ON THE GENERAL ERDŐS-MOSER EQUATION VIA THE NOTION OF OLLOIDS

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Abstract. We introduce and develop the notion of the **olloid**. We apply this notion to study a variant and a generalized version of the Erdős-Moser equation under some special local condition.

1. Introduction

The Erdős-Moser equation is an equation of the form

$$1^k + 2^k + \cdots + m^k = (m+1)^k$$

where m and k are positive integers. The only known solution to the equation is $1^1 + 2^1 = 3^1$ and Paul Erdős is known to have conjectured that the equation has no further solution. The exponent k and the arguments in the the Erdős-Moser equation has also been studied quite extensively. In other words, several contraints on the exponent k and the argument m of the Erdős-Moser equation have been studied under a presumption that other solutions - if any - exists. In particular, it has been shown that k must be divisible by 2 and that there is no solution with $m < 10^{1000000}$ [1]. The methods introduced by Moser were later refined and adapted to show that $m > 1.485 \times 10^{9321155}$ [2]. This was improved to the lower bound $m > 2.7139 \times 10^{1,667,658,416}$ in [5] via large scale computation of $\ln(2)$. It is also shown (see [3]) that $6 \leq k+2 < m < 2k$. It is also known that $lcm(1,2,\cdots,200)$ must divide k and that any prime factor of $m+1$ must be irregular and > 1000 [4]. In 2002, it was shown that all primes $200 < p < 1000$ must divide the exponent k in the Erdős-Moser equation

$$1^k + 2^k + \cdots + m^k = (m+1)^k$$

where m and k are positive integers.

In this paper we introduce and study the notion of the **olloid** and develop a technique for extending the solution of the generalized Erdős-Moser equation upto exponents k under some special local conditions of the underlying generator. In particular, we obtain the following result

Theorem 1.1 (The generalized extension method)**.** *Let $h : \mathbb{N} \longrightarrow \mathbb{R}^+$ have continuous derivative on $[1,s]$ and decreasing on $\mathbb{R}^+$. If the equation*

$$\sum_{i=1}^{s} h(i)^k = h(s+1)^k$$

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for $k>1$ has a solution and there exist some $r \in \mathbb{N}$ such that

$$1-\frac{1}{g(s)^r} > \int\limits_1^s \frac{g'(t)}{g(t)^2}dt + \frac{1}{g(s)}\int\limits_1^s \frac{g'(t)}{g(t)^2}dt + \cdots + \frac{1}{g(s)^{r-1}}\int\limits_1^s \frac{g'(t)}{g(t)^2}dt$$

with

$$g(i) := \frac{h(i)}{h(s+1)}$$

for $1 \leq i \leq s$. Then the equation

$$\sum_{i=1}^{s} h(i)^{k+r} = h(s+1)^{k+r}$$

also has a solution.

This result is a consequence of the more fundamental result using the notion of the **olloid**.

Lemma 1.2 (Expansion principle)**.** *Let $\mathbb{F}_s^k$ be an s-dimensional **olloid** of degree k for a fixed $k \in \mathbb{N}$ with $k>1$. If $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1,s]$ and decreasing on $\mathbb{R}^+$ such that*

$$1-\frac{1}{g(s)^r} > \int\limits_1^s \frac{g'(t)}{g(t)^2}dt + \frac{1}{g(s)}\int\limits_1^s \frac{g'(t)}{g(t)^2}dt + \cdots + \frac{1}{g(s)^{r-1}}\int\limits_1^s \frac{g'(t)}{g(t)^2}dt$$

*for $r \in \mathbb{N}$ then $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is also a generator of the **olloid** $\mathbb{F}_s^{k+r}$ of degree $k+r$.*

2. The notion of the olloid

In this section we launch the notion of the **olloid** and prove a fundamental lemma, which will be relevant for our studies in the sequel.

