Fundamental Physics as the General Solution to a Maximization Problem on the Shannon Entropy of All Measurements

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Abstract

We present a novel approach to quantum theory construction that involves maximizing the Shannon entropy of quantum measurements relative to their initial preparation. By constraining the maximization problem with a vanishing phase (i.e., a phase that vanish under measurements), we obtain quantum mechanics (vanishing U(1)-valued phase), relativistic quantum mechanics (vanishing $\mathrm{Spin}^c(3,1)$-valued phase), and quantum gravity (vanishing $\mathrm{SL}(4,\mathbb{R})$-valued phase). The first two cases are equivalent to established theory, whereas the later case yields a quantum theory of accelerated reference frames, in which a quantized version of the Einstein field equation lives. Specifically, the spacetime interval is promoted to an observable, effectively building the metric tensor from the underlying quantum structure. Subsequently, the Schrödinger equation generates metric tensor diffeomorphisms and SO(3,1) transformations, providing a unified description of quantum mechanics and general relativity. Remarkably, the quantized Einstein Field Equations derived from this framework are provably non-perturbatively finite. Moreover, the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetries of the Standard Model also arise naturally without additional assumptions. Notably, the solution is consistent only with 3+1 spacetime dimensions, as it encounters obstructions in all other dimensional configurations. This framework integrates quantum mechanics, relativistic quantum mechanics, quantum gravity, spacetime dimensionality, and particle physics gauge symmetries from a simple entropy maximization problem constrained by a vanishing phase.

1 Introduction

The canonical formalism of quantum mechanics (QM) is based on five principal axioms[1, 2]:

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(QM) Axiom 1 of 5 **State Space:** Each physical system corresponds to a complex Hilbert space, with the system's state represented by a ray in this space.

(QM) Axiom 2 of 5 **Observables:** Physical observables correspond to Hermitian operators within the Hilbert space.

(QM) Axiom 3 of 5 **Dynamics:** The time evolution of a quantum system is dictated by the Schrödinger equation, where the Hamiltonian operator signifies the system's total energy.

(QM) Axiom 4 of 5 **Measurement:** The act of measuring an observable results in the system's transition to an eigenstate of the associated operator, with the measurement value being one of the eigenvalues.

(QM) Axiom 5 of 5 **Probability Interpretation:** The likelihood of a specific measurement outcome is determined by the squared magnitude of the state vector's projection onto the relevant eigenstate.

Contrastingly, statistical mechanics (SM), the other statistical pillar of physics, derives its probability measures through entropy maximization, constrained by the following expression:

(SM) Constraint 1 of 1: **Average Energy Constraint:** The average of energy measurements of a system at thermodynamic equilibrium converge to a specific value ($\overline{E}$):

$$\overline{E} = \sum_{q \in \mathbb{Q}} \rho(q) E(q) \tag{1}$$

To maximize entropy while satisfying this constraint, the theory uses a Lagrange multiplier approach.

Definition 1 (Fundamental Lagrange Multiplier Equation of SM)**.**

$$\mathcal{L}(\rho, \lambda, \beta) = \underbrace{-k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q)}_{\text{Boltzmann entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q)\right)}_{\text{Average Energy Constraint}} \tag{2}$$

where λ and β are the Lagrange multipliers.

Theorem 1 (Gibbs Measure)**.** *The solution to the Lagrange multiplier equation of SM, is the well-known Gibbs measure.*

$$\rho(q) = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} \exp(-\beta E(r))}}_{\text{Microcanonical Ensemble}} \exp(-\beta E(q)) \tag{3}$$

Proof. This is an well-known result by E. T. Jaynes [3, 4]. As a convenience, we replicate the proof in Annex A. □

As evident from E. T. Jaynes' methodological innovation, SM relies on a single constraint related to the nature of the measurements under consideration, which allows the formulation of an optimization problem sufficient to derive the relevant probability measure. This is an exceptionally parsimonious formulation of a physical theory.

We propose a generalization of E. T. Jaynes' approach to the realms of Quantum Mechanics (QM), Relativistic Quantum Mechanics (RQM), and Quantum Gravity (QG). For each of these three domains, we will introduce a single constraint related to measurements, formulate a corresponding entropy maximization problem, and present a main theorem that fully encapsulates the theory within each realm. This formulation reduces fundamental physics to its simplest and most parsimonious expression, deriving the core theories as optimal solutions to a well-defined entropy maximization problem.

1.1 Quantum Mechanics

To reformulate QM as the solution to an entropy maximization problem, we propose the following constraint:

QM Constraint 1 of 1 **Vanishing Complex-Phase:** Quantum measurements admit a vanishing complex phase. The constraint is:

$$0 = \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{4}$$

which associates to the follow equation:

Definition 2 (Fundamental Lagrange Multiplier Equation of QM)**.**

$$\mathcal{L}(\rho, \lambda, \tau) = \underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\text{Relative Shannon Entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\text{Normalization Constraint}} + \underbrace{\tau \left(-\operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}_{\text{Vanishing Complex-Phase}} \tag{5}$$

where λ and τ are the Lagrange multipliers.

The *relative* Shannon entropy[5, 6] is utilized because we are solving for the least biased theory that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$.

Theorem 2. *The least biased theory that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$, under the constraint of the vanishing complex-phase, is:*

$$\rho(q) = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \|\exp(-itE(r)/\hbar)\|}}_{\text{Unitarily Invariant Ensemble}} \underbrace{\|\exp(-itE(q)/\hbar)\|}_{\text{Born Rule}} \underbrace{p(q)}_{\text{Initial Preparation}} \tag{6}$$

where we have defined $\tau = t/\hbar$ (analogous to $\beta = 1/(k_B T)$ in SM).

The proof of this theorem will be presented in the results section. We will show that this solution entails the five axioms of QM, which are now promoted to theorems, thereby establishing it as the most parsimonious yet complete formulation of QM to date.

1.2 Relativistic Quantum Mechanics

Before we can discuss RQM, we first need to introduce some notation. Let $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector and $\mathbf{b}$ is a pseudo-scalar, be a multivector of the geometric algebra $\mathrm{GA}(3,1)$, and let $\mathbf{M_u}$ be its matrix representation. Then, the fundamental constraint of RQM is:

RQM Constraint 1 of 1 **Vanishing Relativistic Phase:** Our formulation of RQM is based around a vanishing phase spanning the $\mathrm{Spin}^c(3,1)$ group. The constraint is:

$$0 = \mathrm{tr}\,\frac{1}{2}\sum_{q\in\mathbb{Q}} \rho(q)\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{v}\to 0} \tag{7}$$

where $\mathbf{M_u}(q)$ is the matrix representation of the multivector $\mathbf{u}$ of $\mathrm{GA}(3,1)$, using the real Majorana representation of the gamma matrices.

The Lagrange multiplier equation is as follows:

Definition 3 (Fundamental Lagrange Multiplier Equation of RQM)**.**

$$\mathcal{L}(\rho,\lambda,\zeta) = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\textit{Relative Shannon}\\ \textit{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\textit{Normalization}\\ \textit{Constraint}}} + \underbrace{\zeta\left(-\,\mathrm{tr}\,\frac{1}{2}\sum_{q\in\mathbb{Q}}\rho(q)\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{v}\to 0}\right)}_{\textit{Vanishing Relativistic Phase}} \tag{8}$$

where λ and ζ are the Lagrange multipliers.

Theorem 3. *The least biased theory that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$, under the constraint of the vanishing relativistic phase, is:*

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r)\det\exp\left(-\zeta\frac{1}{2}\mathbf{M_u}(r)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right)}}_{\textit{Spin}^c\textit{(3,1) Invariant Ensemble}} \underbrace{\det\exp\left(-\zeta\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right)}_{\textit{Spin}^c\textit{(3,1) Born Rule}} \underbrace{p(q)}_{\textit{Initial Preparation}} \tag{9}$$

In the results section, we aim to demonstrate that this solution represents a quantum mechanical theory of inertial reference frames, where ζ is a "phase-twisted" version of the rapidity. This theory allows for measurements, superpositions, and interference between inertial reference frames, providing the arena in which relativistic quantum mechanics (RQM) operates. While incorporating David Hestenes' results regarding the geometric algebra formulation of RQM, the Dirac current, and the Dirac equation, our approach completes his formulation by introducing missing elements which allow the promotion of the spacetime interval to an observable constructing the metric tensor. The formulation thus lays the foundation for the forthcoming development of quantum gravity through the introduction of quantum frame fields and metric measurements.

1.3 Quantum Gravity

Our formulation of QG is based on a quantum theory of accelerated reference frames. To formulate the maximization problem whose resolution automatically yields the theory, we introduce the following constraint:

QG Constraint 1 of 1 **Vanishing Linear Phase:**

$$\overline{A} = \operatorname{tr} \frac{1}{2} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M}_{\mathbf{u}}(q)|_{a \to 0} \tag{10}$$

where $\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0}$ is the matrix representation of a multivector $\mathbf{u} = \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$ of GA$(3,1)$.

Definition 4 (Fundamental Lagrange Multiplier Equation of QG)**.** *The least biased theory that connects an initial preparation $A_0(q)$ to its final measurement $A(q)$, under the constraint of the special linear phase, is:*

$$\mathcal{L}(A,\lambda,\kappa) = \underbrace{-\sum_{q\in\mathbb{Q}} A(q) \ln \frac{A(q)}{A_0(q)}}_{\substack{\text{Relative Shannon} \\ \text{Entropy}}} + \underbrace{\lambda\left(\mathcal{A} - \sum_{q\in\mathbb{Q}} A(q)\right)}_{\text{Normalization Constraint}} \underbrace{+\kappa\left(-\operatorname{tr}\frac{1}{2}\sum_{q\in\mathbb{Q}} A(q)\mathbf{M}(q)|_{a\to 0}\right)}_{\text{Vanishing Linear Phase}} \tag{11}$$

where λ and κ are the Lagrange multipliers.

The Lagrange multiplier equation reaches a maximum level of generality in terms of wavefunctions living in a vector space; evidently, one cannot be more general than an arbitrary matrix, which exponentiates to the $\mathrm{GL}^+(4,\mathbb{R})$ group, without completely departing from the linear domain.

Theorem 4. *The least biased theory which connects an initial preparation $A_0(q)$ to its final measurement $A(q)$, under the constraint of the vanishing linear phase, is:*

$$A(q) = \underbrace{\frac{\mathcal{A}}{\sum_{r\in\mathbb{Q}} A_0(r) \det\exp\left(-\kappa\frac{1}{2}\mathbf{M}(r)\right)}}_{\text{Geometrically Invariant Ensemble}} \underbrace{\det\exp\left(-\kappa\frac{1}{2}\mathbf{M}(q)\right)}_{\text{Geometric Born Rule}} \underbrace{A_0(q)}_{\text{Initial Preparation}} \tag{12}$$

In the results section, we aim to demonstrate that the solution entails a quantum theory of accelerated reference frames. This theory defines the arena in which QG operates. In the solution, $A(q)$ is not a probability distribution but, as revealed by dimensional analysis, an area distribution. The size of this area serves as the normalization constraint, remains invariant with respect to all transformations, and its entropy is associated with the information required to describe all quantum states of the system. The Lagrange multiplier κ serves as the generator of special linear flow which preserves the area-size associated with the $\mathrm{GL}^+(4,\mathbb{R})$-valued wavefunction. The Schrödinger equation is the active generator of diffeomorphism and SO(3,1) metric transformations, associating with the symmetries of GR. As in the RQM case, the spacetime interval is an observable, enabling the construction of the metric tensor, here valid for metrics of any curvature. Finally, we construct a Fock space in which the metric tensor is promoted to an operator, derive the quantized Einstein field equations, and demonstrate that they are non-perturbatively finite.

1.4 Dimensional Obstructions

We end the result section with a number of theorems showing that the formalism, except for the scalar case of SM and the U(1) case of QM, is found to be consistent only with 3+1-dimensional spacetime, encountering various obstructions in all other dimensional configurations, and we discuss the implications.

2 Results

2.1 Quantum Mechanics

In statistical mechanics, the founding observation is that energy measurements of a thermally equilibrated system tend towards an average value. Comparatively, in QM, the founding observation involves the interplay between the systematic elimination of complex phases in measurement outcomes and the presence of interference effects in repeated measurement outcomes. To represent this observation, we introduce the *Vanishing Complex-Phase* Anti-Constraint:

$$0 = \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{13}$$

where $E(q)$ are scalar-valued functions of $\mathbb{Q}$. The usage of the matrix generates a U(1) phase, and the trace causes it to vanish under specific circumstances (which will correspond to measurements).

At first glance, this expression may seem to reduce to a tautology equating zero with zero, suggesting it imposes no restriction on energy measurements. However, this appearance is deceptive. Unlike a conventional constraint that limits the solution space, this expression serves as a formal device to expand it, allowing for the incorporation of complex phases into the probability measure.

The expression's role in broadening, rather than restricting, the solution space leads to its designation as an "anti-constraint."

In general, usage of anti-constraints expand classical probability distributions into larger domains, such as quantum probabilities.

Its significance will become evident upon the completion of the optimization problem. For the moment, this expression can be conceptualized as the correct expression that, when incorporated as an anti-constraint within an entropy-maximization problem, resolves into the axioms of quantum mechanics.

Our next procedural step involves solving the corresponding Lagrange multiplier equation, mirroring the methodology employed in statistical mechanics by E. T. Jaynes. We utilize the relative Shannon entropy because we wish to solve for the least biased distribution that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$. For that, we deploy the following Lagrange multiplier equation:

$$\mathcal{L} = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text{Relative Shannon} \\ \text{Entropy}}} + \underbrace{\lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right)}_{\substack{\text{Normalization} \\ \text{Constraint}}} + \underbrace{\tau \left(\operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}_{\text{Vanishing Complex-Phase}} \tag{14}$$

Where λ and τ are the Lagrange multipliers.

We solve the maximization problem as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{15}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{16}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{17}$$

$$\implies \rho(q) = p(q) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{18}$$

$$= \frac{1}{Z(\tau)} p(q) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{19}$$

The partition function, is obtained as follows:

$$1 = \sum_{r\in\mathbb{Q}} p(r) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{20}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{21}$$

$$Z(\tau) := \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{22}$$

Finally, the least biased theory that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$, under the constraint of the vanishing complex phase, is:

$$\rho(q) = \frac{1}{\sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \left[\begin{smallmatrix} 0 & -E(r) \\ E(r) & 0 \end{smallmatrix}\right]\right)} \exp\left(-\tau \operatorname{tr} \left[\begin{smallmatrix} 0 & -E(q) \\ E(q) & 0 \end{smallmatrix}\right]\right) p(q) \tag{23}$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

Upon examination, we find that phase elimination is manifestly evident in the probability measure: since the trace evaluates to zero, the probability measure simplifies to classical probabilities, aligning precisely with the Born rule's exclusion of complex phases:

$$\rho(q) = \frac{p(q)}{\sum_{r\in\mathbb{Q}} p(r)} \tag{24}$$

However, the significance of this phase elimination extends beyond this mere simplicity. As we will soon see, the partition function Z gains unitary invariance, allowing for the emergence of interference patterns and other quantum characteristics under appropriate basis changes.

We will begin by aligning our results with the conventional quantum mechanical notation. As such, we transform the representation of complex numbers from $\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ to $a + ib$. For instance, the exponential of a complex matrix is:

$$\exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a \tag{25}$$

Then, we associate the exponential trace to the complex norm using $\exp \operatorname{tr} \mathbf{M} \equiv \det \exp \mathbf{M}$:

$$\begin{aligned}
\exp \operatorname{tr} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} &= r^2 \det \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a &(26)\\
&= r^2(\cos^2(b) + \sin^2(b)) &(27)\\
&= \|r(\cos(b) + i\sin(b))\| &(28)\\
&= \|r \exp(ib)\| &(29)
\end{aligned}$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_B T)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial prepration into :

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r) \|\exp(-itE(r)/\hbar)\|}}_{\text{Unitarily Invariant Partition Function}} \underbrace{\|\exp(-itE(q)/\hbar)\|}_{\text{Born Rule}} \underbrace{p(q)}_{\text{Initial Preparation}} \tag{30}$$

We are now in a position to explore the solution space.

