Proof of the 3n+1 problem for \( n \geq 1 \)

by

Steffen Bode

Albuquerque, New Mexico, USA
email: the3nplus1proof@yahoo.com

Abstract

I establish the existence of a unique binary pattern inherent to the 3n+1 step, and then use this binary pattern to prove the 3n+1 problem for all positive integers.

Introduction

Observe that every positive odd integer \( n \), defined as \( n = \sum_{i=0}^{x} 4^i, \ x \in \mathbb{Z}^+ \)

requires one 3n+1 step and then \( 2(x+1) \) consecutive \( n/2 \) steps to be reduced to 1.

The truth of this statement will become apparent when the 3n+1 step with such an integer is observed in base 2.

Example 1: Let \( n = \sum_{i=0}^{2} 4^i = 21 = 10101_2 \), then

\[
10101_2 \times 10_2 \Rightarrow 101010_2 + 10101_2 \Rightarrow 111111_2 + 1_2 = 1000000_2 = 2^6, \text{ and}
\]

\[
1000000_2/10_2 \Rightarrow 1000000_2/10_2 \Rightarrow 10000_2/10_2 \Rightarrow 1000_2/10_2 \Rightarrow 100_2/10_2 \Rightarrow 10_2/10_2 = 1.
\]

Therefore, the base 2 representation of positive integers furnishes more insight into the 3n+1 problem than their base 10 representation.
Proof

Let \( O^+ \) be the set of positive odd integers, then \( O^+ = \{ x \in \mathbb{Z} | x = 2y + 1, y \geq 0, y \in \mathbb{Z} \} \).

Theorem 1: Let \( P \) designate the 3n+1 problem. Then if \( P \) is true for all positive odd integers, it is true for all positive integers.

\[ \forall a \in O^+: P(a) \Rightarrow \forall b \in \mathbb{Z}^+: P(b) \]

Proof: Case 1: Power of two

Let \( n = 2^x, x \in \mathbb{Z}^+ \). Then \( n \) requires \( x \) consecutive \( n/2 \) steps to be reduced to 1.

Case 2: Odd integer multiplied by a power of two

Let \( y = 2^x n, n \in O^+ \) and \( x \in \mathbb{Z}^+ \). Then \( x \) consecutive \( n/2 \) steps are required to have \( y = n \).

Since these cases are exhaustive, it shows that if the 3n+1 problem is true for all \( a \in O^+ \) it hast to be true for all \( b \in \mathbb{Z}^+ \).

The iteration between the 3n+1 step and the \( n/2 \) step modifies every integer \( n, n \in O^+ \) in such a way that, at some point the integer becomes \( 2^x, x \in \mathbb{Z}^+ \).

However, the process of this transformation is obscured by the \( n/2 \) step. In order to make the process apparent, the \( n/2 \) step is omitted and the addition of 1 in the 3n+1 step is modified to compensate for the omission of the \( n/2 \) step.

Example 2: Let \( n = 9 = 1001_2 \), then \( 3n+2^x \) produces this pattern:

\[
\begin{align*}
1001_2 \times 11_2 & \Rightarrow 11011_2 + 1_2 = 11100_2 \\
11100_2 \times 11_2 & \Rightarrow 1010100_2 + 100_2 = 1011000_2 \\
1011000_2 \times 11_2 & \Rightarrow 100001000_2 + 1000_2 = 100010000_2 \\
100010000_2 \times 11_2 & \Rightarrow 1100110000_2 + 10000_2 = 1101000000_2 \\
1101000000_2 \times 11_2 & \Rightarrow 10011100000_2 + 1000000_2 = 101000000000_2 \\
101000000000_2 \times 11_2 & \Rightarrow 1111000000000_2 + 1000000000_2 = 100000000000000_2 = 2^{13}.
\end{align*}
\]
In example 2, the least significant bit transcends the most significant bit after six $3n+2^k$ steps, transforming $n$ into a power of two.

