A Note on the Mass-Energy Relation

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The famous equation that relates the mass with the energy can be deduced without using the special relativity of Einstein; however, the relation obtained is slightly different.

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The famous equation $E = mc^2$ can be deduced without using the special relativity of Einstein, such as in [1], or from this other simpler form: when an atom absorbs a photon, the energy is converted into matter, that is, into mass. Thus, an atom at rest of mass m_0 recoils with a speed v when it absorbs a photon of an energy E that corresponds to a mass μ. The momentum of the photon would be $p = F\tau = F\lambda/c = W/c = E/c$, where F is the force exerted by the photon, $\tau = \lambda/c$ the duration of the event, λ the wavelength, c the speed of the light in the vacuum and $W = F\lambda$ the work done by the photon (the energy E is converted into the work W during the event). (Note that as $E = h\nu$ and $c = \lambda\nu$, then $p = E/c = h\nu/\lambda\nu = h/\lambda$, where h is the Planck's constant and ν the frequency; and also that $\tau = \lambda/c =\lambda/\lambda\nu = 1/\nu$). From the conservation of the momentum, $(p_1 + p_2)_{final} = (p_1 + p_2)_{initial}$, where the subscript 1 is for the atom and the 2 for the photon; we would have that $mv + 0 = 0 + E/c$, or $mv = E/c = (E/c^2)c = \mu c$, where m is the moving mass of the atom and $\mu = E/c^2 = h\nu/c^2$ the so-called "effective mass" of the photon. If the energy of the photon $E = h\nu$ can be written also like $E = \mu c^2$, we postulate that the rest energy of the atom before of the absorption (its initial energy) is $E_{0a} = m_0c^2$. From the conservation of the energy, $(E_1 + E_2)_{final} = (E_1 + E_2)_{initial}$, we would have that $E_a + 0 = m_0c^2 + \mu c^2$, or $E_a = m_0c^2 + \mu c^2 = (m_0 + \mu)c^2 = mc^2$ with $m = m_0 + \mu$, where $E_a = E_{0a} + T_a$ is the total energy of the atom (its final energy) and $T_a = \mu c^2$ its kinetic energy. If we do $\gamma m_0 = m = m_0 + \mu$, $(\gamma - 1)m_0 = \mu$ and $(\gamma - 1)m_0c^2 = \mu c^2$, the final kinetic energy of the atom, $T_a = E_a - E_{0a} = mc^2 - m_0c^2 = \gamma m_0c^2 - m_0c^2 = (\gamma - 1)m_0c^2$, is the initial kinetic energy of the photon, $T_a = \mu c^2$, with $\gamma = (1 - v/c)^{-1}$ because from $\gamma m_0v = mv = \mu c$, $(\gamma - 1)m_0 = \mu$ and $(\gamma - 1)m_0c = \mu c$ we have that $\gamma m_0v = (\gamma - 1)m_0c$, $\gamma v = (\gamma - 1)c$ and $\gamma = (1 - v/c)^{-1}$. Therefore, for a body of rest and moving masses m_0 and m its energy would be $E = mc^2 = \gamma m_0c^2 = (1 - v/c)^{-1}m_0c^2$, and for $v << c$, $E \approx m_0c^2 + m_0vc + m_0v^2$, which is a balanced expression. However, in the special relativity it is $\gamma = (1 - v^2/c^2)^{-1/2}$ and $E = mc^2 = \gamma m_0c^2 = (1 - v^2/c^2)^{-1/2}m_0c^2$, and for $v^2 << c^2$, $E \approx m_0c^2 + (1/2)m_0v^2$. (Note that in both cases it is $0 \leq v < c$ since $v = c$ implies $\gamma = \infty$). In short, we have deduced the mass-energy relation without using the special relativity; however, the relation obtained is slightly different.

[1] An Elementary Derivation of E = mc^2. This handout is based on the treatment given on pages 283 to 286 of the book, Einstein's Theory of Relativity, by Max Born, Dover Publications, New York (1965).
http://www.personal.psu.edu/pjm11/Einstein.doc