Definition 2.1. Let $\mathbb{F}_s^k := \left\{(u_1, u_2, \ldots, u_s) \in \mathbb{R}^s \mid \sum\limits_{i=1}^{s} u_i^k = 1,\ k>1\right\}$. Then we call $\mathbb{F}_s^k$ an s-dimensional **olloid** of degree $k>1$. We say $g : \mathbb{N} \longrightarrow \mathbb{R}$ is a generator of the s-dimensional olloid of degree k if there exists some vector $(v_1, v_2, \ldots, v_s) \in \mathbb{F}_s^k$ such that $v_i = g(i)$ for each $1 \leq i \leq s$.

Question 2.2. Does there exists a fixed generator $g : \mathbb{N} \longrightarrow \mathbb{R}$ with infinitely many olloids?

Remark 2.3. While it may be difficult to provide a general answer to question 2.2, we can in fact provide an answer by imposition certain conditions for which the generator of the **olloid** must satisfy. In particular, we launch a basic and a fundamental principle relevant for our studies in the sequel.

Lemma 2.4 (Expansion principle)**.** *Let $\mathbb{F}_s^k$ be an s-dimensional **olloid** of degree $k>1$ for a fixed $k \in \mathbb{N}$. If $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a generator with continuous derivative on $[1,s]$ and decreasing on $\mathbb{R}^+$ such that*

$$1-\frac{1}{g(s)^r} > \int\limits_1^s \frac{g'(t)}{g(t)^2}dt + \frac{1}{g(s)}\int\limits_1^s \frac{g'(t)}{g(t)^2}dt + \cdots + \frac{1}{g(s)^{r-1}}\int\limits_1^s \frac{g'(t)}{g(t)^2}dt$$

*for $r \in \mathbb{N}$ then $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is also a generator of the **olloid** $\mathbb{F}_s^{k+r}$ of degree $k+r$.*

Proof. Suppose $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a generator of the **olloid** $\mathbb{F}_s^k$ with continuous derivative on $[1, s]$. Then there exists a vector $(v_1, v_2, \ldots, v_s) \in \mathbb{F}_s^k$ such that $v_i = g(i)$ for each $1 \leq i \leq s$, so that we can write

$$\sum_{i=1}^{s} g(i)^k = 1.$$

Let us assume to the contrary that there exists no $r \in \mathbb{N}$ such that $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is a generator of the **olloid** $\mathbb{F}_s^{k+r}$. By applying the summation by parts, we obtain the inequality

$$\frac{1}{g(s)} \sum_{i=1}^{s} g(i)^{k+1} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt \tag{2.1}$$

by using the inequality

$$\sum_{i=1}^{s} g(i)^{k+1} < \sum_{i=1}^{s} g(i)^k = 1.$$

By applying summation by parts on the left side of (2.1) and using the contrary assumption, we obtain further the inequality

$$\frac{1}{g(s)^2} \sum_{i=1}^{s} g(i)^{k+2} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt. \tag{2.2}$$

By induction we can write the inequality as

$$\frac{1}{g(s)^r} \sum_{i=1}^{s} g(i)^{k+r} \geq 1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \cdots - \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt$$

for any $r \geq 2$ with $r \in \mathbb{N}$. Since $g : \mathbb{N} \longrightarrow \mathbb{R}^+$ is decreasing, it follows that

$$1 - \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt - \cdots - \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt > 1$$

and using the requirement

$$1 - \frac{1}{g(s)^r} > \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_{1}^{s} \frac{g'(t)}{g(t)^2} dt$$

for $r \in \mathbb{N}$, we have the inequality

$$\begin{aligned} 1 &= \sum_{i=1}^{s} g(i)^k \\ &\geq \sum_{i=1}^{s} g(i)^{k+r} > 1 \end{aligned}$$

which is absurd. This completes the proof of the Lemma. $\square$

Lemma 2.4 - albeit fundamental - is ultimately useful for our study of variants and possibly extensions of the Erdős-Moser equation. It can be seen as a tool for extending the solution of equations of the form

$$\sum_{i=1}^{s} g(i)^k = 1$$

for $k > 1$ - under the presumption that it exists - to the solution of equations of the form

$$\sum_{i=1}^{s} g(i)^{k+r} = 1$$

for a fixed $r \in \mathbb{N}$ under some special requirements of the generator $g : \mathbb{N} \longrightarrow \mathbb{R}$.