The wavefunction is delineated by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_{r\in\mathbb{Q}} p(r)\|\exp(-itE(r)/\hbar)\| = Z = \langle\psi|\psi\rangle \tag{31}$$

where

$$\begin{bmatrix}\psi_1(t)\\ \vdots \\ \psi_n(t)\end{bmatrix} = \begin{bmatrix}\exp(-itE(q_1)/\hbar) & & \\ & \ddots & \\ & & \exp(-itE(q_n)/\hbar)\end{bmatrix}\begin{bmatrix}\psi_1(0)\\ \vdots \\ \psi_n(0)\end{bmatrix} \tag{32}$$

We clarify that $p(q)$ represents the probability associated with the initial preparation of the wavefunction, where $p(q_i) = \langle\psi_i(0)|\psi_i(0)\rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle\psi|\psi\rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as the POP formulation of QM associates physical states with its probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates (QM) Axiom 1 of 5.

- In Z, an observable must satisfy:

  $$\overline{O} = \sum_{r\in\mathbb{Q}} p(r)O(r)\|\exp(-itE(r)/\hbar)\| \tag{33}$$

  Since $Z = \langle\psi|\psi\rangle$, then any self-adjoint operator satisfying the condition $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$ will equate the above equation, simply because $\langle\mathbf{O}\rangle = \langle\psi|\,\mathbf{O}\,|\psi\rangle$. This demonstrates (QM) Axiom 2 of 5.

- Upon transforming Equation 32 out of its eigenbasis through unitary operations, we find that the energy, $E(q)$, typically transforms in the manner of a Hamiltonian operator:

  $$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle \tag{34}$$

  The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

  $$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle) \tag{35}$$

  $$= -i\mathbf{H}/\hbar\exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle \tag{36}$$

  $$= -i\mathbf{H}/\hbar\,|\psi(t)\rangle \tag{37}$$

  $$\implies \mathbf{H}\,|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \tag{38}$$

Which is the Schrödinger equation. This demonstrates (QM) Axiom 3 of 5.

- From Equation 32 it follows that the possible microstates $E(q)$ of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from $\rho(q,t)$, with the measured state being the occupied microstate q of $\mathbb{Q}$. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability distribution is:

$$\rho(q,t) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q,t))^\dagger\psi(q,t). \tag{39}$$

  In scenarios where the probability measure $\rho(q,\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \tag{40}$$

  Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observables, this demonstrates (QM) Axiom 4 of 5.

- Finally, since the probability measure (Equation 30) replicates the Born rule, (QM) Axiom 5 of 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the vanishing complex phase constraint (Equation 13) is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. Equation 13 becomes the formulation's new singular foundation, and Axioms 1, 2, 3, 4, and 5 are now theorems.

2.2 RQM in 2D

In this section, we investigate RQM in 2D. Although all dimensional configurations except 3+1D contain obstructions, which will be discussed later in this section, the 2D case provides a valuable starting point before addressing the more complex 3+1D case. In RQM 2D, the fundamental Lagrange Multiplier Equation is:

$$\mathcal{L}(\rho,\lambda,\theta) = \underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\text{Relative Shannon}\\\text{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\text{Normalization}\\\text{Constraint}}} + \underbrace{\theta\left(-\operatorname{tr}\frac{1}{2}\sum_{q\in\mathbb{Q}}\rho(q)\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0}\right)}_{\text{Vanishing Relativistic Phase}} \tag{41}$$

where λ and θ are the Lagrange multipliers, and where $\mathbf{M_u}(q)$ is the matrix representation of a multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ of GA(2), where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ is a bivector:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \tag{42}$$

where the basis elements are defined as:

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{43}$$

If we take $a \to 0, \mathbf{x} \to 0$ then $\mathbf{M_u}$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a\to 0, \mathbf{x}\to 0} = \mathbf{b} \implies \mathbf{M_u}|_{a\to 0, \mathbf{x}\to 0} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \tag{44}$$

The Lagrange multiplier equation can be solved as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \theta)}{\partial \rho(q)} = 0 = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right] \tag{45}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right] \tag{46}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right] \tag{47}$$

$$\implies \rho(q) = p(q) \exp(-1 - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right]\right) \tag{48}$$

$$= \frac{1}{Z(\theta)} p(q) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right]\right) \tag{49}$$

The partition function $Z(\theta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_{r\in\mathbb{Q}} p(r) \exp(-1 - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right]\right) \tag{50}$$

$$\implies (\exp(-1 - \lambda))^{-1} = \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right]\right) \tag{51}$$

$$Z(\theta) := \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right]\right) \tag{52}$$

Consequently, the least biased theory that connects an initial preparation $p(q)$ to a final measurement $\rho(q)$, under the constraint of the vanishing relativistic phase in 2D is:

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r) \det \exp\left(-\frac{1}{2}\theta \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right]\right)}}_{\text{Spin(2) Invariant Ensemble}} \underbrace{\det \exp\left(-\frac{1}{2}\theta \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix}\right]\right)}_{\text{Spin(2) Born Rule}} \underbrace{p(q)}_{\text{Initial Preparation}} \tag{53}$$

where $\det \exp M = \exp \operatorname{tr} M$.

In 2D, the Lagrange multiplier θ correspond to an angle of rotation, and in 1+1D it would correspond to the rapidity ζ:

$$\text{2D}: \quad \exp \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \text{ is the angle of rotation} \tag{54}$$

$$1+1\text{D}: \quad \exp \zeta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{bmatrix} \quad \zeta \text{ is the rapidity} \tag{55}$$

The 2D solution may appear equivalent to the QM case because they are related by an isomorphism $\text{Spin}(2) \cong \text{SO}(2) \cong \text{U}(1)$ and under the replacement $\theta \to \tau$. However, an isomorphism is not an equality, and in Spin(2) we gain extra structures related to a relativistic description, which are not available in the QM case.

To investigate the solution in more detail, we introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 5 (Multivector conjugate (a.k.a Clifford conjugate))**.** *Let* $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ *be a multi-vector of the geometric algebra over the reals in two dimensions* GA(2)*. The multivector conjugate is defined as:*

$$\mathbf{u}^\ddagger = a - \mathbf{x} - \mathbf{b} \tag{56}$$

The determinant of the matrix representation of a multivector can be expressed as a self-product:

Theorem 5 (Determinant as a Multivector Self-Product)**.**

$$\mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M}_\mathbf{u} \tag{57}$$

Proof. Let $\mathbf{u} = a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$, and let $\mathbf{M}_\mathbf{u}$ be its matrix representation $\left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right]$. Then:

$$1: \quad \mathbf{u}^\ddagger \mathbf{u} \tag{58}$$

$$= (a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})^\ddagger (a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \tag{59}$$

$$= (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) \tag{60}$$

$$= a^2 - x^2 - y^2 + b^2 \tag{61}$$

$$2: \quad \det \mathbf{M}_\mathbf{u} \tag{62}$$

$$= \det \left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right] \tag{63}$$

$$= (a + x)(a - x) - (y - b)(y + b) \tag{64}$$

$$= a^2 - x^2 - y^2 + b^2 \tag{65}$$

□

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 6 (Multivector Conjugate Transpose)**.** *Let* $|V\rangle\!\rangle \in (\mathrm{GA}(2))^n$*:*

$$|V\rangle\!\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \tag{66}$$

The multivector conjugate transpose of $|V\rangle\!\rangle$ *is defined as first taking the transpose and then the element-wise multivector conjugate:*

$$\langle\!\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \tag{67}$$

Definition 7 (Bilinear Form)**.** *Let* $|V\rangle\!\rangle$ *and* $|W\rangle\!\rangle$ *be two vectors valued in* $\mathrm{GA}(2)$*. We introduce the following bilinear form:*

$$\langle\!\langle V|W\rangle\!\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \tag{68}$$

Theorem 6 (Inner Product)**.** *Restricted to the even sub-algebra of* $\mathrm{GA}(2)$*, the bilinear form is an inner product.*

Proof.

$$\langle\!\langle V|W\rangle\!\rangle_{\mathbf{x}\to 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \tag{69}$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$. □

Definition 8 (Spin(2)-valued Wavefunction)**.**

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1+\mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n+\mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1}R_1 \\ \vdots \\ \sqrt{\rho_2}R_2 \end{bmatrix} \tag{70}$$

where $\sqrt{\rho_i} = e^{\frac{1}{2}a_i}$ *representing the square root of the probability and* $R_i = e^{\frac{1}{2}\mathbf{b}_i}$ *representing a rotor in 2D (or boost in 1+1D).*

The partition function of the probability distribution can be expressed using the bilinear form applied to the Spin(2)-valued Wavefunction:

Theorem 7 (Partition Function)**.** $Z = \langle\!\langle \psi|\psi\rangle\!\rangle$

Proof.

$$\langle\!\langle\psi|\psi\rangle\!\rangle = \sum_{q\in\mathbb{Q}} \psi(q)^\ddagger\psi(q) = \sum_{q\in\mathbb{Q}} \rho(q)R(q)^\ddagger R(q) = \sum_{q\in\mathbb{Q}} \rho(q) = Z \tag{71}$$

□

Thus, the Spin(2)-valued wavefunction $|\psi\rangle\!\rangle$ is a linear object whose inner product reduces to the partition function.

Definition 9 (Spin(2)-valued Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\theta\mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta\mathbf{b}_n} \end{bmatrix} \tag{72}$$

Theorem 8. *The partition function is invariant with respect to the Spin(2)-valued evolution operator.*

Proof.

$$\langle\!\langle T\psi|T\psi\rangle\!\rangle = \sum_{q\in\mathbb{Q}} \det(T(q)\psi(q)) = \sum_{q\in\mathbb{Q}} \det T(q)\det\psi(q) = \sum_{q\in\mathbb{Q}} \det\psi(q) = \langle\!\langle\psi|\psi\rangle\!\rangle \tag{73}$$

where $\det T(q) = 1$, because $e^{-\frac{1}{2}\theta\mathbf{b}(q)}$ is traceless. □

We note that since the even sub-algebra of GA(2) is closed under addition and multiplication, and the bilinear form constitutes an inner product, it follows that it can be employed to construct a Hilbert space, in this case a Spin(2)-valued Hilbert space. The primary difference between a wavefunction living in a complex Hilbert space and one living in a Spin(2) Hilbert space relates to the subject matter of the theory. In the present case, the subject matter is a quantum theory of inertial reference frames in 2D.

The dynamics of reference frame transformations follow from the Schrödinger equation, which is obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier θ. Each element of the wavefunction represents an inertial reference frame, whose transformation is generated by the θ angle (for instance, the change of angle experienced by an inertial observer).

Definition 10 (Spin(2)-valued Schrödinger Equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix}\psi_1(\theta)\\ \vdots \\ \psi_n(\theta)\end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbf{b}_1 & & \\ & \ddots & \\ & & -\frac{1}{2}\mathbf{b}_n \end{bmatrix}\begin{bmatrix}\psi_1(\theta)\\ \vdots \\ \psi_n(\theta)\end{bmatrix} \tag{74}$$

Definition 11 (Reference Frame Measurement)**.** *The (QM) Axiom 5 of 5, regarding the measurement postulates, is derived as a theorem in the RQM case as well (for the same reason as it is in the QM case). This allows us to measure the wavefunction $|\psi\rangle$ into one of its states q according to probability $\rho(q)$. Here the post-measurement state q corresponds to picking a specific inertial reference frame q from $\mathbb{Q}$.*

We note that, as a linear system, linear combinations of the wavefunction (such as $\psi(q) = \lambda_1\psi_1(q) + \lambda_2\psi_2(q)$) will also be solutions. This can introduce interference patterns between inertial reference frames:

Theorem 9 (Reference Frame Superpositions and Interference)**.**

Proof. Let $T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $|\psi\rangle\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{\rho_1}R_1 \\ \sqrt{\rho_2}R_2 \end{bmatrix}$, then:

$$T\,|\psi\rangle\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{\rho_1}R_1 \\ \sqrt{\rho_1}R_2 \end{bmatrix} \tag{75}$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2 \\ \sqrt{\rho_1}R_1 - \sqrt{\rho_2}R_2 \end{bmatrix} \tag{76}$$

$$= \frac{1}{2}(\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2)\,|0\rangle\rangle + \frac{1}{2}(\sqrt{\rho_1}R_1 - \sqrt{\rho_2}R_2)\,|1\rangle\rangle \tag{77}$$

Then the probability can be computed as follows:

$$|\langle\langle 0|\psi\rangle\rangle|^2 = \frac{1}{2}(\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2)^\ddagger(\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2) \tag{78}$$

$$= \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 + \frac{1}{2}\sqrt{\rho_1\rho_2}(R_1^\ddagger R_2 + R_2^\ddagger R_1) \tag{79}$$

$$= \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 + \underbrace{\frac{1}{2}\sqrt{\rho_1\rho_2}\cos(\theta b_1 - \theta b_2)}_{\text{Spin(2)-valued Interference}} \tag{80}$$

Since Spin(2)≅U(1), then Spin(2)-valued interference is isomorphism to complex interference. □

Definition 12 (David Hestenes' Formulation)**.** *In 3+1D, the David Hestenes' formulation [7] of the wavefunction is $\psi = \sqrt{\rho}Re^{ib/2}$, where $R = e^{\mathbf{f}/2}$ is a Lorentz boost or rotation and where $e^{ib/2}$ is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to $\psi = \sqrt{\rho}R$, which is identical to what we recovered.*

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Definition 13 (Dirac Current)**.** *Given the basis $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the Dirac current is defined as:*

$$J_1 \equiv \psi(q)^\ddagger\hat{\mathbf{x}}\psi(q) = \rho(q)R(q)^\ddagger\hat{\mathbf{x}}(q)R(q) = \rho(q)\mathbf{e}_1 \tag{81}$$

$$J_2 \equiv \psi(q)^\ddagger\hat{\mathbf{y}}\psi(q) = \rho(q)R(q)^\ddagger\hat{\mathbf{y}}(q)R(q) = \rho(q)\mathbf{e}_2 \tag{82}$$

where $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *are a Spin(2) rotated frame field.*

Theorem 10 (Dirac Equation)**.** *The Dirac equation in 2D is a special case of the Spin(2)-valued Schrödinger equation.*

Proof. A number of steps is required to reduce the Spin(2)-valued Schrödinger equation to the Dirac equation in 2D.

1. We pose $\psi(\theta) = \langle\!\langle \theta | \psi \rangle\!\rangle$.
2. We parametrize θ in (x, y), yielding $\theta(x, y)$.
3. We expand the left-side of the Schrödinger equation into a total derivative:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\psi(\theta(x,y)) = \frac{\mathrm{d}}{\mathrm{d}x}\psi(\theta(x,y))\frac{\mathrm{d}x}{\mathrm{d}\theta} + \frac{\mathrm{d}}{\mathrm{d}y}\psi(\theta(x,y))\frac{\mathrm{d}y}{\mathrm{d}\theta} \tag{83}$$

   We set $\theta(x, y)$ to be a constant, entailing a global reference frame. With $\theta(x, y)$ constant, the terms $\hat{\mathbf{x}} = \mathrm{d}x/\mathrm{d}\theta$ and $\hat{\mathbf{y}} = \mathrm{d}y/\mathrm{d}\theta$ are basis vector within the tangent space of $\theta(x, y)$ and correspond to the basis vector of the underlying space (x, y). Thus:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\psi(\theta(x,y)) = \hat{\mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}x}\psi(\theta(x,y)) + \hat{\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}y}\psi(\theta(x,y)) \tag{84}$$

$$= \nabla\psi(\theta(x,y)) \tag{85}$$

4. On the right-side, we note the isomorphism $\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \cong i$, yielding

$$-\frac{1}{2}\mathbf{b}\psi(\theta(x,y)) = -\frac{1}{2}ib\psi(\theta(x,y)) \tag{86}$$

5. Bringing both sides back together we have

$$i\nabla\psi(\theta(x,y)) = \frac{b}{2}\psi(\theta(x,y)) \tag{87}$$

6. Finally, posing $b/2 \equiv mc/\hbar$ and $\varphi(x, y) \equiv \psi(\theta(x, y))$, we get

$$(i\hbar\nabla - mc)\varphi(x,y) = 0 \tag{88}$$

   which is the Dirac equation in 2D.

