicted by the Relativistic Rocket equations:\n")
      fmt.Printf("\tBob ages %0.1f yr\n", tExpected)
      fmt.Printf("\tSue ages %0.1f yr\n", TExpected)

      fmt.Printf("\nPredicted by this code:\n")
      fmt.Printf("\tDuring the experiment in Bob's frame:\n")
      fmt.Printf("\t\tBob ages %0.1f yr\n", t)
      fmt.Printf("\t\tSue ages %0.1f yr\n", T)
      fmt.Printf("\t\tThey move %0.1f ly relative to each other in his
frame\n", dRoundTrip)
      fmt.Printf("\t\tThey move %0.1f ly relative to each other in her
frame\n", D)

      fmt.Printf("\n\tDuring the experiment in Sue's frame:\n")
      fmt.Printf("\t\tBob ages %0.1f yr\n", tb)
      fmt.Printf("\t\tSue ages %0.1f yr\n", T)
      fmt.Printf("\t\tThey move %0.1f ly relative to each other in his
frame\n", db)
      fmt.Printf("\t\tThey move %0.1f ly relative to each other in her
frame", D)

      // fmt.Printf("\n\nThere were %0.0f steps in the numerical
integration\n", t / tSegment)
}
```