3. Application to solutions of the generalized Erdős-Moser equation

In this section we apply the notion of the **olloid** to study solutions of the Erdős-Moser equation. We launch the following method as an outgrowth of Lemma 2.4.

Theorem 3.1 (The generalized extension method)**.** *Let $h : \mathbb{N} \longrightarrow \mathbb{R}^+$ have continuous derivative on $[1, s]$ and decreasing on $\mathbb{R}^+$. If the equation*

$$\sum_{i=1}^{s} h(i)^k = h(s+1)^k$$

for $k > 1$ has a solution and there exist some $r \in \mathbb{N}$ such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

with

$$g(i) := \frac{h(i)}{h(s+1)}$$

for $1 \leq i \leq s$. Then the equation

$$\sum_{i=1}^{s} h(i)^{k+r} = h(s+1)^{k+r}$$

also has a solution.

Proof. Suppose the equation

$$\sum_{i=1}^{s} h(i)^k = h(s+1)^k \tag{3.1}$$

has a solution. Then equation (3.1) can be recast as

$$\sum_{i=1}^{s} \left(\frac{h(i)}{h(s+1)} \right)^k = 1 \tag{3.2}$$

which can also be transformed into the sum

$$\sum_{i=1}^{s} g(i)^k = 1$$

with
$$g(i) := \frac{h(i)}{h(s+1)}.$$
The function
$$g(i) := \frac{h(i)}{h(s+1)}$$
for $1 \leq i \leq s$ is decreasing and has continuous derivative on $[1,s]$ since $h : \mathbb{N} \longrightarrow \mathbb{R}^{+}$ have continuous derivative on $[1,s]$ and decreasing on $\mathbb{R}^{+}$, so that if there exists some $r \in \mathbb{N}$ such that
$$1 - \frac{1}{g(s)^r} > \int\limits_{1}^{s} \frac{g'(t)}{g(t)^2}dt + \frac{1}{g(s)}\int\limits_{1}^{s} \frac{g'(t)}{g(t)^2}dt + \cdots + \frac{1}{g(s)^{r-1}}\int\limits_{1}^{s} \frac{g'(t)}{g(t)^2}dt$$
with
$$g(i) := \frac{h(i)}{h(s+1)}$$
for $1 \leq i \leq s$, then by appealing to Lemma 2.4 the equation
$$\sum_{i=1}^{s} g(i)^{k+r} = 1 \tag{3.3}$$
also has a solution. We note that equation (3.3) can also be transformed to the equation
$$\sum_{i=1}^{s} \left(\frac{h(i)}{h(s+1)}\right)^{k+r} = 1 \tag{3.4}$$
so that it has a solution. Since equation (3.4) can be recast as
$$\sum_{i=1}^{s} h(i)^{k+r} = h(s+1)^{k+r}$$
and the claim follows immediately. $\square$

It is important to note that if the values of h on the positive integers is still a positive integer and have continuous derivative on $[1,s]$ and decreasing on $\mathbb{R}^{+}$, then the integer solution of the more general Erdős-Moser equation
$$\sum_{i=1}^{s} h(i)^{k} = h(s+1)^{k}$$
can be extended to the integer solutions of the equation
$$\sum_{i=1}^{s} h(i)^{k+r} = h(s+1)^{k+r}$$
under the local condition of the normalized values of h on $[1, s+1]$. One could also examine the problem with the sequence $h : \mathbb{N} \longrightarrow \mathbb{R}^{+}$ and ask if it is possible to take h to be an arithmetic progression. A similar question could be ask for sequences $h : \mathbb{N} \longrightarrow \mathbb{R}^{+}$ of general types. It is important to recognize that the tool we have developed only allows us to extend solutions of the general Erdős-Moser equation under a certain local condition of normalized generators of the **olloid**.

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