Koide Mass Formula for Neutrinos

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Since 1982 the Koide mass relation has provided an amazingly accurate relation between the masses of the charged leptons. In this note we show how the Koide relation can be expanded to cover the neutrinos, and we use the relation to predict neutrino masses.

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In 1982, Yoshio Koide [1, 2], discovered a formula relating the masses of the charged leptons:

$$\frac{(\sqrt{m_e}+\sqrt{m_\mu}+\sqrt{m_\tau})^2}{m_e+m_\mu+m_\tau} = \frac{3}{2}. \tag{1}$$

Written in the above manner, this relation removes one degree of freedom from the three charged lepton masses. In this paper, we will first derive this relation as an eigenvalue equation, then obtain information about the other degrees of freedom, and finally speculatively apply the same techniques to the problem of predicting the neutrino masses.

If we suppose that the leptons are composite particles made up of colorless combinations of colored subparticles or preons, we expect that the three colors must be treated equally, and therefore a natural form for a matrix operator that can cross generation boundaries is the *circulant*:

$$\Gamma(A,B,C) = \begin{pmatrix} A & B & C \\ C & A & B \\ B & C & A \end{pmatrix}. \tag{2}$$

where A, B and C are complex constants. Other authors have explored these sorts of matrices in the context of neutrino masses and mixing angles including [3–5]. Such matrices have eigenvectors of the form:

$$|n\rangle = \begin{pmatrix} 1 \\ e^{+2in\pi/3} \\ e^{-2in\pi/3} \end{pmatrix}, \quad n = 1,2,3. \tag{3}$$

This set of eigenvectors are common to more than circulant matrices. Other authors finding uses for these sets of eigenvectors in the problem of neutrino mixing include [6–10].

If we require that the eigenvalues be real, we obtain that A must be real, and that B and C are complex conjugates. This reduces the 6 real degrees of freedom present in the 3 complex constants A, B and C to just 3 real degrees of freedom, the same as the number of eigenvalues for the operators. In order to parameterize these sorts of operators, in a manner only slightly different from that chosen in [4], let us write:

$$\Gamma(\mu,\eta,\delta) = \mu \begin{pmatrix} 1 & \eta\exp(+i\delta) & \eta\exp(-i\delta) \\ \eta\exp(-i\delta) & 1 & \eta\exp(+i\delta) \\ \eta\exp(+i\delta) & \eta\exp(-i\delta) & 1 \end{pmatrix}, \tag{4}$$

where we can assume η to be non negative. Note that while η and δ are pure numbers, μ scales with the eigenvalues. Then the three eigenvalues are given by:

$$\begin{aligned} \Gamma(\mu,\eta,\delta)\,|n\rangle &= \lambda_n\,|n\rangle, \\ &= \mu(1+2\eta\cos(\delta+2n\pi/3))\,|n\rangle. \end{aligned} \tag{5}$$

The sum of the eigenvalues are given by the trace of Γ:

$$\lambda_1+\lambda_2+\lambda_3 = 3\mu, \tag{6}$$

and this allows us to calculate μ from a set of eigenvalues. The sum of the squares of the eigenvalues are given by the trace of Γ^2:

$$\lambda_1^2+\lambda_2^2+\lambda_3^2 = 3\mu^2(1+2\eta^2), \tag{7}$$

and this gives a formula for η^2 in terms of the eigenvalues:

$$\frac{\lambda_1^2+\lambda_2^2+\lambda_3^2}{(\lambda_1+\lambda_2+\lambda_3)^2} = \frac{1+2\eta^2}{3}, \tag{8}$$

The value of δ is then easy to calculate from Eq. (5).

We have restricted Γ to a form where all its eigenvalues are real, but it is still possible that some or all will be negative. For situations where all the eigenvalues are non negative, it is natural to suppose that our values are eigenvalues not of the Γ matrix, but instead of Γ^2.

The masses of the charged leptons are positive, so let us compute the square roots of the masses of the charged leptons, and find the values for μ_1, η_1^2 and δ_1, where the subscript will distinguish the parameters for the masses of the charged leptons from that of the neutral leptons.