□

2.2.1 Obstructions

We identify two obstructions:

1. **In 1+1D:** The 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}})(a + b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}})$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. (This is why we had to use 2D instead of 1+1D in this two-dimensional introduction...)

2. **In 1+1D and in 2D:** The basis vectors ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in 2D, and $\hat{\mathbf{t}}$ and $\hat{\mathbf{x}}$ in 1+1D) are not self-adjoint. Although used in the context defining the Dirac current, their non-self-adjointness prevents the construction of the spacetime interval (or in 2D, the Euclidean distance) as a quantum observable. The benefits of having the basis vectors self-adjoint will become obvious in the 3+1D case, where we will be able to construct the metric tensor from spacetime interval measurements. Specifically, in 2D:

$$(\hat{\mathbf{x}}_\mu \mathbf{u})^\ddagger \mathbf{u} \neq \mathbf{u}^\ddagger \hat{\mathbf{x}}_\mu \mathbf{u} \tag{89}$$

   because $(\hat{\mathbf{x}}_\mu \mathbf{u})^\ddagger \mathbf{u} = \mathbf{u}^\ddagger \hat{\mathbf{x}}_\mu^\ddagger \mathbf{u} = \mathbf{u}^\ddagger(-\hat{\mathbf{x}}_\mu)\mathbf{u}$.

In the following section, we will explore the obstruction-free 3+1D case.

2.3 RQM in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of 3+1D dimensions. The Lagrange multiplier equation is as follows:

$$\mathcal{L}(\rho, \lambda, \tau) = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\text{Relative Shannon Entropy}} + \lambda \underbrace{\left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right)}_{\text{Normalization Constraint}} + \zeta \underbrace{\left(-\operatorname{tr} \frac{1}{2} \sum_{q\in\mathbb{Q}} \rho(q) \mathbf{M}_{\mathbf{u}}(q)|_{a\to 0, \mathbf{x}\to 0, \mathbf{v}\to 0}\right)}_{\text{Vanishing Relativistic-Phase}} \tag{90}$$

The solution (proof in Annex B) is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r) \det \exp\left(-\zeta \frac{1}{2} \mathbf{M}_{\mathbf{u}}(r)|_{a\to 0, \mathbf{x}\to 0, \mathbf{b}\to 0}\right)}}_{\text{Spin}^c(3,1) \text{ Invariant Ensemble}} \underbrace{\det \exp\left(-\zeta \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{a\to 0, \mathbf{x}\to 0, \mathbf{b}\to 0}\right)}_{\text{Spin}^c(3,1) \text{ Born Rule}} \underbrace{p(q)}_{\text{Initial Preparation}} \tag{91}$$

where ζ is a "twisted-phase" rapidity. (If the invariance group was Spin(3,1) instead of Spin^c(3,1), obtainable by posing $\mathbf{b} \to 0$, then it would simply be the rapidity).

Our initial goal will be to express the partition function as a self-product of elements of the vector space. As such, we begin by defining a general multivector in the geometric algebra $\mathrm{GA}(3,1)$.

Definition 14 (Multivector)**.** *Let* $\mathbf{u}$ *be a multivector of* GA(3,1)*. Its general form is:*

$$
\begin{aligned}
\mathbf{u} = {} & a && (92)\\
& + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} + t\hat{\mathbf{t}} && (93)\\
& + f_{01}\hat{\mathbf{t}}\wedge\hat{\mathbf{x}} + f_{02}\hat{\mathbf{t}}\wedge\hat{\mathbf{y}} + f_{03}\hat{\mathbf{t}}\wedge\hat{\mathbf{z}} + f_{12}\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} + f_{13}\hat{\mathbf{x}}\wedge\hat{\mathbf{z}} + f_{23}\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} && (94)\\
& + p\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} + q\hat{\mathbf{t}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} + v\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{z}} + w\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} && (95)\\
& + b\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} && (96)
\end{aligned}
$$

where $\hat{\mathbf{t}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ *are the basis vectors in the real Majorana representation.*

A more compact notation for $\mathbf{u}$ *is*

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{97}$$

where a *is a scalar,* $\mathbf{x}$ *a vector,* $\mathbf{f}$ *a bivector,* $\mathbf{v}$ *is pseudo-vector and* $\mathbf{b}$ *a pseudo-scalar.*

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation:

Definition 15 (Matrix Representation $\mathbf{M_u}$ of $\mathbf{u}$)**.**

$$\mathbf{M_u} = \begin{bmatrix} a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\ -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\ -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\ f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \end{bmatrix} \tag{98}$$

To manipulate and analyze multivectors in GA(3,1), we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 16 (Multivector Conjugate (in 4D))**.**

$$\mathbf{u}^{\ddagger} = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{99}$$

Definition 17 (3,4 Blade Conjugate)**.** *The 3,4 blade conjugate of* $\mathbf{u}$ *is*

$$\lfloor \mathbf{u} \rfloor_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \tag{100}$$

The results of Lundholm[8], demonstrates that the multivector norms in the following definition, are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{uv}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 18. *The self-products associated with low-dimensional geometric algebras are:*

$$\mathrm{GA}(0,1): \qquad \varphi^{\dagger}\varphi \tag{101}$$

$$\mathrm{GA}(2,0): \qquad \varphi^{\ddagger}\varphi \tag{102}$$

$$\mathrm{GA}(3,0): \qquad \lfloor\varphi^{\ddagger}\varphi\rfloor_3\varphi^{\ddagger}\varphi \tag{103}$$

$$\mathrm{GA}(3,1): \qquad \lfloor\varphi^{\ddagger}\varphi\rfloor_{3,4}\varphi^{\ddagger}\varphi \tag{104}$$

$$\mathrm{GA}(4,1): \qquad (\lfloor\varphi^{\ddagger}\varphi\rfloor_{3,4}\varphi^{\ddagger}\varphi)^{\dagger}(\lfloor\varphi^{\ddagger}\varphi\rfloor_{3,4}\varphi^{\ddagger}\varphi) \tag{105}$$

We can now express the determinant of the matrix representation of a multivector via the self-product $\lfloor\varphi^{\ddagger}\varphi\rfloor_{3,4}\varphi^{\ddagger}\varphi$. This choice is not arbitrary, but the unique choice with allows us to represent the determinant of the matrix representation of a multivector within $\mathrm{GA}(3,1)$:

Theorem 11 (Determinant as a Multivector Self-Product)**.**

$$\lfloor\mathbf{u}^{\ddagger}\mathbf{u}\rfloor_{3,4}\mathbf{u}^{\ddagger}\mathbf{u} = \det\mathbf{M}_{\mathbf{u}} \tag{106}$$

Proof. Please find a computer assisted symbolic proof of this equality in Annex C. □

Definition 19 ($\mathrm{GA}(3,1)$-valued Vector)**.**

$$|V\rangle\!\rangle = \begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_n\end{bmatrix} = \begin{bmatrix}a_1+\mathbf{x}_1+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_1\\ \vdots\\ a_n+\mathbf{x}_n+\mathbf{f}_n+\mathbf{v}_n+\mathbf{b}_n\end{bmatrix} \tag{107}$$

These constructions allow us to express the distribution in terms of the multivector self-product.

Definition 20 (Multilinear Form)**.**

$$\langle\!\langle V|V|V|V\rangle\!\rangle = \lfloor\begin{bmatrix}\mathbf{u}_1^{\ddagger} & \dots & \mathbf{u}_n\end{bmatrix}\begin{bmatrix}\mathbf{u}_1 & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \mathbf{u}_n\end{bmatrix}\rfloor_{3,4}\begin{bmatrix}\mathbf{u}_1^{\ddagger} & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \mathbf{u}_n^{\ddagger}\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \vdots\\ \mathbf{u}_n\end{bmatrix} \tag{108}$$

Theorem 12 (Partition Function)**.** $Z = \langle\!\langle V|V|V|V\rangle\!\rangle$

Proof.

$$\langle\langle V|V|V|V\rangle\rangle \tag{109}$$

$$= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \tag{110}$$

$$= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 & \dots & \mathbf{u}_n \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n^\ddagger \mathbf{u}_n \end{bmatrix} \tag{111}$$

$$= \lfloor \mathbf{u}_1^\ddagger \mathbf{u}_1 \rfloor_{3,4} \mathbf{u}_1^\ddagger \mathbf{u}_1 + \dots + \lfloor \mathbf{u}_n^\ddagger \mathbf{u}_n \rfloor_{3,4} \mathbf{u}_n^\ddagger \mathbf{u}_n \tag{112}$$

$$= \sum_{i=1}^{n} \det \mathbf{M}_{\mathbf{u}_i} \tag{113}$$

$$= Z \tag{114}$$

□

Theorem 13 (Non-negative inner product)**.** *The multilinear form, applied to the even sub-algebra of* GA(3, 1) *is awlays non-negative.*

Proof. Let $|V\rangle\rangle = \begin{bmatrix} a_1 + \mathbf{f}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{f}_n + \mathbf{b}_n \end{bmatrix}$. Then,

$$\langle\langle V|V|V|V\rangle\rangle \tag{115}$$

$$= \lfloor \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{116}$$

$$= \lfloor \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \tag{117}$$

$$= \lfloor \begin{bmatrix} a_1^2 + a_1 \mathbf{f}_1 + a_1 \mathbf{b}_1 - \mathbf{f}_1 a_1 - \mathbf{f}_1^2 - \mathbf{f}_1 \mathbf{b}_1 + \mathbf{b}_1 a_1 + \mathbf{b}_1 \mathbf{f}_1 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{118}$$

$$= \lfloor \begin{bmatrix} a_1^2 - \mathbf{f}_1^2 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{119}$$

We note 1) $\mathbf{b}^2 = (bI)^2 = -b^2$ and 2) $\mathbf{f}^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3) & \dots \end{bmatrix} \rfloor_{3,4} \dots \tag{120}$$

We note that the terms are now complex numbers, which we rewrite as $\mathrm{Re}(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $\mathrm{Im}(z) = -4(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} z_1 & \dots & z_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{121}$$

$$= \begin{bmatrix} z_1^\dagger & \dots & z_2^\dagger \end{bmatrix} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{122}$$

$$= z_1^\ddagger z_1 + \dots + z_n^\ddagger z_n \tag{123}$$

Which is always non-negative. □

We now define the $\mathrm{Spin}^{\mathrm{c}}(3,1)$-valued wavefunction, which is valued in the even sub-algebra of $\mathrm{GA}(3,1)$:

Definition 21 ($\mathrm{Spin}^{\mathrm{c}}(3,1)$-valued Wavefunction)**.**

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{f}_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 B_1 \\ \vdots \\ \sqrt{\rho_n} R_n B_n \end{bmatrix} \tag{124}$$

where R_i is a rotor, B_i is a phase, and $\sum_{q \in \mathbb{Q}} \rho(q) = 1$.

The evolution operator, leaving the partition function invariant, becomes:

Definition 22 ($\mathrm{Spin}^{\mathrm{c}}(3,1)$ Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\zeta(\mathbf{f}_1 + \mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} \tag{125}$$

In turn, this leads to a Schrödinger equation obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier ζ:

Definition 23 ($\mathrm{Spin}^{\mathrm{c}}(3,1)$-valued Schrödinger equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\mathbf{f}_1 + \mathbf{b}_1) & & \\ & \ddots & \\ & & -\frac{1}{2}(\mathbf{f}_n + \mathbf{b}_n) \end{bmatrix} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} \tag{126}$$

Definition 24 (David Hestenes' Formulation)**.** *Our* $\mathrm{Spin}^{\mathrm{c}}(3,1)$*-valued wavefunction is identical to David Hestenes' formulation of the wavefunction within GA(3,1). Both contain a rotor $R = e^{-\mathbf{f}/2}$, a phase $B = e^{-\mathbf{b}/2}$ and the probability term $\sqrt{\rho}$.*

Definition 25 (Dirac Current)**.** *The definition employed in the 2D case (same as Hestenes') applies here as well:*

$$J \equiv \psi^\ddagger \gamma_\mu \psi = \rho R^\ddagger B^\ddagger \gamma_\mu B R = \rho R^\ddagger \gamma_\mu B^{-1} B R = \rho \mathbf{e}_\mu \tag{127}$$

Theorem 14 (Dirac Equation)**.** *The Dirac equation is a special case of the Spin^c(3,1)-valued Schrödinger equation, but the derivation contains an extra step compared to the 2D derivation, eliminating* **f** *from the evolution operator leaving only U(1)-valued evolution.*

Proof. A number of steps is required to reduce the Spin^c(3,1)-valued Schrödinger equation to the Dirac equation.

1. We pose $\psi(\zeta) = \langle\!\langle \zeta | \psi | \zeta | \psi \rangle\!\rangle$.
2. We parametrize ζ in (t, x, y, z), yielding $\zeta(t, x, y, z)$, or just $\zeta(\vec{x})$ for short.
3. We expand the left-side of the Schrödinger equation into a total derivative:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\psi(\zeta(\vec{x})) = \frac{\mathrm{d}}{\mathrm{d}t}\psi(\zeta(\vec{x}))\frac{\mathrm{d}t}{\mathrm{d}\zeta} + \frac{\mathrm{d}}{\mathrm{d}x}\psi(\zeta(\vec{x}))\frac{\mathrm{d}x}{\mathrm{d}\zeta} + \frac{\mathrm{d}}{\mathrm{d}y}\psi(\zeta(\vec{x}))\frac{\mathrm{d}y}{\mathrm{d}\zeta} + \frac{\mathrm{d}}{\mathrm{d}z}\psi(\zeta(\vec{x}))\frac{\mathrm{d}z}{\mathrm{d}\zeta} \tag{128}$$

   We set $\zeta(\vec{x})$ to be a constant, entailing a global reference frame. With $\zeta(\vec{x})$ constant, the terms $\hat{\mathbf{t}} = \mathrm{d}x/\mathrm{d}\theta$, $\hat{\mathbf{x}} = \mathrm{d}y/\mathrm{d}\theta$, etc. are basis vector within the tangent space of $\zeta(\vec{x})$, and correspond to the basis vectors of the underlying flat spacetime. Thus:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\psi(\zeta(\vec{x})) = \hat{\mathbf{t}}\frac{\mathrm{d}}{\mathrm{d}t}\psi(\zeta(\vec{x})) + \hat{\mathbf{x}}\frac{\mathrm{d}}{\mathrm{d}x}\psi(\zeta(\vec{x})) + \hat{\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}y}\psi(\zeta(\vec{x})) + \hat{\mathbf{z}}\frac{\mathrm{d}}{\mathrm{d}z}\psi(\zeta(\vec{x})) \tag{129}$$

$$= \nabla\psi(\zeta(\vec{x})) \tag{130}$$

4. On the right-side, we note the appearance of **f** and **b** in the evolution operator. We must pose $\mathbf{f} \to 0$, focusing only on the phase transformations. (We do not pose $\mathbf{f} \to 0$ in the wavefunction, just in the evolution operator.)
5. Then, again on the right-side, we note the isomorphism $\hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \cong i$, yielding

$$-\frac{1}{2}\mathbf{b}\psi(\zeta(\vec{x})) = -\frac{1}{2}ib\psi(\zeta(\vec{x})) \tag{131}$$

6. Bringing both sides together we have

$$i\nabla\psi(\zeta(\vec{x})) = \frac{b}{2}\psi(\zeta(\vec{x})) \tag{132}$$

7. Finally, posing $b/2 \equiv mc/\hbar$ and $\varphi(t,x,y,z) \equiv \psi(\zeta(\vec{x}))$, we get

$$(i\hbar\nabla - mc)\varphi(t,x,y,z) = 0 \tag{133}$$

which is the Dirac equation.

□

In addition to linearity, the multilinear form supports a double-copy wavefunction. Specifically, we note that in the multilinear form, the term $\sqrt{\rho(q)}$ will be multiplied four times, leading to $\rho(q)^2$ as the probability. This is acceptable because the multiplication of two probabilities yields a probability. In fact, the multilinear form supports a double-copy of a wavefunction:

Definition 26 (Double-copy)**.** *Let ψ and φ be two Spin^c(3,1)-valued wavefunctions. Then, the double copy*

$$\underbrace{\lfloor \psi(q)^\ddagger \psi(q) \rfloor_{3,4}}_{\text{copy 1}} \underbrace{\varphi(q)^\ddagger \varphi(q)}_{\text{copy 2}} = \rho_\psi \rho_\varphi = \rho \tag{134}$$

is a valid probability.

