Can von Neumann's theory be consistent when measuring commuting observables?

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Abstract

Based upon our assertion, there is an inconsistency in von Neumann's theory. Barros discusses the inconsistencies do not come from von Neumann's theory, but from extra assumptions about the reality of observables. von Neumann's theory is equivalent to classical theory when we consider only commuting observables. Using this fact, we discuss there is an inconsistency, probably due to the nature of Matrix theory based on commutativeness, within von Neumann's theory. That is, we may omit extra assumptions about the reality of observables. One of the objectives of this paper is for us to remain wondering the extension of von Neumann's axiom to concrete commuting observables themselves.

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I. INTRODUCTION

von Neumann's theory (cf. [1–7]) is a successful physical theory in order to explain quantum experiments. Recently, Nagata and Nakamura claim [8, 9] to derive an inconsistency in von Neumann's theory. Barros discusses [10] that the inconsistencies do not come from von Neumann's theory, but from extra assumptions about the reality of observables. We discuss the inconsistency comes from von Neumann's theory, without extra assumptions about the reality of observables. We show here the inconsistency in an arbitrary dimensional unitary space when measuring commuting observables.

We notice that von Neumann's mathematical model for quantum mechanics is quite logically successful. And the axiomatic system for the mathematical model is a very consistent one. Thus, we cannot say that von Neumann's mathematical model has an inconsistency. What is the inconsistency to be discussed in this paper? We cannot expand the von Neumann's beautiful mathematical model more in handling real experimental data. Mathematically, von Neumann's model is logically very consistent, which fact is true. However, von Neumann's theory is questionable in the sense that the mathematical model does not always expand to real experimental data. And there is the inconsistency if we apply the von Neumann's model to expanding even a simple physical situation. In short, von Neumann's mathematical model might not be useful in that case.

The inconsistency to be discussed in this paper is very impressive. von Neumann's mathematical model has the qualification to be very true axiomatic system for quantum mechanics. Therefore, we cannot modify the axioms based on the nature of Matrix theory. Nevertheless, we encounter an inconsistency, probably due to the nature of Matrix theory based on commutativeness, within von Neumann's theory.

Here, we discuss there is an inconsistency within von Neumann's theory when we limit ourselves to commuting observables. We do not introduce extra assumptions about the reality of observables because we consider only commuting observables. We suppose the two measured observables are commutative. We introduce a supposition that the operation Addition is equivalent to the operation Multiplication and we have an example of an inconsistency, probably due to the nature of Matrix theory based on commutativeness. One of the objectives of this paper is for us to remain wondering the extension of von Neumann's axiom to concrete commuting observables themselves.

We see two patterns in the nature of Matrix theory, that is, commutative or non-commutative. Therefore, we construct two situations as follows:

1. $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$ (General case).
2. $[A_1, A_2] = 0$ (Specific case).

Note that the specific case is a subset of the general case. But, we present two calculation results, which are different with each other. This fact is not agree with that the specific case is a subset of the general case. The two calculation results must be same, but they are not. Thus we can say that one of the reasons of the inconsistency is due to that there are two situations we describe above. In fact, the non-commutative character of the nature of Matrix theory is superficial for the derivation of the inconsistency as we say below.

The inconsistency can be derived only by commuting observables because we may introduce a supposition that the operation Addition is equivalent to the operation Multiplication when considering only commuting observables. And we may derive simultaneously two values for a parameter, that is, the inconsistency is derived only by commuting observables. Thus, the non-commutative character of the nature of Matrix theory is superficial for the derivation of the inconsistency.

We define an inconsistency as follows when considering only two commuting Hermitian matrices:

1. Define two commuting Hermitian matrices A_1, A_2.
2. Define a two-variable function $f(X, Y)$, where f is an appropriate function and X, Y are two variables.
3. Derive a value of $f(A_1, A_2) = a$ by substituting A_1, A_2 into X, Y, respectively.
4. Introduce a supposition that the operation Addition is equivalent to the operation Multiplication. It is possible because of $[A_1, A_2] = 0$.
5. Derive another value of $f(A_1, A_2) = b(\neq a)$ under the supposition that the operation Addition is equivalent to the operation Multiplication.
6. Derive simultaneously two values a, b for the value of $f(A_1, A_2)$.
7. Confirm the inconsistency derived only by the two commuting Hermitian matrices A_1, A_2. We cannot assign simultaneously the truth value "1" for the two suppositions $f(A_1, A_2) = a$ and $f(A_1, A_2) = b$. Otherwise, we cannot define the two-variable function $f(X, Y)$.

The paradox cannot be avoided by the commutative character of the partial case of Matrix theory.

In what follows, we apply such an inconsistency into von Neumann's theory based on the nature of Matrix theory. We repeat that the non-commutative character of the nature of Matrix theory is superficial for the derivation of the inconsistency. The most important is only the commutative character of the nature of Matrix theory for our purpose.

