Abstract
Suppose that $y > 0$, $0 \leq \alpha < 2\pi$ and $0 < K < 1$. Let $P^+$ be the set of primes $p$ such that $\cos(y \ln p + \alpha) > K$ and $P^-$ the set of primes $p$ such that $\cos(y \ln p + \alpha) < -K$. In this paper we prove $\sum_{p \in P^+} \frac{1}{p} = \infty$ and $\sum_{p \in P^-} \frac{1}{p} = \infty$.

2020 Mathematics Subject Classification : 11N05.

1 Introduction
Let $P$ be the set of primes and $\mathbb{N}$ be the set of natural numbers. In 1737, Euler proved the sum of reciprocals of primes is divergent.

$$\sum_{p \in P} \frac{1}{p} = \infty$$

Definition 1.1. Suppose that $y > 0$, $0 \leq \alpha < 2\pi$ and $0 < K < 1$. Let

$$P^+(y, \alpha, K) = \{ p \in P \mid \cos(y \ln p + \alpha) > K \}$$

and

$$P^-(y, \alpha, K) = \{ p \in P \mid \cos(y \ln p + \alpha) < -K \}.$$ We write $P^+$ and $P^-$ for the sake of simplicity.

Throughout this paper we always assume that $y > 0$. In this paper we prove

Theorem 1.2.

$$\sum_{p \in P^+} \frac{1}{p} = \infty \quad \text{and} \quad \sum_{p \in P^-} \frac{1}{p} = \infty.$$
2 Proof of Theorem 1.2

We will use the prime number theorem in the proof of Theorem 1.2.

**Prime Number Theorem** ([1, 3]). Let \( \pi(x) \) be the number of primes less than or equal to \( x \). Then
\[
\lim_{x \to \infty} \frac{\pi(x)}{x/\ln x} = 1.
\]

**Lemma 2.1.** Recall that \( y > 0 \). Let \( 0 \leq \gamma < 2\pi \). There are at most two primes \( p \) such that
\[
y \ln p = 2n\pi + \gamma
\]
for some \( n \in \mathbb{N} \cup \{ 0 \} \).

**Proof.** Suppose that there exist three distinct primes \( p_1 < p_2 < p_3 \) and \( \ell, m, n \in \mathbb{N} \cup \{ 0 \} \) such that
\[
y \ln p_1 = 2\ell\pi + \gamma, \quad y \ln p_2 = 2m\pi + \gamma, \quad y \ln p_3 = 2n\pi + \gamma.
\]
(1)

We will get a contradiction. From eq. (1), we have
\[
y(\ln p_2 - \ln p_1) = 2(m - \ell)\pi, \quad y(\ln p_3 - \ln p_1) = 2(n - \ell)\pi.
\]
(2)

Notice that \( \ell < m < n \). Let \( m - \ell = h \) and \( n - \ell = k \). From eq. (2), we have
\[
\frac{\ln p_3 - \ln p_1}{\ln p_2 - \ln p_1} = \frac{k}{h}.
\]
Therefore
\[
h(\ln p_3 - \ln p_1) = k(\ln p_2 - \ln p_1)
\]
and hence
\[
\left( \frac{p_3}{p_1} \right)^h = \left( \frac{p_2}{p_1} \right)^k.
\]

Thus
\[
p_1^h p_3^h = p_1^k p_2^k.
\]
This contradicts to the uniqueness of prime factorization. \( \square \)

**Definition 2.2.** Recall that \( y > 0 \) and \( 0 < K < 1 \). Let \( \beta \) be the number such that
\[
\cos \beta = K, \quad 0 < \beta < \frac{\pi}{2}
\]
For each \( n \in \mathbb{N} \cup \{ 0 \} \), let
\[
A_n = \{ p \in P \mid 2n\pi - \beta < y \ln p + \alpha \leq 2n\pi + \beta \},
B_n = \{ p \in P \mid (2n+1)\pi - \beta < y \ln p + \alpha \leq (2n+1)\pi + \beta \}
\]
and
\[
A = \bigcup_{n=0}^{\infty} A_n, \quad B = \bigcup_{n=0}^{\infty} B_n.
\]

Proof of Theorem 1.2

Notice that $P^+ \subset A$ and $P^- \subset B$. From Lemma 2.1, we know that $A - P^+$ has at most two elements and $B - P^-$ also has at most two elements. Therefore it is enough to show that

$$\sum_{p \in A} \frac{1}{p} = \infty \quad \text{and} \quad \sum_{p \in B} \frac{1}{p} = \infty.$$  

Recall that $y > 0$. By the prime number theorem, there exists $M > 0$ such that if $x > M$ then

$$e^{-\frac{\beta}{y}} \frac{x}{\ln x} \leq \pi(x) \leq e^{\frac{\beta}{y}} \frac{x}{\ln x}. \quad (3)$$

From Definition 2.2, we have

$$A_n = \left\{ p \in P \mid e^{\frac{2\pi}{y} \beta} < p \leq e^{\frac{2\pi}{y} \beta + \alpha} \right\}$$

and

$$B_n = \left\{ p \in P \mid e^{\frac{(2n+1)\pi}{y} \beta} < p \leq e^{\frac{(2n+1)\pi}{y} \beta + \alpha} \right\}.$$  