This double-copy feature is the reason why the spacetime interval is an observable, and will be important to define the metric tensor, both for quantum inertial reference frames in this section, and quantum accelerated reference frames in the next section.

Theorem 15 (Inner Product as an Observable)**.**

$$\frac{1}{2}\left(\frac{\langle\!\langle \psi|\mathbf{v}\psi|\psi|\mathbf{w}\psi\rangle\!\rangle}{\langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle} + \frac{\langle\!\langle \psi|\mathbf{w}\psi|\psi|\mathbf{v}\psi\rangle\!\rangle}{\langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle}\right) == \mathbf{v}\cdot\mathbf{w} \tag{135}$$

where $\mathbf{v} = t_v\gamma_0 + x_v\gamma_1 + y_v\gamma_2 + z_v\gamma_2$ and $\mathbf{w} = t_w\gamma_0 + x_w\gamma_1 + y_w\gamma_2 + z_w\gamma_3$.

Proof.

$$1 : \frac{1}{2}\left(\frac{\langle\!\langle \psi|\mathbf{v}\psi|\psi|\mathbf{w}\psi\rangle\!\rangle}{\langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle} + \frac{\langle\!\langle \psi|\mathbf{w}\psi|\psi|\mathbf{v}\psi\rangle\!\rangle}{\langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle}\right) \tag{136}$$

$$= \frac{1}{2}(R^\ddagger B\mathbf{v}BRR^\ddagger B\mathbf{w}RB + R^\ddagger B\mathbf{w}BRR^\ddagger B\mathbf{v}RB) \tag{137}$$

$$= \frac{1}{2}(R^\ddagger \mathbf{v}\mathbf{w}R + R^\ddagger \mathbf{w}\mathbf{v}R) \tag{138}$$

$$= \frac{1}{2}(R^\ddagger(\mathbf{v}\mathbf{w} + \mathbf{w}\mathbf{v})R) \tag{139}$$

$$= R^\ddagger(\mathbf{v}\cdot\mathbf{w})R \tag{140}$$

$$= \mathbf{v}\cdot\mathbf{w} \tag{141}$$

We also demonstrate that its self-adjoint yields the same answer:

$$2 : \frac{1}{2}\left(\frac{\langle\!\langle \mathbf{v}\psi|\psi|\mathbf{w}\psi|\psi\rangle\!\rangle}{\langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle} + \frac{\langle\!\langle \mathbf{w}\psi|\psi|\mathbf{v}\psi|\psi\rangle\!\rangle}{\langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle}\right) \tag{142}$$

$$= \frac{1}{2}(R^\ddagger B(-\mathbf{v})BRR^\ddagger B(-\mathbf{w})RB + R^\ddagger B(-\mathbf{w})BRR^\ddagger B(-\mathbf{v})RB) \tag{143}$$

$$= \frac{1}{2}(R^\ddagger B\mathbf{v}BRR^\ddagger B\mathbf{w}RB + R^\ddagger B\mathbf{w}BRR^\ddagger B\mathbf{v}RB) \tag{144}$$

the rest proceeds the same as above...

$$= \mathbf{v} \cdot \mathbf{w} \tag{145}$$

□

This leads us to the metric tensor constructed as an observable. Here, the metric is flat because the wavefunction only contains rotors, but in the next section on quantum gravity, this definition will apply to curved spacetimes:

Theorem 16 (Metric Tensor as an Observable)**.** *The metric tensor is the expectation value of the γ_μ and γ_ν spacetime intervals:*

$$\langle \eta_{\mu\nu} \rangle = \frac{1}{2}\left(\frac{\langle\!\langle \psi|\gamma_\mu\psi|\psi|\gamma_\nu\psi\rangle\!\rangle}{\langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle} + \frac{\langle\!\langle \psi|\gamma_\nu\psi|\psi|\gamma_\mu\psi\rangle\!\rangle}{\langle\!\langle \psi|\psi|\psi|\psi\rangle\!\rangle}\right) \tag{146}$$

where to improve the legibility, we have dropped the explicit parametrization in (q).

Proof.

$$\frac{1}{2}\langle\!\langle \psi|\gamma_\mu\psi|\psi|\gamma_\nu\psi\rangle\!\rangle + \frac{1}{2}\langle\!\langle \psi|\gamma_\nu\psi|\psi|\gamma_\mu\psi\rangle\!\rangle \tag{147}$$

$$= \frac{1}{2}\lfloor R^\ddagger B^\ddagger \gamma_\mu BR\rfloor_{3,4} R^\ddagger B^\ddagger \gamma_\nu BR + \frac{1}{2}\lfloor R^\ddagger B^\ddagger \gamma_\nu BR\rfloor_{3,4} R^\ddagger B^\ddagger \gamma_\mu BR \tag{148}$$

$$= \frac{1}{2}R^\ddagger \lfloor B^\ddagger\rfloor_{3,4}\gamma_\mu\gamma_\nu BR + \frac{1}{2}R^\ddagger \lfloor B^\ddagger\rfloor_{3,4}\gamma_\nu\gamma_\mu BR \tag{149}$$

because $\lfloor BR\rfloor_{3,4} R^\ddagger B^\ddagger = e^{-\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}} = 1$

$$= \frac{1}{2}R^\ddagger\gamma_\mu\gamma_\nu R + \frac{1}{2}R^\ddagger\gamma_\nu\gamma_\mu R \tag{150}$$

because $\lfloor B^\ddagger\rfloor_{3,4} = B^{-1}$ and $B^{-1}\gamma_\mu\gamma_\nu B = \gamma_\mu\gamma_\nu B^{-1}B = \gamma_\mu\gamma_\nu$

$$= \frac{1}{2}(\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu) \tag{151}$$

because $R^\ddagger\gamma_\mu\gamma_\nu R = R^\ddagger\gamma_\mu RR^\ddagger\gamma_\nu R = \mathbf{e}_\mu\mathbf{e}_\nu$, since $R^\ddagger R = 1$.

$$= \eta_{\mu\nu} \tag{152}$$

As one can swap γ_μ with γ_ν and obtain the same metric tensor, the multilinear form guarantees that $\eta_{\mu\nu}$ is symmetric. Finally, since $\langle\gamma_\mu\psi(q)|\psi(q)|\gamma_\nu\psi(q)|\psi(q)\rangle = \langle\psi(q)|\gamma_\mu\psi(q)|\psi(q)|\gamma_\nu\psi(q)\rangle$, then γ_μ and γ_ν are self-adjoint within the multilinear form, entailing the interpretation of $\eta_{\mu\nu}$ as a quantum observable. □

This definition will automatically extend to $g_{\mu\nu}$ in the next section.

We will now demonstrate that the theory contains the U(1), SU(2), and SU(3) gauge symmetries, which play a fundamental role in the standard model of particle physics. Using the γ_0 basis (instead of any of $\gamma_1, \gamma_2, \gamma_3$), means that we are interested in a transformation that preserves a charge density in time, rather than that of a charge current in space.

Theorem 17 (U(1) Invariance)**.** *[9, 10]*

$$\langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle \tag{153}$$

Proof.

$$\langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle \tag{154}$$

$$= \lfloor\psi(q)^\ddagger e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q) \tag{155}$$

$$= \lfloor\psi(q)^\ddagger \gamma_0 e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger \gamma_0 e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}}\psi(q) \tag{156}$$

$$= \lfloor\psi(q)^\ddagger\gamma_0\psi(q)\rfloor_{3,4}\psi(q)^\ddagger\gamma_0\psi(q) \tag{157}$$

$$= \langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle \tag{158}$$

□

Theorem 18 (SU(2) Invariance)**.** *[9, 10]*

$$\langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle \tag{159}$$

implies $\mathbf{f} = \theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_2 + \theta_3\gamma_0\gamma_3$*, which generates* SU(2)*.*

Proof.

$$\langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle \tag{160}$$

$$= \lfloor\psi(q)^\ddagger e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q) \tag{161}$$

We can now identify that the condition to preserve the equality reduces to this expression:

$$e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} = \gamma_0 \tag{162}$$

We further note that moving the left most term to the right yields:

$$e^{-\theta_1\gamma_0\gamma_1-\theta_2\gamma_0\gamma_2-\theta_3\gamma_0\gamma_3-B_1\gamma_2\gamma_3-B_2\gamma_1\gamma_3-B_3\gamma_1\gamma_2}\gamma_0 e^{\frac{1}{2}\mathbf{f}} \tag{163}$$

$$= \gamma_0 e^{-\theta_1\gamma_0\gamma_1-\theta_2\gamma_0\gamma_2-\theta_3\gamma_0\gamma_3+B_1\gamma_2\gamma_3+B_2\gamma_1\gamma_3+B_3\gamma_1\gamma_2} e^{\frac{1}{2}\mathbf{f}} \tag{164}$$

Therefore, the product $e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}$ reduces to γ_0 if and only if $B_1 = B_2 = B_3 = 0$, leaving $\mathbf{f} = \theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_2 + \theta_3\gamma_0\gamma_3$:

Finally, we note that $e^{\theta_1\gamma_0\gamma_1+\theta_2\gamma_0\gamma_2+\theta_3\gamma_0\gamma_3}$ generates SU(2). □

Theorem 19 (SU(3) invariance)**.** *[9, 10]*

$$\langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)|\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)\rangle \tag{165}$$

Proof. From the above relation, we identify that the following expression must remain invariant: $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$. Now, let $\mathbf{f} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$. Then:

$$-(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\mathbf{f} \tag{166}$$

The first three terms anticommute with γ_0, while the last three commute with γ_0:

$$= \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2\gamma_1\gamma_3 - B_3\gamma_1\gamma_2)\mathbf{f} \tag{167}$$

This can be written as:

$$\gamma_0(\mathbf{E} - \mathbf{B})(\mathbf{E} + \mathbf{B}) \tag{168}$$

$$= \gamma_0(\mathbf{E}^2 + \mathbf{E}\mathbf{B} - \mathbf{B}\mathbf{E} - \mathbf{B}^2) \tag{169}$$

where $\mathbf{E} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3$ and $\mathbf{B} = B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$.

Thus, for $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$, we require: 1) $\mathbf{E}^2 - \mathbf{B}^2 = 1$ and 2) $\mathbf{E}\mathbf{B} = \mathbf{B}\mathbf{E}$. The second requirement means that $\mathbf{E}$ and $\mathbf{B}$ must commute (and thus be isomorphic to three complex numbers), and the first implies:

$$\mathbf{E}^2 - \mathbf{B}^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \tag{170}$$

which are the defining conditions for the SU(3) symmetry group. □

We have now demonstrated that the solution to the entropy maximization problem offers a powerful framework that naturally incorporates $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge symmetries, retains invariance with respect to the $\mathrm{Spin}^c(3,1)$ group, includes the Dirac current and equation, and introduces the notion of the metric tensor via spacetime interval measurements. The specificity of these gauges is attributable to the set of all time-invariant gauges supported by the multilinear form in $\mathrm{GA}(3,1)$, and cannot be different.

2.4 Quantum Gravity in 3+1D

In the previous section, we developed a quantum theory of inertial reference frames valued in Spin^c(3,1), in which RQM lives. Our goal in this section is to extend the methodology to accelerated frame fields, in which *General Relativity* (GR) lives. Specifically, we will investigate a wavefunction that includes all multivector grades $a, \mathbf{x}, \mathbf{f}, \mathbf{v}$, and $\mathbf{b}$ of $\mathrm{GA}(3,1)$.

In general, an arbitrary 4×4 matrix $\mathbf{M_u}(q)$ exponentiates to a surjection of the $\mathrm{GL}^+(4,\mathbb{R})$ group. Thus, the wavefunction becomes a $\mathrm{GL}^+(4,\mathbb{R})$-valued vector, which we call a world vector:

Definition 27 (World Vector)**.**

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1+\mathbf{x}_1+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n+\mathbf{x}_n+\mathbf{f}_n+\mathbf{v}_n+\mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} e^{\frac{1}{2}a_1}e^{\frac{1}{2}(\mathbf{x}'_1+\mathbf{v}'_1+\mathbf{b}'_1)}e^{\frac{1}{2}\mathbf{f}'_1} \\ \vdots \\ e^{\frac{1}{2}a_n}e^{\frac{1}{2}(\mathbf{x}'_n+\mathbf{v}'_n+\mathbf{b}'_n)}e^{\frac{1}{2}\mathbf{f}'_n} \end{bmatrix} = \begin{bmatrix} Q_1R_1 \\ \vdots \\ Q_nR_n \end{bmatrix} \tag{171}$$

where $e^{\frac{1}{2}a_i}e^{\frac{1}{2}(\mathbf{x}'_i+\mathbf{v}'_i+\mathbf{b}'_i)}e^{\frac{1}{2}\mathbf{f}'_i}$ *is the QR decomposition of* $e^{\frac{1}{2}(a_i+\mathbf{x}_i+\mathbf{f}_i+\mathbf{v}_i+\mathbf{b}_i)}$*, where* $Q_i = e^{\frac{1}{2}a_i}e^{\frac{1}{2}(\mathbf{x}'_i+\mathbf{v}'_i+\mathbf{b}'_i)}$ *and* $R_i = e^{\frac{1}{2}\mathbf{f}'_i}$*.*

The frame field then becomes an object transformed by said wavefunction. Specifically, the basis elements are given by its adjoint action on the gamma matrices:

$$\lfloor\psi^\ddagger\rfloor_{3,4}\gamma_\mu\psi \tag{172}$$

$$= e^{-\frac{1}{2}\mathbf{f}'}\underbrace{e^{\frac{1}{2}a}e^{-\frac{1}{2}(\mathbf{x}'+\mathbf{v}'+\mathbf{b}')}\gamma_\mu e^{\frac{1}{2}a}e^{\frac{1}{2}(\mathbf{x}'+\mathbf{v}'+\mathbf{b}')}}_{\text{FX}^+/\text{SO(3,1)}}e^{\frac{1}{2}\mathbf{f}'} \tag{173}$$

$$= \underbrace{e^{-\frac{1}{2}\mathbf{f}'}\mathbf{e}_\mu e^{\frac{1}{2}\mathbf{f}'}}_{\text{SO(3,1)}} \tag{174}$$

where a $\text{FX}^+/\text{SO}(3,1)$-valued transformation was applied to the frame field, yielding an arbitrary curvilinear basis, as well as a $\text{SO}(3,1)$ transformation constituting the invariant symmetry of the metric tensor.