II. VON NEUMANN'S THEORY FOR TWO COMMUTING OBSERVABLES

Though doing later, we dare to introduce firstly a supposition that the sum rule is equivalent to the product rule for the purpose of showing our interesting objective obtained here [11]. The supposition that the sum rule is equivalent to the product rule means a supposition that the operation Addition is equivalent to the operation Multiplication.

Let A_1, A_2 be two Hermitian operators, where they are also supposed to be commutative. They could be defined respectively as follows:

$$A_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Let us consider a simultaneous eigenstate of A_1, A_2, that is, $|\Psi\rangle$, such that

$$\langle\Psi|A_1|\Psi\rangle = +1, \quad \langle\Psi|A_2|\Psi\rangle = -1. \tag{2}$$

Thus, the measured results of trials are either $+1$ or -1.

First, we define the functional rule as follows:

$$f(g(O)) = g(f(O)), \tag{3}$$

where O is a Hermitian operator and f, g are appropriate functions. Second, the sum rule is defined as follows:

$$f(A_1 + A_2) = f(A_1) + f(A_2). \tag{4}$$

Finally, the product rule is defined as follows:

$$f(A_1 \cdot A_2) = f(A_1) \cdot f(A_2). \tag{5}$$

This fact above is based on the property of these two Hermitian operators themselves. This leads to the propositions that they are valid even for the real numbers of the diagonal elements of the two Hermitian operators.

We may have [3] the following relation between the three rules which are valid for the commuting observables:

$$\begin{aligned} &\text{The functional rule} \\ &\Leftrightarrow \text{The sum rule} \\ &\Leftrightarrow \text{The product rule} \end{aligned} \tag{6}$$

For example, let us derive the sum rule and the product rule from the functional rule. Suppose now that A and B are two commuting Hermitian operators. Since A and B commute they can be diagonalized simultaneously. This means that there exists a basis $\{P_i\}$ by which we can expand $A = \sum_i a_i P_i$ and such that B can also be expanded in the form $B = \sum_i b_i P_i$. Now construct a Hermitian operator $O := \sum_i o_i P_i$ with real values o_i, which are all different. Here O is assumed to be non-degenerate by construction. Let us define respectively functions j and k by $j(o_i) := a_i$ and $k(o_i) := b_i$. Then we can see that if A and B commute, there exists a non-degenerate Hermitian operator O such that $A = j(O)$ and $B = k(O)$. Therefore, we can introduce a function h such that $A \cdot B = h(O)$ where $h := j \cdot k$. Thus we have

$$\begin{aligned} &f(A \cdot B) = f(h(O)) = h(f(O)) = j(f(O)) \cdot k(f(O)) \\ &f(j(O)) \cdot f(k(O)) = f(A) \cdot f(B), \end{aligned} \tag{7}$$

where we use the functional rule. We can introduce also a function l such that $A + B = l(O)$ where $l := j + k$. Thus we have

$$\begin{aligned} &f(A + B) = f(l(O)) = l(f(O)) = j(f(O)) + k(f(O)) \\ &f(j(O)) + f(k(O)) = f(A) + f(B), \end{aligned} \tag{8}$$

where we use the functional rule. In fact, the sum rule is equivalent to the product rule for commuting observables.

III. VON NEUMANN'S THEORY IS NOT CONSISTENT

Let us consider a simultaneous eigenstate of A_1, A_2. We might be in an inconsistency when the first result is $+1$ by the measured observable A_1, the second result is -1 by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on commutativeness.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either $+1$ or -1. We suppose the number of -1 is equal to the number of $+1$. If the number of trials is two, then we have

$$V = (+1) + (-1) = 0. \tag{9}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$(V \times V)_{\max} = 0. \tag{10}$$

This is the general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. (The non-commutative character of the nature of Matrix theory is superficial for the derivation of the inconsistency. Thus, we might omit $[A_1, A_2] \neq 0$ case.)

We can depict experimental data r_1, r_2 as follows: $r_1 = +1$ and $r_2 = -1$. Let us write V as follows:

$$V = \sum_{l=1}^{2} r_l. \tag{11}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1, A_2] = 0$.

We introduce a supposition that the sum rule is equivalent to the product rule. The supposition that the sum rule is equivalent to the product rule means a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$\begin{aligned} & V \times V \\ & = (\sum_{l=1}^{2} r_l)^2 \\ & = (\sum_{l'=1}^{2} r_{l'}) \times (\sum_{l=1}^{2} r_l) \\ & = \sum_{l'=1}^{2} \sum_{l=1}^{2} r_{l'} r_l \\ & \leq \sum_{l'=1}^{2} \sum_{l=1}^{2} |r_{l'} r_l| \\ & = 4. \end{aligned} \tag{12}$$

The inequality (12) can be saturated because the following case is possible:

$$\begin{aligned} & \|\{l | r_l = +1\}\| = \|\{l' | r_{l'} = +1\}\|, \\ & \|\{l | r_l = -1\}\| = \|\{l' | r_{l'} = -1\}\|. \end{aligned} \tag{13}$$

Thus,

$$(V \times V)_{\max} = 4. \tag{14}$$

This proposition is true for the specific case $[A_1, A_2] = 0$. We cannot assign simultaneously the truth value "1" for the two suppositions (10) and (14) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is $+1$, the second result is -1, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2.