Notice that $A_1, B_1, A_2, B_2, \cdots$ are mutually disjoint. There exists $N \in \mathbb{N}$ such that if $n > N$ then

$$e^{\frac{2n\pi}{y} \beta} > M.$$  

From now on, we assume that $n > N$. By eq. (3), we can find the lower bounds of the number of elements of $A_n$ and $B_n$. We have

$$|A_n| \geq e^{-\frac{\beta}{y}} \frac{e^{\frac{2\pi}{y} \beta}}{2n\pi + \beta - \alpha} - e^{-\frac{\beta}{y}} \frac{e^{\frac{2\pi}{y} \beta}}{2n\pi - \beta + \alpha}$$

$$= ye^{\frac{2\pi}{y} \beta} \frac{e^{\frac{2\pi}{y} \beta - 2\alpha}}{2n\pi + \beta - \alpha} - ye^{\frac{2\pi}{y} \beta} \frac{e^{\frac{2\pi}{y} \beta + 2\alpha}}{2n\pi - \beta - \alpha} \quad (4)$$

and

$$|B_n| \geq e^{-\frac{\beta}{y}} \frac{e^{\frac{(2n+1)\pi}{y} \beta}}{(2n+1)\pi + \beta - \alpha} - e^{-\frac{\beta}{y}} \frac{e^{\frac{(2n+1)\pi}{y} \beta}}{(2n+1)\pi - \beta + \alpha}$$

$$= ye^{\frac{(2n+1)\pi}{y} \beta} \frac{e^{\frac{(2n+1)\pi}{y} - 2\alpha}}{(2n+1)\pi + \beta - \alpha} - ye^{\frac{(2n+1)\pi}{y} \beta} \frac{e^{\frac{(2n+1)\pi}{y} + 2\alpha}}{(2n+1)\pi - \beta - \alpha}. \quad (5)$$

Notice that if $p \in A_n$ then

$$\frac{1}{p} \geq e^{\frac{2\pi}{y} \beta} - \frac{\beta - \alpha}{y} \quad (6)$$

and if $p \in B_n$ then

$$\frac{1}{p} \geq e^{\frac{(2n+1)\pi}{y} \beta} - \frac{\beta - \alpha}{y}. \quad (7)$$

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From eq. (4) and (6), we have
\[
\sum_{p \in A_n} \frac{1}{p} \geq \left( \frac{ye^{\frac{2\pi n + \beta - \alpha}{2}}}{2n\pi + \beta - \alpha} - \frac{ye^{\frac{2\pi n - \beta + \alpha}{2}}}{2n\pi - \beta + \alpha} \right) e^{-\frac{2\pi n - \beta - \alpha}{2}} - \frac{ye^{\frac{\beta}{2}}}{2n\pi + \beta - \alpha} - \frac{ye^{\frac{3\beta}{2}}}{2n\pi - \beta + \alpha}
\]
\[
= \frac{y(e^{\frac{\beta}{2}} - e^{\frac{3\beta}{2}})}{(2n\pi - \beta + \alpha)^2 - \beta^2} - \frac{y(e^{\frac{\beta}{2}} - e^{\frac{3\beta}{2}})}{(2n\pi - \beta + \alpha)^2 - \beta^2}
\]
\[
= \frac{2cn - d}{(2n\pi - \beta + \alpha)^2 - \beta^2},
\]
where
\[
c = y\pi \left( e^{\frac{\beta}{2}} - e^{\frac{3\beta}{2}} \right) > 0
\]
and
\[
d = y\alpha \left( e^{\frac{\beta}{2}} - e^{\frac{3\beta}{2}} \right) + y\beta \left( e^{\frac{\beta}{2}} + e^{\frac{3\beta}{2}} \right).
\]

Similarly from eq. (5) and (7), we have
\[
\sum_{p \in B_n} \frac{1}{p} \geq \left( \frac{ye^{\frac{(2n+1)\pi + \beta - \alpha}{2}}}{(2n+1)\pi + \beta - \alpha} - \frac{ye^{\frac{(2n+1)\pi - \beta + \alpha}{2}}}{(2n+1)\pi - \beta + \alpha} \right) e^{-\frac{(2n+1)\pi - \beta - \alpha}{2}} - \frac{ye^{\frac{\beta}{2}}}{(2n+1)\pi + \beta - \alpha} - \frac{ye^{\frac{3\beta}{2}}}{(2n+1)\pi - \beta + \alpha}
\]
\[
= \frac{y(e^{\frac{\beta}{2}} - e^{\frac{3\beta}{2}})}{(2n+1)\pi + \beta - \alpha} - \frac{y(e^{\frac{\beta}{2}} - e^{\frac{3\beta}{2}})}{(2n+1)\pi - \beta + \alpha}
\]
\[
= \frac{(2n+1)\pi - \beta + \alpha \left( e^{\frac{\beta}{2}} - e^{\frac{3\beta}{2}} \right)}{(2n+1)\pi + \beta - \alpha)^2 - \beta^2} - \beta \left( e^{\frac{\beta}{2}} + e^{\frac{3\beta}{2}} \right)
\]
\[
= \frac{c(2n+1) - d}{(2n+1)\pi + \beta - \alpha)^2 - \beta^2},
\]
Recall eq. (8). Since \(c > 0\), we have
\[
\sum_{p \in A} \frac{1}{p} \geq \sum_{n=N+1}^{\infty} \sum_{p \in A_n} \frac{1}{p} \geq \sum_{n=N+1}^{\infty} \frac{2cn - d}{(2n\pi - \beta + \alpha)^2 - \beta^2} = \infty
\]
and
\[
\sum_{p \in B} \frac{1}{p} \geq \sum_{n=N+1}^{\infty} \sum_{p \in B_n} \frac{1}{p} \geq \sum_{n=N+1}^{\infty} \frac{c(2n+1) - d}{(2n+1)\pi + \beta - \alpha)^2 - \beta^2} = \infty.
\]
Thus
\[
\sum_{p \in A} \frac{1}{p} = \infty \quad \text{and} \quad \sum_{p \in B} \frac{1}{p} = \infty.
\]

References