The construction of the metric tensor as a quantum observable relies on the inner product being an observable within the multilinear form (Theorem 15):

Theorem 20 (Metric Measurement)**.** *The metric measurement is the expectation value of the* γ_μ *and* γ_ν *operators, applied to a* $\text{GL}^+(4,\mathbb{R})$*-valued wavefunction:*

$$\langle g_{\mu\nu}\rangle = \frac{1}{2}\Big(\langle\!\langle\psi|\gamma_\mu\psi|\psi|\gamma_\nu\psi\rangle\!\rangle + \langle\!\langle\psi|\gamma_\nu\psi|\psi|\gamma_\mu\psi\rangle\!\rangle\Big) \tag{175}$$

where to improve the legibility, we have dropped the explicit parametrization in (q)*.*

Proof.

$$\frac{1}{2}\langle\!\langle\psi|\gamma_\mu\psi|\psi|\gamma_\nu\psi\rangle\!\rangle + \frac{1}{2}\langle\!\langle\psi|\gamma_\nu\psi|\psi|\gamma_\mu\psi\rangle\!\rangle \tag{176}$$

$$= \frac{1}{2}\lfloor\tilde{R}\tilde{Q}\gamma_\mu QR\rfloor_{3,4}\tilde{R}\tilde{Q}\gamma_\nu QR + \frac{1}{2}\lfloor\tilde{R}\tilde{Q}\gamma_\nu QR\rfloor_{3,4}\tilde{R}\tilde{Q}\gamma_\mu QR \tag{177}$$

where $\tilde{R} = e^{-\frac{1}{4}\mathbf{f}}$ and where $\tilde{Q} = e^{\frac{1}{4}a}e^{\frac{1}{4}(-\mathbf{x}+\mathbf{v}+\mathbf{b})}$.

$$= \frac{1}{2}\sqrt{\rho}\tilde{R}\lfloor\tilde{Q}\rfloor_{3,4}\gamma_\mu\gamma_\nu QR + \frac{1}{2}\sqrt{\rho}\tilde{R}\lfloor\tilde{Q}\rfloor_{3,4}\gamma_\nu\gamma_\mu QR \tag{178}$$

because $\lfloor QR \rfloor_{3,4} \tilde{R}\tilde{Q} = e^{\frac{1}{4}a} e^{\frac{1}{4}(\mathbf{x}-\mathbf{v}-\mathbf{b})} e^{\frac{1}{4}\mathbf{f}} e^{-\frac{1}{4}\mathbf{f}} e^{\frac{1}{4}a} e^{\frac{1}{4}(-\mathbf{x}+\mathbf{v}+\mathbf{b})} = e^{\frac{1}{2}a} = \sqrt{\rho}$

$$= \frac{1}{2}(\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu) \tag{179}$$

$$= g_{\mu\nu} \tag{180}$$

As one can swap γ_μ with γ_ν and obtain the same metric tensor, the multilinear form guarantees that $g_{\mu\nu}$ is symmetric. Finally, since $\langle \gamma_\mu \psi(q) | \psi(q) | \gamma_\nu \psi(q) | \psi(q) \rangle = \langle \psi(q) | \gamma_\mu \psi(q) | \psi(q) | \gamma_\nu \psi(q) \rangle$, then γ_μ and γ_ν are self-adjoint within the multilinear form, entailing the interpretation of $g_{\mu\nu}$ as a quantum observable. $\square$

2.5 The Lagrange Multiplier Equation

Following this initial heuristic investigation, we now define the problem formally via a Lagrange multiplier equation. First, we raise an interpretational observation regarding the scalar term $e^{\frac{1}{2}a}$ of ψ. In the previous sections on QM and RQM, this term was associated with the square root of the probability $e^{\frac{1}{2}a} = \sqrt{\rho}$. However, as we noted in Equation 173, it now associates with a dilation factor. The frame field absorbs the term into its curvilinear transformation. Hence, the world vector cannot be a statement regarding probabilities.

The breakthrough in understanding the precise role of $e^{\frac{1}{2}a}$ came from dimensional analysis. Specifically, to construct the entries of the metric tensor from the world vector, the factor $e^{\frac{1}{2}a}$ ends up being multiplied four times with itself (twice per gamma matrix). The 4-volume density of the metric, given by the square root of the metric determinant $\sqrt{-|g|}$, thus scales as e^{4a}. Significantly, e^{2a} is the square root of the 4-volume e^{4a}, indicating that the distribution grows with the area associated with the metric it defines.

This area, given as the sum total of the distribution, will replace the typical normalization constraint of a probability measure, and thus, will remain invariant with respect to all geometric transformations of the system. It will bear an entropy proportional to its size, and its size will be proportional to the information required to encode the states of the quantum system. A candidate for such an area in GR is, of course, the area of a horizon boundary to a system, as its size remains invariant under the transformations of GR, and it has already been associated with entropy in the physics literature on multiple occasions[11, 12]. However, we intuit that it is a property related to the area density (i.e. the square root of the 4-volume density) defined by the metric over all spacetime, irrespective of the presence of absence of horizons.

Consequently, the solution will not be a probability distribution; rather, it will be a distribution of entropy-bearing oriented areas. In the case of a world manifold (such as required by GR), differently oriented areas will not enter the picture because they would flip the orientation of parts of the manifold, and world manifolds are orientable. Thus, GR will automatically force all areas to be similarly oriented. Similar orientedness draws a parallel with the requirement that probabilities are always positive in a probability measure.

In line of this interpretation, the Lagrange multiplier equation is as follows:

Definition 28 (The Fundamental Lagrange Multiplier Equation of QG)**.**

$$\mathcal{L}(A,\lambda,\kappa) = \underbrace{-\sum_{q\in\mathbb{Q}} A(q)\ln\frac{A(q)}{A_0(q)}}_{\textit{Relative Shannon Entropy}} + \underbrace{\lambda\left(\mathcal{A}-\sum_{q\in\mathbb{Q}} A(q)\right)}_{\textit{Normalization Constraint}} + \underbrace{\kappa\left(-\frac{1}{2}\operatorname{tr}\sum_{q\in\mathbb{Q}} A(q)\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0}\right)}_{\textit{Vanishing Linear Phase}} \tag{181}$$

where $A(q)$ is the distribution, $A_0(q)$ is the initial preparation, $\mathcal{A}$ is the total area, $\mathbf{M}_{\mathbf{u}}(q)$ maps q to a 4×4 matrix, and where κ is the Lagrange multiplier which will generate general linear flow on frame fields.

Theorem 21. *The least biased theory which connects an initial preparation $A_0(q)$ to its final measurement $A(q)$, under the constraint of the vanishing linear phase, is:*

$$A(q) = \frac{\mathcal{A}}{\sum_{r\in\mathbb{Q}} A_0(r)\det\exp\left(-\frac{1}{2}\kappa\mathbf{M}_{\mathbf{u}}(r)\right)}\det\exp\left(-\frac{1}{2}\kappa\mathbf{M}_{\mathbf{u}}(q)\right)A_0(q) \tag{182}$$

Proof.

$$\frac{\partial\mathcal{L}(A,\lambda,\kappa)}{\partial A(q)} = 0 = -\ln\frac{A(q)}{A_0(q)} - 1 - \lambda - \kappa\operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q) \tag{183}$$

$$0 = \ln\frac{A(q)}{A_0(q)} + 1 + \lambda + \kappa\operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q) \tag{184}$$

$$\implies \ln\frac{A(q)}{A_0(q)} = -1 - \lambda - \kappa\operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q) \tag{185}$$

$$\implies A(q) = A_0(q)\exp(-1-\lambda)\exp\left(-\kappa\operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)\right) \tag{186}$$

$$= \frac{1}{Z(\kappa)}A(q)\exp\left(-\kappa\operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)\right) \tag{187}$$

The partition function $Z(\kappa)$, serving as a normalization constant, is determined as follows:

$$\mathcal{A} = \sum_{r\in\mathbb{Q}} p(r)\exp(-1-\lambda)\exp\left(-\kappa\operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)\right) \tag{188}$$

$$\implies (\exp(-1-\lambda))^{-1} = \frac{1}{\mathcal{A}}\sum_{r\in\mathbb{Q}} p(r)\exp\left(-\kappa\operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)\right) \tag{189}$$

$$Z(\kappa) := \frac{1}{\mathcal{A}}\sum_{r\in\mathbb{Q}} p(r)\exp\left(-\kappa\operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)\right) \tag{190}$$

□

Theorem 22 (Area-Entropy Relation)**.** *The Shannon entropy $S = -\sum_{q\in\mathbb{Q}} A(q)\ln A(q)$ leads to a thermodynamic law relating the entropy to the area.*

Proof.

$$S = -\sum_{q\in\mathbb{Q}} A(q)\ln A(q) \tag{191}$$

$$= -\sum_{q\in\mathbb{Q}} \frac{1}{Z(\kappa)}\exp\left(-\frac{1}{2}\operatorname{tr}\kappa\mathbf{M_u}(q)\right)A_0(q)\ln\frac{1}{Z(\kappa)}\exp\left(-\frac{1}{2}\kappa\operatorname{tr}\mathbf{M_u}(q)\right)A_0(q) \tag{192}$$

$$= -\sum_{q\in\mathbb{Q}} \left(\frac{1}{Z(\kappa)}\exp\left(-\frac{1}{2}\operatorname{tr}\kappa\mathbf{M_u}(q)\right)A_0(q)\right)\left(\ln\exp\left(-\frac{1}{2}\kappa\operatorname{tr}\mathbf{M_u}(q)\right)+\ln\frac{A_0(q)}{Z(\kappa)}\right) \tag{193}$$

$$= -\sum_{q\in\mathbb{Q}} \left(\frac{1}{Z(\kappa)}\exp\left(-\frac{1}{2}\operatorname{tr}\kappa\mathbf{M_u}(q)\right)A_0(q)\right)\left(\ln\frac{A_0(q)}{Z(\kappa)}\right) \tag{194}$$

Since $\operatorname{tr}\mathbf{M_u}(q) = 0$, then

$$= -\sum_{q\in\mathbb{Q}} \frac{A_0(q)}{Z(\kappa)}\ln\frac{A_0(q)}{Z(\kappa)} \tag{195}$$

This mid result is not surprising, because the evolution operator preserves the probability. Continuing...

$$= -\sum_{q\in\mathbb{Q}} \frac{A_0(q)}{Z(\kappa)}(\ln A_0(q) - \ln Z(\kappa)) \tag{196}$$

Since $A_0(q) = e^{-\frac{1}{2}\operatorname{tr}(a(q)+\mathbf{x}(q)+\mathbf{f}(q)+\mathbf{v}(q)+\mathbf{b}(q)} = e^{-2a}$, then

$$= -\sum_{q\in\mathbb{Q}} \frac{e^{-2a}}{Z(\kappa)}(\ln e^{-2a} - \ln Z(\kappa)) \tag{197}$$

$$= \sum_{q\in\mathbb{Q}} \frac{e^{-2a}}{Z(\kappa)}2a + \sum_{q\in\mathbb{Q}} \frac{e^{-2a}}{Z(\kappa)}\ln Z(\kappa) \tag{198}$$

Since $\frac{1}{Z(\kappa)} = \frac{\mathcal{A}}{\sum_{r\in\mathbb{Q}} A_0(r)\det\exp\left(-\frac{1}{2}\kappa\mathbf{M_u}(r)\right)} = \frac{\mathcal{A}}{\sum_{r\in\mathbb{Q}} A_0(r)} = \frac{\mathcal{A}}{\sum_{r\in\mathbb{Q}} e^{-2a(r)}}$, then

$$= \mathcal{A}\sum_{q\in\mathbb{Q}} \frac{e^{-2a(q)}}{Z(\kappa)}2a(q) + \sum_{q\in\mathbb{Q}} \frac{e^{-2a(q)}}{Z(\kappa)}\ln Z(\kappa) \tag{199}$$

Since $\sum_{q\in\mathbb{Q}} \frac{e^{-2a(q)}}{Z(\kappa)}2a(q)$ is the definition of the average, it yields $2\overline{a}$. Furthermore, $\sum_{q\in\mathbb{Q}} e^{-2a(q)}$ is the definition $Z(\kappa)$. Then:

$$= \mathcal{A}2\overline{a} + \ln Z(\kappa) \tag{200}$$

□

This result connects the entropy to the area $\mathcal{A}$. The terms $2a(q)$ form the Lie algebra of the dilation group, which are applied to the gamma matrices as an adjoint action: $e^{\frac{1}{2}a(q)}\gamma_\mu e^{\frac{1}{2}a(q)}$. As such, they determine the scale factor for the area defined by the metric. As an example, an area scaling factor with value of $2\overline{a} = 1/4l_p^2$ leads to the Bekenstein-Hawking entropy[12].

$$S = k_B\frac{1}{4l_p^2}\mathcal{A} + k_B\ln Z(\kappa) \tag{201}$$

where the additional logarithmic term is there to satisfy the third law of thermodynamics.

The dynamics are governed by the general linear Schrödinger equation, or as we prefer to call it, the world generator. It is able to generate all possible worlds (all metrics) whose entropy is consistent with the size of the surface (the normalization constraint). The equation is obtained by taking the derivative of the world vector with respect to the Lagrange multiplier κ:

Definition 29 (World Generator)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}\begin{bmatrix}\psi_1(\kappa)\\ \vdots\\ \psi_n(\kappa)\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}\mathbf{x}_1+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_1 & & \\ & \ddots & \\ & & \mathbf{x}_n+\mathbf{f}_1+\mathbf{v}_1+\mathbf{b}_n\end{bmatrix}\begin{bmatrix}\psi_1(\kappa)\\ \vdots\\ \psi_n(\kappa)\end{bmatrix} \tag{202}$$

Let us investigate a special case of interest where both the wavefunction and the Schrödinger equation are valued in $\mathbf{x}$. The diffeomorphism-generating part of the Schrödinger equation, where $\mathbf{f}, \mathbf{v}, \mathbf{b} \to 0$ (leaving only $\mathbf{x}$), bears a strong resemblance to the equation that generates infinitesimal diffeomorphisms from a point p on a manifold X, commonly used in differential geometry:

$$\frac{d}{dt}\varphi_p(t) = \mathbf{x}\varphi_p(t), \text{ with initial condition } \varphi_p(0) = p \tag{203}$$

Specifically, the multivector Schrödinger equation ($\mathbf{f}, \mathbf{v}, \mathbf{b} \to 0, \mathbf{x} \neq 0$) for state $\psi_i(\kappa)$ reduces to:

$$\frac{d}{d\kappa}\psi_i(\kappa) = -\frac{1}{2}\mathbf{x}_i\psi_i(\kappa), \text{ with initial condition } \psi_i(0) = e^{\frac{1}{2}\mathbf{x}_i} \tag{204}$$

where $e^{\frac{1}{2}\mathbf{x}_i}$, in geometric algebra, represents a point on the manifold, obtained by applying the exponential map to the vector $\frac{1}{2}\mathbf{x}_i$ in the tangent space at some origin p. The $-1/2$ factor is a choice of convention that does not change the meaning of the equation.

Thus, the Schrödinger equation is the generator of active diffeomorphisms. Furthermore, as the probability measure is invariant with respect to the action of the Schrödinger equation, it follows that the theory is invariant under active diffeomorphisms.

In the general case, the multivectorial Schrödinger equation governs the dynamics that enable the active generation of all possible metric transformations, not just diffeomorphisms. In fact, each geometric "block" is represented:

$$\frac{d}{d\kappa}\psi_i(\kappa) = -\frac{1}{2}\mathbf{f}_i\psi_i(\kappa), \text{ with initial condition } \psi_i(0) = e^{\frac{1}{2}\mathbf{f}_i} \tag{205}$$

generates Spin(3,1) transformations.

$$\frac{d}{d\kappa}\psi_i(\kappa) = -\frac{1}{2}\mathbf{b}_i\psi_i(\kappa), \text{ with initial condition } \psi_i(0) = e^{\frac{1}{2}\mathbf{b}_i} \tag{206}$$

generates phase transformations

$$\frac{d}{d\kappa}\psi_i(\kappa) = -\frac{1}{2}\mathbf{v}_i\psi_i(\kappa), \text{ with initial condition } \psi_i(0) = e^{\frac{1}{2}\mathbf{v}_i} \tag{207}$$

generates volume shears.

The evolution operator can be applied directly to the spacetime interval observable to generate both diffeomorphisms and SO(3,1) transformations. Due to the structure of the multilinear form, only diffeomorphisms and SO(3,1) transformations of the metric tensor are possible, which are precisely the symmetry invariances of GR. First, let us see how these transformations are applied:

1. Diffeomorphism transformations of the metric tensor by action of the Schrödinger equation:

$$\frac{1}{2}\left(\left\langle\!\left\langle e^{\frac{\mathbf{x}}{2}}\psi \middle| \gamma_\mu e^{\frac{\mathbf{x}}{2}}\psi \middle| e^{\frac{\mathbf{x}}{2}}\psi \middle| \gamma_\nu e^{\frac{\mathbf{x}}{2}}\psi\right\rangle\!\right\rangle + \left\langle\!\left\langle e^{\frac{\mathbf{x}}{2}}\psi \middle| e^{\frac{\mathbf{x}}{2}}\gamma_\nu\psi \middle| e^{\frac{\mathbf{x}}{2}}\psi \middle| e^{\frac{\mathbf{x}}{2}}\gamma_\mu\psi\right\rangle\!\right\rangle\right) \tag{208}$$

$$= \lfloor \psi^\ddagger e^{-\kappa\frac{\mathbf{x}}{2}}\gamma_\mu e^{\frac{\mathbf{x}}{2}}\psi \rfloor_{3,4} \psi^\ddagger e^{-\kappa\frac{\mathbf{x}}{2}}\gamma_\nu e^{\kappa\frac{\mathbf{x}}{2}}\psi + \dots \tag{209}$$

$$\vdots \qquad \text{(sames steps as in Theorem 20)} \tag{210}$$

$$= g'_{\mu\nu} \tag{211}$$

where the relation $v' = e^{-\kappa\frac{\mathbf{x}}{2}} v e^{\kappa\frac{\mathbf{x}}{2}}$ transports the vector v across the manifold. This leads to a metric tensor $g'_{\mu\nu}$ related to $g_{\mu\nu}$ by a diffeomorphism.