Given the latest PDG[11] data (MeV):

$$\begin{aligned} m_1 &= m_e = 0.510998918(44) \\ m_2 &= m_\mu = 105.6583692(94) \\ m_3 &= m_\tau = 1776.99+0.29-0.26 \end{aligned} \tag{9}$$

and ignoring, for the moment, the error bars, and keeping 7 digits of accuracy, we obtain

$$\begin{aligned} \mu_1 &= 17.71608\ \text{MeV}^{0.5} \\ \eta_1^2 &= 0.5000018 \\ \delta_1 &= 0.2222220 \end{aligned} \tag{10}$$

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The fact that η_1^2 is very close to 0.5 was noticed in 1982 by Yoshio Koide[2] at a time when the τ mass had not been experimentally measured to anywhere near its present accuracy. Also see [12]. That δ_1 is close to 2/9 went unnoticed until this author discovered it in 2005.[13]

The present constraints on the electron, muon and tau masses exclude the possibility that δ_1 is exactly 2/9, while on the other hand, $\eta_1^2 = 0.5$ fits the data close to the middle of the error bars. If we interpret Γ as a matrix of coupling constants, $\eta^2 = 1/2$ is a probability. Thus if the preons are to be spin$-1/2$ states, the $P = (1+\cos(\theta))/2$ rule for probabilities implies that the three coupled states are perpendicular, a situation that would be more natural for classical waves than quantum states.

By making the assumption that η_1^2 is precisely 0.5, one obtains a prediction for the mass of the tau. Since the best measurements for the electron and muon masses are in atomic mass units, we give the predicted τ mass in both those units and in MeV:

$$\begin{aligned} m_\tau &= 1776.968921(158) \quad \text{MeV} \\ &= 1.907654627(46) \quad \text{AMU}. \end{aligned} \tag{11}$$

The error bars in the above, and in later calculations in this paper, come from assuming that the electron and muon masses are anywhere inside the error bars given by Eq. (9).

This gives us the opportunity to fine tune our estimate for μ_1 and δ_1. Since the electron and muon data are the most exact, we assume the Koide relation and compute the tau mass from them. Then we compute μ_1 and then δ_1 over the range of electron and muon masses, obtaining:

$$\begin{aligned} \delta_1 &= .22222204715(311) \quad \text{from MeV data} \\ &= .22222204717(48) \quad \text{from AMU data.} \end{aligned} \tag{12}$$

If δ_1 were zero, the electron and muon would have equal masses, while if δ_1 were $\pi/12$, the electron would be massless. Instead, δ_1 is close to a rational fraction, while the other terms inside the cosine are rational multiples of pi. Using the more accurate AMU data, the difference between δ_1 and 2/9 is:

$$2/9 - \delta_1 = 1.7505(48) \times 10^{-7}, \tag{13}$$

and we can hope that a deeper theory will allow this small difference to be computed. This difference could be written as

$$1.75 \times 10^{-7} = \frac{4\pi}{3^{12}} \left(\alpha + \mathcal{O}(\alpha^2)\right) \tag{14}$$

for example.

Note that δ_1 is close to a rational number, while the other terms that are added to it inside the cosine of Eq. (5) are rational multiples of pi. This distinction follows our parameterization of the eigenvalues in that the rational fraction part comes from the operator Γ, while the $2n\pi/3$ term comes from the eigenvectors. Rather than depending on the details of the operator, the $2n\pi/3$ depends only on the fact that the operator has the symmetry of a circulant matrix.

We will use m_1, m_2 and m_3 to designate the masses of the neutrinos. The experimental situation with the neutrinos is primitive at the moment. The only accurate measurements are from oscillation experiments, and are for the absolute values of the differences between squares of neutrino masses. Recent 2σ data from [14] are:

$$\begin{aligned} |m_2^2 - m_1^2| &= 7.92(1 \pm .09) \times 10^{-5} \quad \text{eV}^2 \\ |m_3^2 - m_2^2| &= 2.4(1 + 0.21 - 0.26) \times 10^{-3} \quad \text{eV}^2 \end{aligned} \tag{15}$$

Up to this time, attempts to apply the unaltered Koide mass formula to the neutrinos have failed,[15–17] but these attempts have assumed that the square roots of the neutrino masses must all be positive.[1] Without loss of generality, we will assume that μ_0 and η_0 are both positive, thus there can be at most one square root mass that is negative, and it can be only the lowest or central mass.

