ON A VARIANT OF BROCARD'S PROBLEM VIA THE DIAGONALIZATION METHOD

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Abstract. In this paper we introduce and develop the method of diagonalization of functions $f : \mathbb{N} \longrightarrow \mathbb{R}$. We apply this method to show that the equations of the form $\Gamma_r(n) + k = m^2$ has a finite number of solutions $n \in \mathbb{N}$ with $n > r$ for a fixed $k, r \in \mathbb{N}$, where $\Gamma_r(n) = n(n-1)\cdots(n-r)$ is the r^{th} truncated Gamma function.

1. Introduction and problem statement

Brocard's problem is an unsolved problem which - roughly speaking - asks if the set of integers whose factorials are unit left translate of a square is either finite or infinite. As the origin of the problem suggests, it was first formulated by the French mathematician Henri Brocard around 1876 and 1885 and subsequently rediscovered by the Indian mathematician Srinivasa Ramanujan in 1913. More formally the problem states

Problem 1.1. Does the equation $n!+1 = m^2$ has integer solutions other than $4, 5, 7$?

It is widely believed any other solutions to Brocard's equation - if they exist - must be finite. In fact, Paul Erdős conjectured that no further solution to Brocard's equation exists. It has also been verified computationally for numbers up to 10^9 for a posssible solution to the equation and within this threshold no further solution to Brocard's problem has been found [4]. Albeit the problem remains unsolved, a good number of theoritical progress has been made. The first major progress has been made by M. Overholt, who showed that there are only a finite number of integer solutions to the Brocard equation $n! + 1 = m^2$ by assuming the ABC conjecture [1]. These result has also been extended to equations with arbitrary shifts of the form $n! + A = m^2$ [2]. A further extension has been made in [3] where it is shown that the equation $n! = P(x)$, where $P(x)$ is a polynomial of degree at least two, also has a finite number of solution by assuming the ABC conjecture.

In this paper we apply the method of **diagonalization** of functions to show that the equation $\Gamma_r(n) + k = n^2$ has a finite number of solutions for $n \in \mathbb{N}$.

1.1. **Notations.** In this paper, we will write $f(n) \ll g(n)$ to mean there exists an absolute constant $c > 0$ such that for all sufficiently large n, then $f(n) \leq c|g(n)|$. Conversely, we will write $f(n) \gg g(n)$ if the reverse inequality holds for all sufficiently large values of n. If both inequalities hold then we write in simple terms $f(n) \asymp g(n)$.

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2. **The notion of diagonalization**

In this section we introduce and study the notion of **diagonalization** of a function. We study this notion together with associated statistics and explore some applications.

Definition 2.1. Let $f : \mathbb{N} \longrightarrow \mathbb{R}$. Then we say f is k - step **diagonalizable** at the spot $n \in \mathbb{N}$ if there exists some $m \in \mathbb{N}$ such that

$$f(n) + k = m^2.$$

We call the set of all spots $n \in \mathbb{N}$ such that f is k - step diagonalizable the k^{th} - step diagonal of f and denote by $\mathcal{D}_k(f)$. We call the set of all truncated spots $\mathcal{D}_k(f) \cap \mathbb{N}_s := \mathcal{D}_k(f, s)$ the s^{th} scale diagonal. We call the set of all squares

$$\mathbb{B}_k(f) := \left\{m^2 \in \mathbb{N} \mid f(n) + k = m^2\right\}$$

the k^{th}-step diagonal squares. We write the length of this diagonal as

$$|\mathcal{D}_k(f, s)| := \#\{n \leq s \mid f(n) + k = m^2\}.$$

It is easy to see that $|\mathcal{D}_k(f, s)| < s$.

2.1. **The s-level trace of the diagonal.**

In this section we introduce the notion of the trace of the diagonal. We launch and examine the following languages.