2. Lorentz transformations of the metric tensor by action of the Schrödinger equation:

$$\frac{1}{2}\left(\left\langle\!\left\langle e^{\frac{\mathbf{f}}{2}}\psi \middle| \gamma_\mu e^{\frac{\mathbf{f}}{2}}\psi \middle| e^{\frac{\mathbf{f}}{2}}\psi \middle| \gamma_\nu e^{\frac{\mathbf{f}}{2}}\psi\right\rangle\!\right\rangle + \left\langle\!\left\langle e^{\frac{\mathbf{f}}{2}}\psi \middle| e^{\frac{\mathbf{f}}{2}}\gamma_\nu\psi \middle| e^{\frac{\mathbf{f}}{2}}\psi \middle| e^{\frac{\mathbf{f}}{2}}\gamma_\mu\psi\right\rangle\!\right\rangle\right) \tag{212}$$

$$= \lfloor \psi^\ddagger e^{-\kappa\frac{\mathbf{f}}{2}}\gamma_\mu e^{\frac{\mathbf{f}}{2}}\psi \rfloor_{3,4} \psi^\ddagger e^{-\kappa\frac{\mathbf{f}}{2}}\gamma_\nu e^{\kappa\frac{\mathbf{f}}{2}}\psi + \dots \tag{213}$$

$$\vdots \qquad \text{(sames steps as in Theorem 20)} \tag{214}$$

$$= g'_{\mu\nu} \tag{215}$$

where the relation $v' = e^{-\kappa\frac{\mathbf{f}}{2}} v e^{\kappa\frac{\mathbf{f}}{2}}$ boosts or rotates the vector within the SO(3,1) group. This leads to a metric tensor $g'_{\mu\nu}$ related to $g_{\mu\nu}$ by a Lorentz transformation.

Both of these transformations when applied to the metric tensor, as they are the symmetries of GR, will leave the Einstein tensor invariant. It is interesting to note that pseudo-vectors $\mathbf{v}$ are applied as $e^{\kappa\mathbf{v}/2} v e^{\kappa\mathbf{v}/2}$ by the multilinear form (notice the absence of a minus sign on the first term), and this is not a valid adjoint action for a volume shear transformation. Thus, volume shears are not accepted by the multilinear form to transform the metric tensor. Furthermore, applying a phase transformation automatically causes it to vanish: $v = e^{\kappa\mathbf{b}/2} v e^{\kappa\mathbf{b}/2} = v e^{-\kappa\mathbf{b}/2} e^{\kappa\mathbf{b}/2} = v$. Consequently, phase transformations are negated by the multilinear form when applied to the metric tensor. Thus, when constructing the metric tensor, the multilinear form allows precisely the invariant transformations of GR.

2.5.1 Fock Space

Fock spaces in this context are to be interpretated as allowing the creation and annihilation of worlds.

The elements of a Fock space can be constructed from individual world vectors by taking the symmetric or antisymmetric tensor:

$$|\psi_1, \psi_2\rangle\!\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle\!\rangle \otimes |\psi_2\rangle\!\rangle + |\psi_2\rangle\!\rangle \otimes |\psi_1\rangle\!\rangle \right) \qquad \text{Symmetric} \tag{216}$$

$$|\psi_1, \psi_2\rangle\!\rangle = \frac{1}{\sqrt{2}} \left(|\psi_1\rangle\!\rangle \otimes |\psi_2\rangle\!\rangle - |\psi_2\rangle\!\rangle \otimes |\psi_1\rangle\!\rangle \right) \qquad \text{Anti-Symmetric} \tag{217}$$

This allows the construction of a Fock space:

$$|\phi\rangle\!\rangle = \alpha_0 \, |0\rangle\!\rangle + \sum_i \alpha_i \, |\psi_i\rangle\!\rangle + \sum_{i,j} \alpha_{ij} \, |\psi_i, \psi_j\rangle\!\rangle + \sum_{i,j,k} \alpha_{ijk} \, |\psi_i, \psi_j, \psi_k\rangle\!\rangle + \ldots \tag{218}$$

where $\alpha_0, \alpha_i, \alpha_{ij}, \alpha_{ijk}, \ldots$ are multi-vector valued.

Expressed with world creation and world annihilation operators, we get:

$$|\phi\rangle\!\rangle = \alpha_0 \, |0\rangle\!\rangle + \sum_i \alpha_i \hat{a}_i^\dagger \, |0\rangle\!\rangle + \sum_{i,j} \alpha_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger \, |0\rangle\!\rangle + \sum_{i,j,k} \alpha_{ijk} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k^\dagger \, |0\rangle\!\rangle + ... \tag{219}$$

where $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ (symmetric) or $\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}$ (anti-symmetric).

We expand the metric measurements (Theorem 20) to an operator:

Definition 30 (Metric Operator)**.**

$$\langle \hat{g}_{\mu\nu} \rangle = \frac{1}{2} \Big(\langle\!\langle \phi | \gamma_\mu \phi | \phi | \gamma_\nu \phi \rangle\!\rangle + \langle\!\langle \phi | \gamma_\nu \phi | \phi | \gamma_\mu \phi \rangle\!\rangle \Big) \tag{220}$$

where $|\phi\rangle\!\rangle$ is a element of the Fock space.

Metric fluctuations are defined using the standard definition of fluctuations in statistical mechanics:

Definition 31 (Metric Fluctuations)**.**

$$\sigma(\hat{g}_{\mu\nu})^2 = \langle \hat{g}_{\mu\nu}^2 \rangle - \langle \hat{g}_{\mu\nu} \rangle^2 \tag{221}$$

2.5.2 Quantized Einstein Field Equations

Since the multilinear form allows the application of both diffeomorphisms and SO(3,1) transformations, then it follows that the Einstein tensor, which admits the same invariant symmetries, will remain invariant under action by the world generator on the metric tensor.

To study the EFE within the present framework, we must express the Einstein tensor in terms of the metric operator $\hat{g}_{\mu\nu}$ (Definition 30), yielding $\hat{G}_{\mu\nu}$, instead of the classical metric tensor $g_{\mu\nu}$.

Definition 32 (Quantum EFE). *The quantum version of the Einstein Field Equation becomes:*

$$\langle \hat{G}_{\mu\nu} \rangle = \langle \hat{T}_{\mu\nu} \rangle \tag{222}$$

- *where*

$$\langle \hat{G}_{\mu\nu} \rangle = \langle \hat{R}_{\mu\nu} \rangle - \frac{1}{2} \langle \hat{g}_{\mu\nu} \rangle \langle \hat{R} \rangle \tag{223}$$

- *where*

$$\langle \hat{R}_{\mu\nu} \rangle = (1/2) \langle \hat{g}^{\lambda\sigma} \rangle (\partial_\lambda \partial_\nu \langle \hat{g}_{\mu\sigma} \rangle + \partial_\lambda \partial_\mu \langle \hat{g}_{\nu\sigma} \rangle - \partial_\lambda \partial_\sigma \langle \hat{g}_{\mu\nu} \rangle - \partial_\nu \partial_\mu \langle \hat{g}_{\lambda\sigma} \rangle) \tag{224}$$

$$+ \langle \hat{g}^{\lambda\sigma} \rangle \langle \hat{g}^{\rho\tau} \rangle (\langle \hat{\Gamma}_{\lambda\rho\mu} \rangle \langle \hat{\Gamma}_{\sigma\tau\nu} \rangle - \langle \hat{\Gamma}_{\lambda\rho\nu} \rangle \langle \hat{\Gamma}_{\sigma\tau\mu} \rangle) \tag{225}$$

- *where*

$$\langle \hat{\Gamma}_{\lambda\rho\mu} \rangle = (1/2)(\partial_\rho \langle \hat{g}_{\lambda\mu} \rangle + \partial_\lambda \langle \hat{g}_{\rho\mu} \rangle - \partial_\mu \langle \hat{g}_{\lambda\rho} \rangle) \tag{226}$$

- *where*

$$\langle \hat{R} \rangle = \langle \hat{R}_{\mu\nu} \rangle \langle \hat{g}^{\mu\nu} \rangle \tag{227}$$

With this in hand, we can now demonstrate that the quantized Einstein tensor is, in this framework, non-perturbatively finite.

Theorem 23 (QG is non-perturbatively finite). $\langle \hat{G}_{\mu\nu} \rangle$ *is finite for all possible* ψ.

Proof. The proof is in two parts.

1. First, we show that the elements of the metric tensor are real-valued. As such, they contain no singularities.

   This is because the metric is engendered by the joint action of the wavefunction on the basis vectors $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$, which is valued in $\mathrm{GL}^+(4, \mathbb{R})$. Any element of $\mathrm{GL}^+(4, \mathbb{R})$, applied to the gamma basis to yield the metric tensor, will yield a real number:

$$\frac{1}{2}(\lfloor \psi^\ddagger \gamma_\mu \psi \rfloor_{3,4} \psi^\ddagger \gamma_\nu \psi + \lfloor \psi^\ddagger \gamma_\mu \psi \rfloor_{3,4} \psi^\ddagger \gamma_\nu \psi) = g_{\mu\nu} \in \mathbb{R}^{4\times 4} \tag{228}$$

   Thus, metric tensors that contain, say, a term in $1/r$, yielding a singularity at $r = 0$, cannot be constructed from the wavefunction, as we would need to pick an element from $\mathrm{GL}^+(4, \mathbb{R})$ that contains ∞ at $r = 0$, and no such element exists in $\mathrm{GL}^+(4, \mathbb{R})$.

2. Second, the finiteness of $\langle \hat{g}_{\mu\nu} \rangle$ implies the finiteness of $\langle \hat{G}_{\mu\nu} \rangle$ if $\langle \hat{g}_{\mu\nu} \rangle$ is twice differentiable. Since $\langle \hat{g}_{\mu\nu} \rangle$, as a metric tensor, is smooth, it is at least twice differentiable.

□

While we concede that this proof does not automatically provide the most efficient algorithm for perturbatively calculating graviton amplitudes, it nonetheless constitutes a valid proof of the claim. That is, $\langle \hat{G}_{\mu\nu} \rangle$ is finite for all possible ψ, and consequently the solution yields a non-perturbatively finite theory of quantum gravity

2.5.3 Observables

We recall that in a complex Hilbert space an observable is given as: $\langle A\psi | \phi \rangle = \langle \psi | A\phi \rangle \implies A^\dagger = A$.

Here, we investigate the general self-adjoint equation for the multilinear form.

Theorem 24 (World Observable)**.**

$$\langle\!\langle \mathbf{O}\psi | \phi | \varphi | \xi \rangle\!\rangle = \langle\!\langle \psi | \mathbf{O}\phi | \varphi | \xi \rangle\!\rangle = \langle\!\langle \psi | \phi | \mathbf{O}\varphi | \xi \rangle\!\rangle = \langle\!\langle \psi | \phi | \varphi | \mathbf{O}\xi \rangle\!\rangle \implies \mathbf{O}^T = \mathbf{O} \tag{229}$$

where the elements of ψ, ϕ, φ and ξ are valued in GA(3, 1)*.*

This relation implies the eigenvalues of **O** *are real-valued and that its eigenvectors are orthogonal, allowing for proper treatment of observables in 3+1D.*

Proof. Let us show the theorem for a two-state system. The observable **O** is represented by a 2×2 matrix:

$$\mathbf{O} = \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix} \tag{230}$$

where $\mathbf{o}_{00}$, $\mathbf{o}_{01}$, $\mathbf{o}_{10}$, and $\mathbf{o}_{11}$ are multivectors, encapsulating the components of the observable.

Let us calculate each part of the equality:

1.

$$\lfloor \begin{bmatrix} \psi_1^\ddagger & \psi_2^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{o}_{00}^\ddagger & \mathbf{o}_{10}^\ddagger \\ \mathbf{o}_{01}^\ddagger & \mathbf{o}_{11}^\ddagger \end{bmatrix} \begin{bmatrix} \phi_1 & \\ & \phi_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \varphi_1^\ddagger & \\ & \varphi_2^\ddagger \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{231}$$

$$= \begin{bmatrix} \lfloor \psi_1^\ddagger \rfloor_{3,4} & \lfloor \psi_2^\ddagger \rfloor_{3,4} \end{bmatrix} \begin{bmatrix} \lfloor \mathbf{o}_{00}^\ddagger \rfloor_{3,4} & \lfloor \mathbf{o}_{10}^\ddagger \rfloor_{3,4} \\ \lfloor \mathbf{o}_{01}^\ddagger \rfloor_{3,4} & \lfloor \mathbf{o}_{11}^\ddagger \rfloor_{3,4} \end{bmatrix} \begin{bmatrix} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \xi_1 \\ \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \xi_2 \end{bmatrix} \tag{232}$$

$$= \begin{bmatrix} \lfloor \psi_1^\ddagger \rfloor_{3,4} & \lfloor \psi_2^\ddagger \rfloor_{3,4} \end{bmatrix} \begin{bmatrix} \lfloor \mathbf{o}_{00}^\ddagger \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \xi_1 + \lfloor \mathbf{o}_{10}^\ddagger \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \xi_2 \\ \lfloor \mathbf{o}_{01}^\ddagger \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \xi_1 + \lfloor \mathbf{o}_{11}^\ddagger \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \xi_2 \end{bmatrix} \tag{233}$$

$$\begin{aligned} &= \lfloor \psi_1^\ddagger \rfloor_{3,4} \lfloor \mathbf{o}_{00}^\ddagger \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \xi_1 + \lfloor \psi_2^\ddagger \rfloor_{3,4} \lfloor \mathbf{o}_{10}^\ddagger \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \xi_2 \\ &\quad + \lfloor \psi_1^\ddagger \rfloor_{3,4} \lfloor \mathbf{o}_{01}^\ddagger \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \xi_1 + \lfloor \psi_2^\ddagger \rfloor_{3,4} \lfloor \mathbf{o}_{11}^\ddagger \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \xi_2 \end{aligned} \tag{234}$$

2.

$$\lfloor \begin{bmatrix} \psi_1^\ddagger & \psi_2^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix} \begin{bmatrix} \phi_1 & \\ & \phi_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \varphi_1^\ddagger & \\ & \varphi_2^\ddagger \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{235}$$

$$\begin{aligned} &= \lfloor \psi_1^\ddagger \rfloor_{3,4} \lfloor \mathbf{o}_{00} \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \xi_1 + \lfloor \psi_2^\ddagger \rfloor_{3,4} \lfloor \mathbf{o}_{01} \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \xi_2 \\ &\quad + \lfloor \psi_1^\ddagger \rfloor_{3,4} \lfloor \mathbf{o}_{10} \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \xi_1 + \lfloor \psi_2^\ddagger \rfloor_{3,4} \lfloor \mathbf{o}_{11} \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \xi_2 \end{aligned} \tag{236}$$

3.

$$\lfloor \begin{bmatrix} \psi_1^\ddagger & \psi_2^\ddagger \end{bmatrix} \begin{bmatrix} \phi_1 & \\ & \phi_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \varphi_1^\ddagger & \\ & \varphi_2^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{o}_{00}^\ddagger & \mathbf{o}_{10}^\ddagger \\ \mathbf{o}_{01}^\ddagger & \mathbf{o}_{11}^\ddagger \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{237}$$

$$\begin{aligned} &= \lfloor \psi_1^\ddagger \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \mathbf{o}_{00}^\ddagger \xi_1 + \lfloor \psi_2^\ddagger \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \mathbf{o}_{10}^\ddagger \xi_2 \\ &\quad + \lfloor \psi_1^\ddagger \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \mathbf{o}_{01}^\ddagger \xi_1 + \lfloor \psi_2^\ddagger \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \mathbf{o}_{11}^\ddagger \xi_2 \end{aligned} \tag{238}$$