Of these two cases, having the central mass with a negative square root is incompatible with the oscillation data, but we can obtain $\eta_0^2 = 1/2$ with masses around:

$$\begin{aligned} m_1 &= 0.0004 \ \text{eV}, \\ m_2 &= 0.009 \ \text{eV}, \\ m_3 &= 0.05 \ \text{eV}. \end{aligned} \tag{16}$$

These masses approximately satisfy the squared mass differences of Eq. (15) as well as the Koide relation as follows:

$$\frac{m_1 + m_2 + m_3}{(-\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2} = \frac{2}{3}. \tag{17}$$

The above neutrino masses were chosen to fix the value of $\eta_0^2 = 1/2$. As such, the fact that this can be done is of little interest, at least until absolute measurements of the neutrino masses are available. However, given these values, we can now compute the μ_0 and δ_0 values for the neutrinos. The results give that, well within experimental error:

$$\begin{aligned} \delta_0 &= \delta_1 + \pi/12, \\ \mu_1/\mu_0 &= 3^{11}. \end{aligned} \tag{18}$$

Recalling the split between the components of the cosine in the charged lepton mass formula, the fact that $\pi/12$ is a rational multiple of pi suggests that it should be related to a symmetry of the eigenvectors rather than the operator. One possible explanation is that in transforming from a right handed particle to a left handed particle, the neutrino (or a portion of it) pick up a phase difference that is a fraction of pi. As a result the neutrino requires

[1] This restriction is difficult to understand given several papers that have assumed that the neutrino masses themselves may be negative.[18, 19]

12 stages to make the transition from left handed back to left handed (with the same phase), while the electron requires only a single stage, and it is natural for the mass hierarchy between the charged and neutral leptons to be a power of $12 - 1 = 11$.

We will therefore assume that the relations given in Eq. (18) are exact, and can then use the measured masses of the electron and muon to write down predictions for the absolute masses of the neutrinos. The parameterization of the masses of the neutrinos is as follows:

$$m_n = \frac{\mu_1}{3^{11}}(1 + \sqrt{2}\cos(\delta_1 + \pi/12 + 2n\pi/3)), \tag{19}$$

where μ_1 and δ_1 are from the charged leptons. Substituting in the the measured values for the electron and muon masses, we obtain extremely precise predictions for the neutrino masses:

$$\begin{aligned} m_1 &= 0.000383462480(38) && \text{eV} \\ &= 0.4116639106(115) \times 10^{-12} && \text{AMU} \end{aligned} \tag{20}$$

$$\begin{aligned} m_2 &= 0.00891348724(79) && \text{eV} \\ &= 9.569022271(246) \times 10^{-12} && \text{AMU} \end{aligned} \tag{21}$$

$$\begin{aligned} m_3 &= 0.0507118044(45) && \text{eV} \\ &= 54.44136198(131) \times 10^{-12} && \text{AMU} \end{aligned} \tag{22}$$

Similarly, the predictions for the differences of the squares of the neutrino masses are:

$$\begin{aligned} m_2^2 - m_1^2 &= 7.930321129(141) \times 10^{-5} && \text{eV}^2 \\ &= .913967200(47) \times 10^{-24} && \text{AMU}^2 \end{aligned} \tag{23}$$

$$\begin{aligned} m_3^2 - m_2^2 &= 2.49223685(44) \times 10^{-3} && \text{eV}^2 \\ &= 2872.295707(138) \times 10^{-24} && \text{AMU}^2 \end{aligned} \tag{24}$$

As with the Koide predictions for the tau mass, these predictions for the squared mass differences are dead in the center of the error bars. We can only hope that the future will show our calculations to be as prescient as Koide's.

That the masses of the leptons should have these sorts of relationships is particularly mysterious in the context of the standard model.[20] It is hoped that this paper will stimulate thought among theoreticians. Perhaps the fundamental fermions are bound states of deeper objects.

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