Definition 2.2. By the s^{th} level trace of the diagonal $\mathcal{D}_k(f)$, denoted $\mathbb{T}_f(s, k)$, we mean the partial sum

$$\mathbb{T}_f(s, k) := \sum_{\substack{n \leq s \\ n \in \mathcal{D}_k(f)}} f(n).$$

Let us suppose that f is a function with continuous derivative on $[1, s]$ for $s \geq 1$ with $s \in \mathbb{R}$, then by applying the Stieltjes integration by parts, we can write the s^{th} level trace of the diagonal in the form

$$\begin{aligned}\mathbb{T}_f(s, k) : &= \sum_{\substack{n \leq s \\ n \in \mathcal{D}_k(f)}} f(n) \\ &= \int_{1^-}^{s} f(t) d|\mathcal{D}_k(f, t)| \\ &= f(s)|\mathcal{D}_k(f, s)| - \int_{1}^{s} f'(t)|\mathcal{D}_k(f, t)| dt.\end{aligned}$$

Theorem 2.3 (Diagonal inequality)**.** *Let f be a function with continuous derivative on $[1, s]$ for $s \geq 1$ with $s \in \mathbb{R}$. If*

$$\mathcal{D}_k(f, s) - \frac{1}{f(s)}\left(\sqrt{\int_{1}^{s} |f'(t)|^2 dt}\right) \times \left(\int_{1}^{s} |\mathcal{D}_k(f, t)|^2 dt\right)^{\frac{1}{2}} \geq 0$$

for all $s \geq 1$ then the inequality holds

$$(\int\limits_{1}^{s} |\mathcal{D}_k(f,t)|^2 dt)^{\frac{1}{2}} \ll \left(\frac{1}{f(s)}\sum_{n\leq s} f(n)\right)\left(1-\frac{1}{f(s)}\sqrt{\int\limits_{1}^{s}|f'(t)|^2 dt}\right)^{-1}.$$

Proof. By appealing to the ensuing discussion, we obtain the upper bound

$$|\mathcal{D}_k(f,s)| \leq \frac{1}{f(s)}\sum_{n\leq s} f(n) + \frac{1}{f(s)}\int\limits_{1}^{s} f'(t)|\mathcal{D}_k(f,t)|dt$$

so that by appealing to the Cauchy-Schwartz inequality, we obtain further the upper bound

$$|\mathcal{D}_k(f,s)| \leq \frac{1}{f(s)}\sum_{n\leq s} f(n) + \frac{1}{f(s)}\left(\int\limits_{1}^{s}|f'(t)|^2 dt\right)^{\frac{1}{2}} \times \left(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2 dt\right)^{\frac{1}{2}}.$$

By rearranging terms, appealing to the condition

$$\mathcal{D}_k(f,s) - \frac{1}{f(s)}\left(\sqrt{\int\limits_{1}^{s}|f'(t)|^2 dt}\right) \times \left(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2 dt\right)^{\frac{1}{2}} \geq 0$$

and noting that

$$(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2 dt)^{\frac{1}{2}} \geq |\mathcal{D}_k(f,s)|$$

for all $s \geq 1$, then the claimed inequality holds. □

Proposition 2.3 supplies a useful inequality to study Brocard's problem, which ask whether there exists a finite number of solutions to the equation $n! + 1 = m^2$. The current development can be leveraged to study a much more general version of the problem. By applying the **Diagonal** inequality, we can obtain further the result

Proposition 2.1 (The Diagonal method)**.** *Let*

$$|\mathcal{D}_k(f,s)| - \frac{1}{f(s)}\left(\sqrt{\int\limits_{1}^{s}|f'(t)|^2 dt}\right) \times \left(\int\limits_{1}^{s}|\mathcal{D}_k(f,t)|^2 dt\right)^{\frac{1}{2}} \geq 0$$

for all $s \geq 1$. If

$$\lim_{s\longrightarrow\infty}\left(\frac{1}{f(s)}\sum_{n\leq s} f(n)\right)\left(1-\frac{1}{f(s)}\sqrt{\int\limits_{1}^{s}|f'(t)|^2 dt}\right)^{-1} < \infty$$

then the equation $f(n) + k = m^2$ has only a finite number of solutions in $\mathbb{N}$ for a fixed $k \in \mathbb{N}$.