4.

$$\lfloor \begin{bmatrix} \psi_1^\ddagger & \psi_2^\ddagger \end{bmatrix} \begin{bmatrix} \phi_1 & \\ & \phi_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \varphi_1^\ddagger & \\ & \varphi_2^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{o}_{00} & \mathbf{o}_{01} \\ \mathbf{o}_{10} & \mathbf{o}_{11} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{239}$$

$$\begin{aligned} &= \lfloor \psi_1^\ddagger \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \mathbf{o}_{00} \xi_1 + \lfloor \psi_2^\ddagger \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \mathbf{o}_{01} \xi_2 \\ &\quad + \lfloor \psi_1^\ddagger \rfloor_{3,4} \lfloor \phi_1 \rfloor_{3,4} \varphi_1^\ddagger \mathbf{o}_{10} \xi_1 + \lfloor \psi_2^\ddagger \rfloor_{3,4} \lfloor \phi_2 \rfloor_{3,4} \varphi_2^\ddagger \mathbf{o}_{11} \xi_2 \end{aligned} \tag{240}$$

For the equality to be realized, it must be the case that the elements of $\mathbf{O}$ commute with with the elements of ψ, ϕ, φ and ξ, because we must move them between the elements of the self-products; for instance the observable elements in 3) and 4) must be move to the left by 2 places to realize the equality. The relations are then:

$$\lfloor \mathbf{o}_{00}^\ddagger \rfloor_{3,4} = \lfloor \mathbf{o}_{00} \rfloor_{3,4} = \mathbf{o}_{00}^\ddagger = \mathbf{o}_{00} \tag{241}$$

$$\lfloor \mathbf{o}_{10}^\ddagger \rfloor_{3,4} = \lfloor \mathbf{o}_{01} \rfloor_{3,4} = \mathbf{o}_{10}^\ddagger = \mathbf{o}_{01} \tag{242}$$

$$\lfloor \mathbf{o}_{01}^\ddagger \rfloor_{3,4} = \lfloor \mathbf{o}_{10} \rfloor_{3,4} = \mathbf{o}_{01}^\ddagger = \mathbf{o}_{10} \tag{243}$$

$$\lfloor \mathbf{o}_{11}^\ddagger \rfloor_{3,4} = \lfloor \mathbf{o}_{11} \rfloor_{3,4} = \mathbf{o}_{11}^\ddagger = \mathbf{o}_{11} \tag{244}$$

which reduces to

$$\mathbf{o}_{10} = \mathbf{o}_{01} \tag{245}$$

$$\mathbf{o}_{01} = \mathbf{o}_{10} \tag{246}$$

implying simply that $\mathbf{O}^T = \mathbf{O}$ and that the elements of $\mathbf{O}$ are valued in the reals (so that the commute with all grades of a multivector). The eigenvalues of a symmetric matrix are real-valued, and its eigenvectors are orthogonal, allowing the consistent description of observables within the theory. □

A general observable for a two-state system would therefore be expressed as follows:

$$A = \begin{bmatrix} a_{00} & a \\ a & a_{11} \end{bmatrix} \tag{247}$$

for a three-state system, as follows:

$$B = \begin{bmatrix} b_{00} & b_1 & b_2 \\ b_1 & b_{11} & b_3 \\ b_2 & b_3 & b_{22} \end{bmatrix} \tag{248}$$

and so on.

We can notice that such matrices spawns the set of all possible inner product for a n-dimensional quantum system (i.e. O defines an inner product as $v^T O v$). Thus observables in our theory associates to the set of all possible inner products on the vector space.

Finally, since we utilize a multilinear form (and not just a bilinear form), we repeat that we also have access to another kind of observables, relying on the double copy structure, and already mentioned for the spacetime interval as an observable in Theorem 15.

2.5.4 A Geometric Twist on Einstein's Dice

Einstein famously remarked, "God does not play dice." It appears that Einstein may have been right: God plays with disks, not dice.

The entropy in 4D spacetime is associated with oriented area elements, or "disks." This arises from the fact that the determinant of the metric tensor, as produced by the general linear wavefunction, in 4D contains 16 products of $e^{\frac{1}{2}a}$, yielding e^{8a}. The square root of the determinant of the metric tensor, which gives the 4-volume density, scales as e^{4a}. The square root of this 4-volume density scaling, e^{2a}, corresponds to the scaling of an area element and matches the factor found in the multilinear form. Thus, entropy-bearing oriented disks are the geometric objects that solves the problem of maximizing the entropy of all possible measurements in 4D spacetime.

But the game changes in different dimensions. In 2D space, God trades disks for sticks. The determinant of the metric tensor in 2D contains 4 products of $e^{\frac{1}{2}a}$, yielding e^{2a}. The square root of this expression, e^{a}, corresponds to the scaling of a line element, matching the factor in the theory's bilinear form in 2D. Therefore, in 2D space, entropy-bearing oriented line elements, or "sticks," solves the problem of maximizing the entropy of all possible geometric measurements.

Moving up to 6D space, God finally picks up the dice. The determinant of the metric tensor in 6D contains 24 products of $e^{\frac{1}{2}a}$, yielding e^{12a}. The 6D hyper-volume scaling is given by the square root of this expression, e^{6a}. The square root of this 6D hyper-volume scaling, e^{3a}, corresponds to the scaling of a 3D volume element, matching the factor in the determinant of a 6x6 matrix in the theory. Thus, in 6D space, entropy-bearing oriented 3D volume elements, or "dice," are the geometric objects that solves the problem of maximizing the entropy of all possible geometric measurements.

In summary, while Einstein was right that God does not play dice in 4D spacetime, the multivector-valued quantum mechanics theory suggests that the divine game varies across dimensions. God flips the sticks in 2D, spins the disks in 4D, and finally rolls the dice in 6D.

2.6 Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to extend the multivector amplitude formalism to other dimensional configurations. We found that all dimensional configurations except those explored in this paper (e.g. GA(0), GA(0, 1) and GA(3, 1)) are obstructed:

Dimensions	*Obstruction*	
GA(0)	**Unobstructed** $\implies$ statistical mechanics	(249)
GA(0, 1)	**Unobstructed** $\implies$ quantum mechanics	(250)
GA(1, 0)	Negative probabilities in the RQM	(251)
GA(2, 0)	No metric measurement $\implies$ Geometry not observationally complete	(252)
GA(1, 1)	Negative probabilities in the RQM	(253)
GA(0, 2)	Not isomorphic to a real matrix algebra	(254)
GA(3, 0)	Not isomorphic to a real matrix algebra	(255)
GA(2, 1)	Not isomorphic to a real matrix algebra	(256)
GA(1, 2)	Not isomorphic to a real matrix algebra	(257)
GA(0, 3)	Not isomorphic to a real matrix algebra	(258)
GA(4, 0)	Not isomorphic to a real matrix algebra	(259)
GA(3, 1)	**Unobstructed** $\implies$ quantum gravity $\wedge$ SU(3) $\times$ SU(2) $\times$ U(1)	(260)
GA(2, 2)	Negative probabilities in the RQM	(261)
GA(1, 3)	Not isomorphic to a real matrix algebra	(262)
GA(0, 4)	Not isomorphic to a real matrix algebra	(263)
GA(5, 0)	Not isomorphic to a real matrix algebra	(264)
$\vdots$	$\vdots$	
GA(6, 0)	No multilinear form as a self-product	(265)
$\vdots$	$\vdots$	
∞		(266)

Let us now demonstrate the obstructions mentioned above.

Theorem 25 (Not isomorphic to a real matrix algebra)**.** *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

Proof. The geometric algebras are classified as follows:

$$\mathrm{GA}(0,2) \cong \mathbb{H} \tag{267}$$
$$\mathrm{GA}(3,0) \cong \mathbb{M}_2(\mathbb{C}) \tag{268}$$
$$\mathrm{GA}(2,1) \cong \mathbb{M}_2^2(\mathbb{R}) \tag{269}$$
$$\mathrm{GA}(1,2) \cong \mathbb{M}_2(\mathbb{C}) \tag{270}$$
$$\mathrm{GA}(0,3) \cong \mathbb{H}^2 \tag{271}$$
$$\mathrm{GA}(4,0) \cong \mathbb{M}_2(\mathbb{H}) \tag{272}$$
$$\mathrm{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H}) \tag{273}$$
$$\mathrm{GA}(0,4) \cong \mathbb{M}_2(\mathbb{H}) \tag{274}$$
$$\mathrm{GA}(5,0) \cong \mathbb{M}_2^2(\mathbb{H}) \tag{275}$$

The determinant of these objects, when such a thing exists, is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. □

Theorem 26 (Negative Probabilities in the RQM)**.** *The even sub-algebra, which associates to the RQM part of the theory, of these dimensional configurations allows for negative probabilities, making them unsuitable as a RQM.*

Proof. We note three cases:

GA(1, 0): Let $\psi(q) = a + be_1$, then:

$$(a+be_1)^\ddagger(a+be_1) = (a-be_1)(a+be_1) = a^2 - b^2e_1e_1 = a^2 - b^2 \tag{276}$$

which is valued in $\mathbb{R}$.

GA(1, 1): Let $\psi(q) = a + be_0e_1$, then:

$$(a+be_0e_1)^\ddagger(a+be_0e_1) = (a-be_0e_1)(a+be_0e_1) = a^2 - b^2e_0e_1e_0e_1 = a^2 - b^2 \tag{277}$$

which is valued in $\mathbb{R}$.

GA(2, 2): Let $\psi(q) = a + be_0e_\emptyset e_1e_2$, where $e_0^2 = -1, e_\emptyset^2 = -1, e_1^2 = 1, e_2^2 = 1$, then:

$$\lfloor (a+\mathbf{b})^\ddagger(a+\mathbf{b})\rfloor_{3,4}(a+\mathbf{b})^\ddagger(a+\mathbf{b}) \tag{278}$$
$$= \lfloor a^2 + 2a\mathbf{b} + \mathbf{b}^2\rfloor_{3,4}(a^2 + 2a\mathbf{b} + \mathbf{b}^2) \tag{279}$$

We note that $\mathbf{b}^2 = b^2e_0e_\emptyset e_1e_2e_0e_\emptyset e_1e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \tag{280}$$
$$= (a^2 + b^2)^2 - 4a^2\mathbf{b}^2 \tag{281}$$
$$= (a^2 + b^2)^2 - 4a^2b^2 \tag{282}$$

which is valued in $\mathbb{R}$.

In all of these cases the RQM probability can be negative. ☐

We repeat the following self-products[8] (Definition 18), which will help us demonstrate the next theorem:

$$\text{GA}(0,1): \qquad \varphi^\dagger\varphi \tag{283}$$
$$\text{GA}(2,0): \qquad \varphi^\ddagger\varphi \tag{284}$$
$$\text{GA}(3,0): \qquad \lfloor\varphi^\ddagger\varphi\rfloor_3\varphi^\ddagger\varphi \tag{285}$$
$$\text{GA}(3,1): \qquad \lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi \tag{286}$$
$$\text{GA}(4,1): \qquad (\lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi)^\dagger(\lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi) \tag{287}$$

Theorem 27 (No Metric Measurements)**.** *This obstruction applies to* GA(2, 0). *Multilinear forms of at least four self-products are required for the theory to be observationally complete with respect to the geometry.*

Proof. A metric measurement requires a multilinear form of 4 self products because the metric tensor is defined using 2 self-products of the gamma matrices:

$$g_{\mu\nu} = \frac{1}{2}(\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu) \tag{288}$$

Each pair of wavefunction products fixes one basis elements. Thus, two pairs of wavefunction products are required to fix the geometry from the wavefunction. As multilinear forms of four self-products begin to appear in 3D, then the GA(2, 0) cannot produce a metric measurement as a quantum observable, thus its geometry is not observationally complete. ☐

Conjecture 1 (No multilinear form as a self-product (in 6D))**.** *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 18 collapse. The research by Acus et al.[13] in 6D geometric algebra demonstrates that the determinant, so far defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2 B g_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{289}$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[13].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2a_{47}^2 + b_2a_0^2a_{47}^2p_{412}p_{422} + \langle 72 \text{ monomials}\rangle = 0 \tag{290}$$
$$b_1a_0^3a_{52} + 2b_2a_0a_{47}^2a_{52}p_{412}p_{422}p_{432}p_{442}p_{452} + \langle 72 \text{ monomials}\rangle = 0 \tag{291}$$
$$\langle 74 \text{ monomials}\rangle = 0 \tag{292}$$
$$\langle 74 \text{ monomials}\rangle = 0 \tag{293}$$

From Equation 289, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1\mathbf{O}Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2Bg_5(g_4(B)g_3(g_2(B)g_1(B))) = b_1Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2\mathbf{O}Bg_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{294}$$

Any equality of the above type between $b_1\mathbf{O}$ and $b_2\mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory quantum theory (with observables) be constructible in 6D. □

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a consistent quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No multilinear form as a self-product (above 6D))**.** *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $\text{GA}(1,3)$ even if in relativity, the signature of the metric $(+,-,-,-)$ versus $(-,-,-,+)$ does not influence the physics. However, in geometric algebra, $\text{GA}(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $\text{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

Consequently, 3+1D is the only dimensional configuration (other than the "non-geometric" configurations of $\text{GA}(0) \cong \mathbb{R}$ and $\text{GA}(0,1) \cong \mathbb{C}$) in which a 'least biased' solution to the problem of maximizing the Shannon entropy of quantum measurements relative to an initial preparation, exists. This is an extremely strong claim regarding the possible spacetime configurations of the universe, and our ability (or inability) to construct an objective theory to explain it.

3 Discussion

3.1 Maximizing The Relative Shannon Entropy

The principle of maximum entropy[3] states that the probability distribution that best represents the current state of knowledge about a system is the one with the largest entropy, constrained by prior data.

In QM, an experiment begins with an initial preparation, followed by some transformations, and concludes with a final measurement of the system, yielding the result of the experiment. Consistent with the maximum entropy principle, our aim is to derive the 'least biased' theory that connects the initial preparation $p(q)$ to its final measurement $\rho(q)$, thereby formulating the theory as a solution to a maximization problem, rather than merely by axiomatic stipulation.

Using this methodology, fundamental physics can be formulated as the general solution to a maximization problem involving the Shannon entropy of all possible measurements of an arbitrary system relative to its initial preparation, under the constraint of a vanishing phase. As such, the structure of the inferred theory is determined by the nature and generality of the employed constraint. In this paper, we have investigated these four entropy maximization problems:

Vanishing Phase	*Constraint*	*Inferred Theory*	*Wavefunction*
none	$\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q)E(q)$	SM	$\mathbb{R}^{\geq 0}$
U(1)	$0 = \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q) \left[\begin{smallmatrix} 0 & -E(q) \\ E(q) & 0 \end{smallmatrix}\right]$	QM	$\mathbb{C}$
$\mathrm{Spin}^c(3,1)$	$0 = \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q)\mathbf{M_u}\vert_{a\to 0,\mathbf{x}\to 0,\mathbf{v}\to 0}$	RQM	$\mathbb{R}\times\mathrm{Spin}^c(3,1)$
$\mathrm{SL}(4,\mathbb{R})$	$0 = \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q)\mathbf{M_u}\vert a \to 0$	QG	$\mathrm{GL}^+(4,\mathbb{R})$

Despite the differences in constraints, all four theories hereso formulated share a common logical genesis, adhere to the same principle of maximum entropy, and qualify as the least biased theory for their given constraint.

3.2 The Multilinear Form

David Hestenes' work on the representation of the relativistic wavefunction within GA(3,1) was instrumental in the development of this research. His results served as a milestone, confirming the validity of our approach at various stages. Hestenes' wavefunction, $\psi = e^{\frac{1}{2}(a+\mathbf{f}+\mathbf{b})} = \sqrt{\rho}Re^{-ib/2}$, contains the same geometric structures as the $\mathrm{Spin}^c(3,1)$ wavefunction in our theory.