IV. GENERAL CASE

Let us move ourselves into the more general case. Let us consider a simultaneous eigenstate of A_1, A_2. We might be in an inconsistency when the first result is x by the measured observable A_1, the second result is not x by the measured observable A_2, and then $[A_1, A_2] = 0$. In general, the physical situation is either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. However we may be in the inconsistency when we suppose $[A_1, A_2] = 0$, probably due to the nature of Matrix theory based on commutativeness.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either x or $y(\neq x)$. We suppose the number of x is equal to the number of y. If the number of trials is two, then we have

$$V = x + y. \tag{15}$$

We derive a general necessary condition of the product $V \times V$ of the value V. In this general case, we have

$$(V \times V)_{\max} = (x + y)^2. \tag{16}$$

This is the general necessary condition for either $[A_1, A_2] \neq 0$ or $[A_1, A_2] = 0$. (The non-commutative character of the nature of Matrix theory is superficial for the derivation of the inconsistency. Thus, we might omit $[A_1, A_2] \neq 0$ case.)

We can depict experimental data r_1, r_2 as follows: $r_1 = x$ and $r_2 = y$. Let us write V as follows:

$$V = \sum_{l=1}^{2} r_l. \tag{17}$$

In the following, we evaluate a value $(V \times V)$ and derive a specific necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1, A_2] = 0$.

We introduce a supposition that the operation Addition is equivalent to the operation Multiplication. Then, we have

$$
\begin{aligned}
& V \times V \\
& = (\sum_{l=1}^{2} r_l)^2 \\
& = (\sum_{l'=1}^{2} r_{l'}) \times (\sum_{l=1}^{2} r_l) \\
& = \sum_{l'=1}^{2} \sum_{l=1}^{2} r_{l'} r_l \\
& \leq \sum_{l'=1}^{2} \sum_{l=1}^{2} |r_{l'} r_l| \\
& \leq 2(x^2 + y^2).
\end{aligned} \tag{18}
$$

The first inequality of (18) can be saturated because the following case is possible:

$$r_{l'} r_l = |r_{l'} r_l|. \tag{19}$$

The second inequality of (18) can be saturated because the following case is possible:

$$
\begin{aligned}
& \|\{l | r_l = x\}\| = \|\{l' | r_{l'} = x\}\|, \\
& \|\{l | r_l = y\}\| = \|\{l' | r_{l'} = y\}\|.
\end{aligned} \tag{20}
$$

Especially, we see the following case is possible:

$$r_{l'} = r_l, \tag{21}$$

where we notice the following formula:

$$\frac{1}{2}(|r_{l'}|^2 + |r_l|^2) \geq |r_{l'}||r_l|. \tag{22}$$

Thus,

$$(V \times V)_{\max} = 2(x^2 + y^2). \tag{23}$$

This is possible for the specific case $[A_1, A_2] = 0$. We cannot assign simultaneously the truth value "1" for the two suppositions (16) and (23) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is x, the second result is not x, and then $[A_1, A_2] = 0$, where the quantum state is a simultaneous eigenstate of A_1, A_2.

V. CONCLUSIONS AND DISCUSSIONS

In conclusions, Nagata and Nakamura have claimed [8, 9] to derive an inconsistency in von Neumann's theory. Barros has discussed [10] as follows: The inconsistencies do not have come from von Neumann's theory, but from extra assumptions about the reality of observables. Here we have discussed there is an inconsistency, probably due to the nature of Matrix theory based on commutativeness, within von Neumann's theory, that is, within commuting observables. We do not have introduced extra assumptions about the reality of observables because we consider only two commuting observables. One of the objectives of this paper has been for us to remain wondering the extension of von Neumann's axiom to concrete commuting observables themselves.

The inconsistency can be derived only by commuting observables because we may introduce a supposition that the operation Addition is equivalent to the operation Multiplication when considering only commuting observables. And we may derive simultaneously two values for a parameter, that is, the inconsistency is derived only by commuting observables. Thus, the non-commutative character of the nature of Matrix theory is superficial for the derivation of the inconsistency. We repeat that the non-commutative character of the nature of Matrix theory is superficial for the derivation of the inconsistency. The most important is only the commutative character of the nature of Matrix theory for our purpose.

If the problem were simply an inconsistency, there are multiple logical systems that can cope with such a problem with robustness (see [12]).

Generally Multiplication is completed by Addition. Therefore, we think that Addition of the starting point may be superior to any other case. In this consideration, the supposition that the operation Addition is equivalent to the operation Multiplication may be not true.

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DECLARATIONS

Ethical Approval

The authors are in an applicable thought to Ethical Approval.

Competing Interests

The authors state that there is no conflict of interest.

Authors' Contributions

Koji Nagata and Tadao Nakamura wrote and read the manuscript.

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