Proof. Appealing to Proposition 2.3, it follows under the requirements that $|\mathcal{D}_k(f)| < \infty$, and the claim follows immediately. □

Remark 2.4. The upper bound derived in Proposition 2.3 supplies a somewhat useful tool to study the size of the quantity

$$\#\{n \leq s \mid f(n) + k = m^2\}$$

and in particular Brocard's problem which asks if the set of integers whose factorials are unit left translate of a square is either an infinite set or a finite set. It is worth noting that the upper bounds we have derived do not depend on the size of the shift but on the underlying function. This uniformity does suggest the actual size of the quantity

$$\#\{n \leq s \mid f(n) + k = m^2\}$$

will mostly be influenced by the function under consideration. In various circumstances the ease with which to verify the underlying conditions will inform the category of bounds to exploit. Now we apply the *Diagonal method* to study a slight variant of Brocard's problem.

Lemma 2.5. *The estimate holds*

$$\bigg(\int\limits_{1}^{s} |\mathcal{D}_k(f,t)|^2 dt\bigg)^{\frac{1}{2}} \ll |\mathcal{D}_k(f,s)|^{\frac{3}{2}}.$$

Remark 2.6. The upper bound in Lemma 2.5 can easily be obtained by exploiting the methods of integrating a function in elementary calculus.

Definition 2.7 (The r^{th} truncated Gamma function)**.** Let $r \in \mathbb{N}$ be fixed. Then by the r^{th} **truncated** Gamma function Γ_r, we mean the function

$$\Gamma_r(n) := \begin{cases} n(n-1)\cdots(n-r) \ \textbf{if} \ n > r \\ 0 \ \textbf{otherwise}. \end{cases}$$

Lemma 2.8. *For all $s > r$, we have*

$$\frac{1}{\Gamma_r(s)}\sqrt{\int\limits_{1}^{s} |\Gamma_r'(t)|^2 dt} \asymp \frac{1}{s^{\frac{1}{2}}}.$$

Proof. It follows naturally from the definition of the r^{th} **truncated** Gamma function that $\Gamma_r(s) \asymp s^r$ so that

$$\int\limits_{1}^{s} |\Gamma_r'(t)|^2 dt \asymp s^{2r-1}$$

and the claimed upper bound is an easy consequence. □

Theorem 2.9 (variational Brocard)**.** *The equation $\Gamma_r(s) + k = m^2$ has finitely many solutions $s \in \mathbb{N}$ with $s > r$ for a fixed $k, r \in \mathbb{N}$, where Γ_r is the r^{th} truncated Euler Gamma function.*

Proof. We first apply Lemma 2.8 and notice that

$$\left(\int_1^s |\mathcal{D}_k(\Gamma_r,t)|^2 dt\right)^{\frac{1}{2}} \ll |\mathcal{D}_k(\Gamma_r,s)|^{\frac{3}{2}} \ll |\mathcal{D}_k(\Gamma_r,s)|\sqrt{s}$$

since $|\mathcal{D}_k(\Gamma_r,s)| < s$. It suffices to check that

$$\lim_{s\longrightarrow\infty} \frac{1}{\Gamma_r(s)} \sum_{n\leq s} \Gamma_r(n) < \infty$$

and that

$$\lim_{s\longrightarrow\infty} \frac{1}{\Gamma_r(s)} \sqrt{\int_1^s |\Gamma_r'(t)|^2 dt} < \infty$$

and using the inequality

$$(\int_1^s |\mathcal{D}_k(\Gamma_r,t)|^2 dt)^{\frac{1}{2}} \geq |\mathcal{D}_k(\Gamma_r,s)|.$$

□

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