**Definition 1:** Let LSB be the least significant bit of $s \in \mathbb{Z}^+$, then

$$\text{LSB} = \{2^r, r \geq 0, r \in \mathbb{Z} \mid 2^r = s/t, t \in \mathbb{O}^+\}.$$  

**Theorem 2:** The $3n$+LSB step and the $3n+1$ step are isomorphic.

**Proof:** Suppose $n_0 \in \mathbb{O}^+$. Let $n_1 = 3n_0 + 1$ and $n_2 = n_1$/LSB, then

$$\frac{3n_1+\text{LSB}}{3n_2+1} = \frac{3n_1+\text{LSB}}{3(\frac{n_1}{\text{LSB}})+1} = \frac{3n_1+\text{LSB}}{\text{LSB}} = \text{LSB}.$$  

$\therefore 3n+$LSB $\equiv 0 \pmod{3n+1}.$

Because a modular congruence exists between the $3n+$LSB step and the $3n+1$ step, they are therefore isomorphic.

The pattern in example 2 is composed of two functions. The first function increases the most significant power of two or most significant bit of $n$, and the second function increases the least significant power of two or least significant bit of $n$.

Let $m(x)$ be the function for repeated multiplication of $n$ by 3 in terms of $x$, $x \in \mathbb{Z}^+$. Then $m(x) = 3^{x+5}n$.

Let $\text{lsb}(x)$ be the function for repeated multiplication by 4 ($3(\text{LSB})+$LSB) of the least significant bit of $n$ in terms of $x$, $x \in \mathbb{Z}^+$. Then $\text{lsb}(x) = 4^{x+5}$.

**Definition 2:** Let $f(x)$ be the function for the $3n+$LSB step for $n \in \mathbb{O}^+$ in terms of $x$, $x \in \mathbb{Z}^+$. Then

$$f(x) = m(x) + \text{lsb}(x) = 3^{x+5}n + 4^{x+5}.$$
Suppose that Tlsb(x) is the function that gives the true position of the least significant bit of the 3n+LSB step for \( n \in \mathbb{O^+} \) in terms of \( x, x \in \mathbb{Z^+} \). Then

\[
\delta = \sum_{1}^{x} Tlsb(x) - \text{lsb}(x).
\]

**Example 3:** \( Tlsb(x) > \text{lsb}(x) \)

Assume that multiplying \( n_k \) by 3 produces \( \cdots 001111100 \cdots \) somewhere in the binary representation of the result; and that the rightmost 1 is \( \text{LSB} = 2^x \). Let \( \text{lsb}(x) = Tlsb(x) \). Adding LSB to \( n_k \) yields \( \cdots 010000000 \cdots \), then

\[
\delta = \sum_{1}^{x} Tlsb(x) - \text{lsb}(x) = \sum_{1}^{x} 2^{x+5} - 2^{x+2} = \sum_{1}^{x} x+5-x-2 = \sum_{1}^{x} 3 = 3.
\]

**Example 4:** \( Tlsb(x) < \text{lsb}(x) \)

Assume that multiplying \( n_k \) by 3 and adding LSB produces \( \cdots 001111100 \cdots \) somewhere in the binary representation of the result; and that the rightmost 1 is \( \text{LSB} = 2^x \). Let \( \text{lsb}(x) = Tlsb(x) \). Then repeated multiplication by 3 and addition of LSB will produce this pattern:

\[
\cdots 001111100 \cdots \text{ times } 3 \text{ plus } 2^x \\
\cdots 101111000 \cdots \text{ times } 3 \text{ plus } 2^{x+1} \\
\cdots 001110000 \cdots \text{ times } 3 \text{ plus } 2^{x+2} \\
\cdots 101100000 \cdots \text{ times } 3 \text{ plus } 2^{x+3} \\
\cdots 001000000 \cdots , \text{ then}
\]

\[
\delta = \sum_{1}^{x+3} Tlsb(x) - \text{lsb}(x) = \sum_{1}^{x+3} 2^{x+1} - 2^{x+2} = \sum_{1}^{x+3} x+1-x-2 = \sum_{1}^{x+3} 1 = -4.
\]

\[
\therefore (\delta < 0) \lor (\delta = 0) \lor (\delta > 0)
\]

