Assuming $c < rad^2(abc)$, The abc Conjecture is True

Abdelmajid Ben Hadj Salem[1*]

[1*]Residence Bousten 8, Bloc B, Street Mosquée Raoudha, Soukra, 1181 Soukra Raoudha, Tunisia.

Corresponding author(s). E-mail(s): abenhadjsalem@gmail.com;

Abstract

In this paper, we consider the $\boldsymbol{abc}$ conjecture. Assuming that $\boldsymbol{c < rad^2(abc)}$ is true, we give the proof of the $\boldsymbol{abc}$ conjecture for $\boldsymbol{\epsilon \geq 1}$, then for the case $\boldsymbol{\epsilon \in]0,1[}$, we consider that the $\boldsymbol{abc}$ conjecture is false, from the proof, we arrive in a contradiction.

Keywords: Elementary number theory, real functions of one variable, transcendental number

MSC Classification: 11AXX , 26AXX , 11JXX

To the memory of my Father who taught me arithmetic,
To my Wife, my Daughter and my Son.

1 *Introduction and notations*

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University

(Paris 6) [4]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1 (***abc* Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication in September 2018, of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [4]. A conjecture was proposed that $c < rad^2(abc)$ [3]. It is the key to resolve the *abc* conjecture. In my paper, I assume that the conjecture $c < rad^2(abc)$ holds, I propose an elementary proof of the *abc* conjecture.

2 *The Proof of the abc conjecture*

Proof We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$. We assume that $c < R^2$ is true.

2.1 Case : $\epsilon \geq 1$

Assuming that $c < R^2$ is true, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with\ \ K(\epsilon) = e,\ \epsilon \geq 1 \tag{4}$$

Then the *abc* conjecture is true.

2.2 Case: $\epsilon < 1$

From the statement of the *abc* conjecture 1, we want to give a proof that $c < K(\epsilon)R^{1+\epsilon} \Longrightarrow LogK(\epsilon) + (1+\epsilon)LogR - Logc > 0$.

For our proof, we proceed by contradiction of the abc conjecture. We suppose that the *abc* conjecture is false:

$$\begin{gathered} \exists\, \epsilon_0 \in]0,1[, \forall\, K(\epsilon) > 0, \quad \exists\, c_0 = a_0 + b_0; \quad a_0, b_0, c_0 \text{ coprime so that} \\ c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \text{ and } \forall \epsilon \in]0,1[,\ c_0 > K(\epsilon)R_0^{1+\epsilon} \end{gathered} \tag{5}$$

We choose the constant $K(\epsilon) = e^{\frac{1}{\epsilon^2}}$. Let :

$$Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1+\epsilon)LogR_0 - Logc_0, \epsilon \in]0,1[\tag{6}$$

From the above explications, if we will obtain $\forall\epsilon \in]0,1[, Y_{c_0}(\epsilon) > 0 \Longrightarrow c_0 < K(\epsilon)R_0^{1+\epsilon} \Longrightarrow c_0 < K(\epsilon_0)R_0^{1+\epsilon_0}$, then the contradiction with (5).

About the function Y_{c_0}, we have:

$$lim_{\epsilon\longrightarrow 1}Y_{c_0}(\epsilon) = 1 + Log(R_0^2/c_0) = \lambda > 0$$

$$lim_{\epsilon\longrightarrow 0}Y_{c_0}(\epsilon) = +\infty$$

The function $Y_{c_0}(\epsilon)$ has a derivative for $\forall\epsilon \in]0,1[$, we obtain:

$$Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + LogR_0 = \frac{\epsilon^3 LogR_0 - 2}{\epsilon^3} \tag{7}$$

$Y'_{c_0}(\epsilon) = 0 \Longrightarrow \epsilon = \epsilon' = \sqrt[3]{\dfrac{2}{LogR_0}} \in]0,1[$ for $R_0 \geq 8$.