However, it is noteworthy that Hestenes' work does not include a fully satisfactory probability measure. He proposes multiplying the wavefunction with its reverse:

$$\tilde{\psi}\psi = \rho\tilde{R}e^{-ib/2}Re^{-b/2} = \rho e^{-ib} \tag{295}$$

The result ρe^{-ib} does contains ρ, but it also includes a phase factor e^{-ib}. As such, it is not a proper probability measure.

Subsequently, Hestenes proposes sandwiching the γ_μ basis to obtain the Dirac current:

$$J = \tilde{\psi}\gamma_\mu\psi = \rho e_\mu \tag{296}$$

This approach eliminates the phase contribution because $e^{-ib/2}\gamma_\mu e^{-ib/2} = \gamma_\mu e^{ib/2}e^{-ib/2} = \gamma_\mu$. Likewise, the Dirac current is not a proper probability measure (nor is it designed to be) as it contains a basis e_μ.

To construct an adapted Born rule that directly yields the probability when applied to the wavefunction, one might be tempted to apply the conjugate to ψ in addition to the reverse:

$$\tilde{\psi}^\ddagger\psi = \rho\tilde{R}e^{ib/2}Re^{-ib/2} = \rho \tag{297}$$

In this case one indeeds maps ψ to ρ, however, this approach disrupts the definition of the Dirac current: $\tilde{\psi}^\ddagger\gamma_\mu\psi = \rho\tilde{R}\gamma_\mu e^{ib/2}Re^{-ib/2} = \rho e_\mu e^{-ib/2} \neq J$.

To correctly incorporate all the necessary features, including both the Dirac current and a probability measure yielding the probability density, the multilinear form must be employed. Transitioning from bilinear forms to multilinear forms involving <u>four</u> self-products of ψ represents a significant conceptual leap. The strength of the entropy maximization problem lies in its ability to automatically reveal the appropriate form to use. Specifically:

1. The multilinear form maps ψ to a probability measure:

$$\begin{aligned}
\lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi &= \lfloor\sqrt{\rho}\tilde{R}e^{-ib/2}\sqrt{\rho}Re^{-ib/2}\rfloor_{3,4}\sqrt{\rho}\tilde{R}e^{-ib/2}\sqrt{\rho}Re^{-ib/2} && (298)\\
&= \rho^2\tilde{R}R\tilde{R}Re^{ib/2}e^{ib/2}e^{-ib/2}e^{-ib/2} && (299)\\
&= \rho^2 && (300)
\end{aligned}$$

2. The definition of the Dirac current is retained:

$$\begin{aligned}
\psi^\ddagger\gamma_\mu\psi &= \sqrt{\rho}R^\ddagger e^{-ib/2}\gamma_\mu\sqrt{\rho}e^{ib/2}R && (301)\\
&= \rho\tilde{R}\gamma_\mu R && (302)\\
&= \rho e_\mu && (303)\\
&= J && (304)
\end{aligned}$$

3. In the multilinear form the "Dirac current" (i.e. sandwiching the gamma matrices within the form) is upgraded to a metric measurement:

$$\frac{1}{2}\left(\frac{\lfloor\psi^\ddagger\gamma_\mu\psi\rfloor_{3,4}\psi^\ddagger\gamma_\nu\psi}{\lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi} + \frac{\lfloor\psi^\ddagger\gamma_\nu\psi\rfloor_{3,4}\psi^\ddagger\gamma_\mu\psi}{\lfloor\psi^\ddagger\psi\rfloor_{3,4}\psi^\ddagger\psi}\right) = \langle\eta_{\mu\nu}\rangle \tag{305}$$

4. In the context of quantum gravity with the $\mathrm{GL}^+(4,\mathbb{R})$-valued wavefunction, the multilinear form leads to metric measurements (Theorem 20):

$$\frac{1}{2}\Big(\lfloor\psi^\ddagger\gamma_\mu\psi\rfloor_{3,4}\psi^\ddagger\gamma_\nu\psi + \lfloor\psi^\ddagger\gamma_\nu\psi\rfloor_{3,4}\psi^\ddagger\gamma_\mu\psi\Big) = \langle g_{\mu\nu}\rangle \tag{306}$$

5. And even to a metric operator over the Fock space (Definition 30):

$$\frac{1}{2}\Big(\lfloor\phi^\ddagger\gamma_\mu\phi\rfloor_{3,4}\phi^\ddagger\gamma_\nu\phi+\lfloor\phi^\ddagger\gamma_\nu\phi\rfloor_{3,4}\phi^\ddagger\gamma_\mu\phi\Big)=\langle\hat{g}_{\mu\nu}\rangle \tag{307}$$

6. In general the multilinear form permits a spacetime interval measurement:

$$\frac{1}{2}\Big(\lfloor\psi^\ddagger\mathbf{v}\psi\rfloor_{3,4}\psi^\ddagger\mathbf{w}\psi+\lfloor\psi^\ddagger\mathbf{w}\psi\rfloor_{3,4}\psi^\ddagger\mathbf{v}\psi\Big)=\mathbf{v}\cdot\mathbf{w} \tag{308}$$

3.3 The Double-Copy Gauge Theory

In recent years, a remarkable connection between gauge theories and gravity has been discovered, known as the "double-copy" relationship. This relationship, first proposed by Bern, Carrasco, and Johansson (BCJ) [14], states that the scattering amplitudes of certain gravity theories can be expressed as a "double-copy" of the scattering amplitudes of gauge theories, such as Yang-Mills theory.

The BCJ double-copy is based on the observation that the scattering amplitudes of gauge theories can be written in a form where the kinematic numerators obey the same algebraic relations as the color factors. This is known as the "color-kinematics duality." By replacing the color factors with another copy of the kinematic numerators, one obtains the scattering amplitudes of a related gravity theory.

Our multilinear form is able to engender its own version of a double-copy of gauge theories. It would be interesting to establish if this relates to the BCJ double copy, or if it is a different double-copy effect.

Theorem 28 (Double-Copy Gauge Theory)**.** *Let* $|\psi_1\rangle\!\rangle$ *and* $|\psi_2\rangle\!\rangle$ *be two* $\mathrm{Spin}^c(3,1)$*-valued wavefunction, and let* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *be two bivectors of* $\mathrm{GA}(3,1)$*. Then:*

$$\langle\!\langle\mathbf{f}_1\psi_1(q)|\gamma_0\mathbf{f}_1\psi_1(q)|\mathbf{f}_2\psi_2(q)|\gamma_0\mathbf{f}_1\psi_2(q)\rangle\!\rangle=\langle\!\langle\psi_1(q)|\gamma_0\psi_1(q)|\psi_2(q)|\gamma_0\psi_2(q)\rangle\!\rangle \tag{309}$$

implies two copies of a $\mathrm{SU}(3)$ *gauge theory, satisfying the invariance of the multilinear form.*

Proof. The relation $\lfloor(\mathbf{f}_1\psi_1(q))^\ddagger\gamma_0\mathbf{f}_1\psi_1(q)\rfloor_{3,4}(\mathbf{f}_2\psi_2(q))^\ddagger\mathbf{f}_2\psi_2(q)$ remains invariant if

$$-\mathbf{f}_1\gamma_0\mathbf{f}_1=\gamma_0 \qquad \text{(copy 1)} \tag{310}$$

$$-\mathbf{f}_2\gamma_0\mathbf{f}_2=\gamma_0 \qquad \text{(copy 2)} \tag{311}$$

$$\tag{312}$$

which according to Theorem 19, each copy yields a realization of the SU(3) gauge; in the present case, yielding two distinct copies. Any perturbative expansion of the metric operator will be formulated in terms of these wavefunction double-copies which can be related to gauge theory. We are not sure if this can be connected to the BCJ double-copy conjecture, but we think it may be an interesting avenue for future research. □

3.4 Density and Continuum

Merely for completeness, let us now extend the entropy maximization problem from the discreet Σ to the continuum $\int$, using a Riemann sum. We will take the quantum mechanics Lagrange multiplier equation an example, but the method can be applied to any of the three Lagrange multiplier equations we introduced.

$$\mathcal{L} = \lim_{n\to\infty} \left(-\sum_{i=1}^{n} \rho(x_i) \ln \frac{\rho(x_i)}{\rho_0(x_i)} + \lambda \left(1 - \sum_{x=1}^{n} \rho(x_i) \right) + \kappa \left(-\operatorname{tr} \frac{1}{2} \sum_{i=1}^{n} \rho(x_i) \frac{1}{m(x_i)} \mathbf{E}(x_i) \right) \right) \Delta x \tag{313}$$

where $\mathbf{E}(x_i) := \begin{bmatrix} 0 & -E(x_i) \\ E(x_i) & 0 \end{bmatrix}$, and where

- n is the number of subintervals,
- $\Delta x = (b-a)/n$ is the width of each subinterval,
- x_i is a point within the i-th subinterval $[x_{i-1}, x_i]$, often chosen to be the midpoint $(x_{i-1} + x_i)/2$.
- $1/m(x_i)$ is a factor required to transform the components of the matrix $\mathbf{M}(x_i)$ into a density, required for integration.

which yields an integral:

$$\mathcal{L} = -\int_a^b \rho(x) \ln \frac{\rho(x)}{\rho_0(x)} \mathrm{d}x + \lambda \left(1 - \int_a^b \rho(x) \mathrm{d}x \right) + \kappa \left(-\operatorname{tr} \frac{1}{2} \int_a^b \rho(x) \frac{1}{m(x)} \mathbf{E}(x) \mathrm{d}x \right) \tag{314}$$

The solution to this optimization problem is a distribution density:

$$\frac{\partial \mathcal{L}}{\partial \rho} = 0 \implies \rho(x) = \underbrace{\frac{1}{\int_a^b \rho_0(r) \exp\left(-\frac{1}{2}\kappa \frac{1}{m(r)} \operatorname{tr} \mathbf{E}(r) \right) \mathrm{d}r}}_{\text{Unitarily Invariant Ensemble}} \underbrace{\exp\left(-\frac{1}{2}\kappa \frac{1}{m(x)} \operatorname{tr} \mathbf{E}(x) \right)}_{\text{Born Rule}} \underbrace{p(x)}_{\text{Initial Preparation}} \tag{315}$$

This formulation extends the framework to the continuum, allowing for the description of continuous systems while preserving the geometric structure and invariance properties of the theory.

4 Conclusion

In conclusion, this paper presents a novel approach to physical theory construction by solving a maximization problem on the Shannon entropy of all possible measurements of a system relative to its initial preparation, under the constraint of a vanishing phase. By appropriately selecting the group of the vanishing phase, the solution resolves to quantum mechanics, relativistic quantum mechanics, or a theory of quantum gravity. Our findings reveal the exceptional ability of this approach to generate a mathematically well-behaved theory that generalizes quantum probabilities through the introduction of vanishing phases. The resulting measure is invariant under a wide range of geometric transformations, including those generated by the gauge groups of the Standard Model, those associated to general relativity, and leads to the metric tensor as a quantum mechanical observable, without the need for additional assumptions beyond the vanishing phase. This finding aligns with the observed dimensionality and gauge symmetries of the universe and suggests a possible explanation for its specificity.

This research represents a significant step in reconciling quantum mechanics with general relativity, challenging and expanding conventional methodologies in theoretical physics, and potentially paving the way for new insights in the field. By reducing fundamental physics to its simplest and most parsimonious expression, deriving the core theories as optimal solutions to a well-defined entropy maximization problem, we offer a unified framework that integrates statistical mechanics, quantum mechanics, relativistic quantum mechanics, and quantum gravity, while also accounting for the dimensionality of spacetime and the gauge symmetries of particle physics.

Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

- Data Availability Statement: No datasets were generated or analyzed during the current study.

- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

A SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L}(\rho,\lambda,\beta) = \underbrace{-k_B\sum_{q\in\mathbb{Q}}\rho(q)\ln\rho(q)}_{\substack{\text{Boltzmann}\\ \text{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\text{Normalization}\\ \text{Constraint}}} + \underbrace{\beta\left(\overline{E}-\sum_{q\in\mathbb{Q}}\rho(q)E(q)\right)}_{\text{Average Energy Constraint}} \tag{316}$$

We solve the maximization problem as follows:

$$\frac{\partial\mathcal{L}(\rho,\lambda,\beta)}{\partial\rho(q)} = 0 = -\ln\rho(q) - 1 - \lambda - \beta E(q) \tag{317}$$

$$0 = \ln\rho(q) + 1 + \lambda + \beta E(q) \tag{318}$$

$$\implies \ln\rho(q) = -1 - \lambda - \beta E(q) \tag{319}$$

$$\implies \rho(q) = \exp(-1-\lambda)\exp\left(-\beta E(q)\right) \tag{320}$$

$$= \frac{1}{Z(\tau)}\exp\left(-\beta E(q)\right) \tag{321}$$

The partition function, is obtained as follows:

$$1 = \sum_{r\in\mathbb{Q}}\exp(-1-\lambda)\exp\left(-\beta E(q)\right) \tag{322}$$

$$\implies \left(\exp(-1-\lambda)\right)^{-1} = \sum_{r\in\mathbb{Q}}\exp\left(-\beta E(q)\right) \tag{323}$$

$$Z(\tau) := \sum_{r\in\mathbb{Q}}\exp\left(-\beta E(q)\right) \tag{324}$$

Finally, the probability measure is:

$$\rho(q) = \frac{1}{\sum_{r\in\mathbb{Q}}\exp\left(-\beta E(q)\right)}\exp\left(-\beta E(q)\right) \tag{325}$$

B RQM in 3+1D

$$\mathcal{L}(\rho,\lambda,\tau) = \underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\text{Relative Shannon}\\ \text{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\text{Normalization}\\ \text{Constraint}}} + \underbrace{\zeta\left(-\operatorname{tr}\frac{1}{2}\sum_{q\in\mathbb{Q}}\rho(q)\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{v}\to 0}\right)}_{\substack{\text{Vanishing Relativistic-Phase}\\ \text{Anti-Constraint}}} \tag{326}$$

The solution is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r) \det\exp\left(-\zeta \frac{1}{2}\mathbf{M_u}(r)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right)}}_{\text{Spin}^c(3,1)\text{ Invariant Ensemble}} \underbrace{\det\exp\left(-\zeta\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right)}_{\text{Spin}^c(3,1)\text{ Born Rule}} \underbrace{p(q)}_{\text{Initial Preparation}} \tag{327}$$

Proof. The Lagrange multiplier equation can be solved as follows:

$$\frac{\partial\mathcal{L}(\rho,\lambda,\zeta)}{\partial\rho(q)} = 0 = -\ln\frac{\rho(q)}{p(q)} - 1 - \lambda - \zeta\operatorname{tr}\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0} \tag{328}$$

$$0 = \ln\frac{\rho(q)}{p(q)} + 1 + \lambda + \zeta\operatorname{tr}\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0} \tag{329}$$

$$\implies \ln\frac{\rho(q)}{p(q)} = -1 - \lambda - \zeta\operatorname{tr}\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0} \tag{330}$$

$$\implies \rho(q) = p(q)\exp(-1-\lambda)\exp\left(-\zeta\operatorname{tr}\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{331}$$

$$= \frac{1}{Z(\zeta)}p(q)\exp\left(-\zeta\operatorname{tr}\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{332}$$

The partition function $Z(\zeta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_{r\in\mathbb{Q}} p(r)\exp(-1-\lambda)\exp\left(-\zeta\operatorname{tr}\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{333}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_{r\in\mathbb{Q}} p(r)\exp\left(-\zeta\operatorname{tr}\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{334}$$

$$Z(\zeta) := \sum_{r\in\mathbb{Q}} p(r)\exp\left(-\zeta\operatorname{tr}\frac{1}{2}\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{335}$$

□

C SageMath program showing $\lfloor \mathbf{u}^\ddagger\mathbf{u}\rfloor_{3,4}\mathbf{u}^\ddagger\mathbf{u} = \det\mathbf{M_u}$

```
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])
```

```
# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()

# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

u2 = u.clifford_conjugate()*u

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4


I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, 1, 0, 0],
                 [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
                 [0, 0, 0, -1],
```

```
                    [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, -1, 0, 0],
                 [1, 0, 0, 0]])

y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())
```

The program outputs

```
True
```

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the multilinear form: $\det \mathbf{M_u} = \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u}$.

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