Assume \( x \in \mathbb{Z^+} \), then \( m(x) < \text{lsb}(x) \) implies that a single power of two is larger than a sum of powers of two.
Using example 2 as an illustration:

\[ m(x) - \text{lsb}(x) = 9(3^{x^2}) - 4^{x^2} = 0 \quad \text{for} \quad x \approx 5.6377. \]

The integer after the root necessitates that \( m(x) < \text{lsb}(x) \). In other words, it requires six \( 3n+\text{LSB} \) steps for 9 to converge to \( 2^{13} \).

**Theorem 3:** For all positive odd integers \( n \), there exists a positive integer \( x \) such that \( m(x) < \text{lsb}(x) \).

\[ \forall n \in \mathbb{O}^+ \exists x \in \mathbb{Z}^+ (m(x) < \text{lsb}(x)) \]

**Proof:**

**Case 1:** \( \delta \leq -1, \delta \in \mathbb{Z} \)

Assume \( n \in \mathbb{O}^+ \) and let \( m(x) - \text{lsb}(x) = 3^{x^2}n - 4^{x^2} = 0 \).

Then \( x = \frac{\log(1/n)}{\log(3/4)} + \delta. \)

\[ \therefore \exists! x \in \mathbb{R}^+ (3^{x^2}n - 4^{x^2} = 0) \Rightarrow \exists x \in \mathbb{Z}^+ (m(x) < \text{lsb}(x)) \]

**Case 2:** \( \delta = 0 \)

Assume \( n \in \mathbb{O}^+ \) and let \( m(x) - \text{lsb}(x) = 3^x - 4^x = 0. \)

Then \( x = \frac{\log(1/n)}{\log(3/4)}. \)

\[ \therefore \exists! x \in \mathbb{R}^+ (3^x - 4^x = 0) \Rightarrow \exists x \in \mathbb{Z}^+ (m(x) < \text{lsb}(x)) \]

**Case 3:** \( \delta \geq 1, \delta \in \mathbb{Z} \)

Assume \( n \in \mathbb{O}^+ \) and let \( m(x) - \text{lsb}(x) = 3^{x^2}n - 4^{x^2} = 0. \)

Then \( x = \frac{\log(1/n)}{\log(3/4)} - \delta. \)

\[ \therefore \exists! x \in \mathbb{R}^+ (3^{x^2}n - 4^{x^2} = 0) \Rightarrow \exists x \in \mathbb{Z}^+ (m(x) < \text{lsb}(x)) \]

Because these cases are exhaustive, it shows that

\[ \forall n \in \mathbb{O}^+ \exists x \in \mathbb{Z}^+ (m(x) < \text{lsb}(x)). \]
For all \( n \in \mathbb{O}^+ \) there exists an \( x \in \mathbb{Z}^+ \) such that \( m(x) < \text{lsb}(x) \) (Theorem 3), therefore \( f(x) \) converges to \( 2^y, y \in \mathbb{Z}^+ \). And since the \( 3n+\text{LSB} \) step and the \( 3n+1 \) step are isomorphic (Theorem 2), it can be concluded that if \( a_0 = n, n \in \mathbb{O}^+ \), then

\[
a_{i+1} = \begin{cases} 
    a_i / 2 & \text{for even } a_i \\
    3a_i + 1 & \text{for odd } a_i
\end{cases},
\]

converges to 1.

Because the \( 3n+1 \) problem is true for all positive odd integers, then by Theorem 1 the truth extends to all positive integers. Therefore, if \( a_0 = n, n \in \mathbb{Z}^+ \), then

\[
a_{i+1} = \begin{cases} 
    a_i / 2 & \text{for even } a_i \\
    3a_i + 1 & \text{for odd } a_i
\end{cases},
\]

converges to 1.

Q.E.D