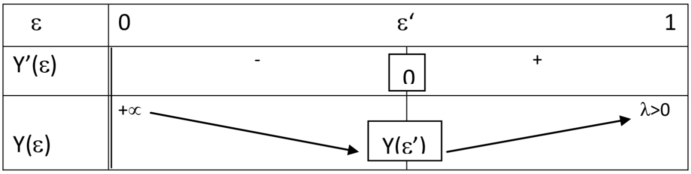

Fig. 1 Table of variations

Discussion from the table (Fig.: 1):

- If $Y_{c_0}(\epsilon') \geq 0$, it follows that $\forall\epsilon \in]0,1[, Y_{c_0}(\epsilon) \geq 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \Longrightarrow c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$ and the supposition that the *abc* conjecture is false can not hold. Hence the *abc* conjecture is true for $\epsilon \in]0,1[$.

- If $Y_{c_0}(\epsilon') < 0 \Longrightarrow \exists\, 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2}$. We recall the following definition:

Definition 2 The number ξ is called algebraic number if there is at least one polynomial:

$$l(x) = l_0 + l_1x + \cdots + a_mx^m, \quad a_m \neq 0 \tag{8}$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality :

$$c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \Longrightarrow \frac{c_0}{R_0} = \frac{\mu_{c_0}}{rad(a_0b_0)} = e^{\frac{1}{\epsilon_1^2}}R_0^{\epsilon_1} \tag{9}$$

i) - We suppose that $\epsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\epsilon_1^2$ and $\alpha_1 = R_0$ are also algebraic numbers. We obtain:

$$\frac{c_0}{R_0} = \frac{\mu_{c_0}}{rad(a_0b_0)} = e^{\frac{1}{\epsilon_1^2}}R_0^{\epsilon_1} = e^{\beta_0}.\alpha_1^{\beta_1} \tag{10}$$

From the theorem (see theorem 3, page 196 in [1]):

Theorem 3 *$e^{\beta_0}\alpha_1^{\beta_1}\dots\alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1,\dots,\alpha_n,\beta_0,\dots,\beta_n$.*

we deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (10) is transcendental, but the term $\dfrac{\mu_{c_0}}{rad(a_0b_0)}$ is an algebraic number, then the contradiction and the case $Y_{c_0}(\epsilon')<0$ is impossible. It follows $Y_{c_0}(\epsilon')\geq 0$ then the *abc* conjecture is true.

ii) - We suppose that ϵ_1 is transcendental, then $1/(\epsilon_1^2), e^{1/(\epsilon_1^2)}$ and $R_0^{\epsilon_1}=e^{\epsilon_1 Log R_0}$ are also transcendental, we obtain that c_0/R_0 is transcendental, then the contradiction with c_0/R_0 an algebraic number. It follows that $Y_{c_0}(\epsilon')\geq 0$ and the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished. Assuming $c<R^2$ true, we obtain that $\forall\epsilon>0$, $\exists K(\epsilon)>0$, if $c=a+b$ with a,b,c positive integers relatively coprime, then :

$$c<K(\epsilon).rad^{1+\epsilon}(abc) \tag{11}$$

and the constant $K(\epsilon)$ depends only of ϵ.

Q.E.D

Ouf, end of the mystery!

□

3 *Conclusion*

Assuming $c<R^2$ is true, we have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 4 *Assuming the conjecture $c<R^2$ true, the abc conjecture is true:*

For each $\epsilon>0$, there exists $K(\epsilon)>0$ such that if a,b,c positive integers relatively prime with $c=a+b$, then:

$$c<K(\epsilon).rad^{1+\epsilon}(abc) \tag{12}$$

where K is a constant depending of ϵ.

Acknowledgments

The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the *abc* conjecture.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] A. Baker, Effective Methods in Diophantine Problems. Proceedings of Symposia in Pure Mathematics, Volume **XX**, 1969 Number Theory Institute. AMS. (1971) 451 pages, pp 195–205.

[2] K. Kremmerz, Titans of Mathematics Clash Over Epic Proof of ABC Conjecture. The Quanta Newsletter, 20 September 2018. www.quantamagazine.org. (2018)

[3] P. Mihăilescu, Around ABC. European Mathematical Society Newsletter, $N°$ **93**, September 2014, (2014) pp 29–34.

[4] M. Waldschmidt, On the abc Conjecture and some of its consequences, presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9. (2013)