Quaternionic continuity equation for charges

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Abstract

The continuity equation is specified in quaternionic format. It means that the density and current of the considered "charge" is combined in a quaternionic probability amplitude distribution (PAD). Next, the Dirac equation is also put in quaternionic format. It is shown that it is a special form of continuity equation. Further it is shown that two other quaternionic continuity equations can be derived from the quaternionic Dirac equation. The square and the squared modulus of the PAD play an essential role in these new equations. Further, the difference between the quaternionic Dirac equation and the quaternionic Majorana equation is explained. The interpretation of these extra equations leads to the insight that when fermions take a new position, they must step over a forbidden region. Finally, the role of the quaternionic covariant derivative is explained.

Continuity equation

When $\rho_0(q)$ is interpreted as a probability density distribution, then the conservation of the corresponding charge is given by the continuity equation:

Total change within *V* = flow into *V* + production inside *V* (1)

$$\frac{d}{dt}\int_V \rho_0 \, dV = \oint_S \hat{\boldsymbol{n}}\rho_0 \frac{\boldsymbol{v}}{c} \, dS + \int_V s_0 \, dV \tag{2}$$

$$\int_V \nabla_0 \rho_0 \, dV = \int_V \langle \boldsymbol{\nabla}, \boldsymbol{\rho} \rangle \, dV + \int_V s_0 \, dV \tag{3}$$

Here $\hat{\boldsymbol{n}}$ is the normal vector pointing outward the surrounding surface *S*, $\boldsymbol{v}(t, \boldsymbol{q})$ is the velocity at which the charge density $\rho_0(t, \boldsymbol{q})$ enters volume *V* and s_0 is the source density inside *V*. In the above formula $\boldsymbol{\rho}$ stands for

$$\boldsymbol{\rho} = \rho_0 \boldsymbol{v}/c \tag{4}$$

It is the flux (flow per unit area and unit time) of ρ_0 .

The combination of $\rho_0(t, \boldsymbol{q})$ and $\boldsymbol{\rho}(t, \boldsymbol{q})$ is a quaternionic skew field $\rho(t, \boldsymbol{q})$ and can be seen as a probability amplitude distribution (PAD).

$$\rho \stackrel{\text{def}}{=} \rho_0 + \boldsymbol{\rho} \tag{5}$$

$\rho(t,\boldsymbol{q})\rho^*(t,\boldsymbol{q})$ can be seen as an overall probability density distribution (PDD). $\rho_0(t,\boldsymbol{q})$ is a charge density distribution. $\boldsymbol{\rho}(t,\boldsymbol{q})$ is the current density distribution.

Depending on their sign selection, quaternions come in four flavors. In a PAD the quaternion flavors do not mix. So, there are four PAD flavors.
Still these flavors can combine in pairs or in quadruples.

The quaternionic field $\rho(t,\boldsymbol{q})$ contains information on the distribution $\rho_0(t,\boldsymbol{q})$ of the considered charge density as well as on the current density $\boldsymbol{\rho}(t,\boldsymbol{q})$, which represents the transport of this charge density.

Where $\rho(t,\boldsymbol{q})\rho^*(t,\boldsymbol{q})$ can be seen as a probability density of finding the center of charge at position $\boldsymbol{q}$, the probability density distribution $\tilde{\rho}(t,\boldsymbol{p})\tilde{\rho}^*(t,\boldsymbol{p})$ can be seen as the probability density of finding the center of the corresponding wave package at location $\boldsymbol{p}$. $\tilde{\rho}(t,\boldsymbol{p})$ is the Fourier transform of $\rho(t,\boldsymbol{q})$.

The dimension of $\rho_0, \boldsymbol{\rho}$ and ρ is $[XTL^{-3}]$, the dimension of s_0 is $[XL^{-3}]$. The factor c has dimension $[T^{-1}L]$. $[X]$ is an arbitrary dimension. It attaches to the charge.

The conversion from formula (2) to formula (3) uses the Gauss theorem[1]. This results in the law of charge conservation

$$\begin{aligned} s_0(t,\boldsymbol{q}) &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \left(\rho_0(t,\boldsymbol{q})\boldsymbol{v}(t,\boldsymbol{q}) + \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})\right)\rangle \\ &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{\nabla}, \boldsymbol{\rho}(t,\boldsymbol{q}) + \boldsymbol{A}(t,\boldsymbol{q})\rangle \\ &= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle \boldsymbol{v}(t,\boldsymbol{q}), \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \mp \langle \boldsymbol{\nabla}, \boldsymbol{v}(t,\boldsymbol{q})\rangle\, \rho_0(t,\boldsymbol{q}) \\ &\qquad \mp\langle \boldsymbol{\nabla}, \boldsymbol{A}(t,\boldsymbol{q})\rangle \end{aligned} \tag{6}$$

The blue colored ± indicates quaternionic sign selection through conjugation of the field $\rho(t,\boldsymbol{q})$. The field $\boldsymbol{a}(t,\boldsymbol{q})$ is an arbitrary differentiable vector function.

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})\rangle = 0 \tag{7}$$

$\boldsymbol{A}(t,\boldsymbol{q}) \stackrel{\text{def}}{=} \boldsymbol{\nabla}\times\boldsymbol{a}(t,\boldsymbol{q})$ is always divergence free. In the following we will neglect $\boldsymbol{A}(t,\boldsymbol{q})$.
In Fourier space the continuity equation becomes:

$$\tilde{s}_0(t,\boldsymbol{p}) = p_0\tilde{\rho}_0(t,\boldsymbol{p}) \mp \langle \boldsymbol{p}, \tilde{\boldsymbol{\rho}}(t,\boldsymbol{p})\rangle \tag{8}$$

This equation represents a balance equation for charge (or mass) density.
Here $\rho_0(q)$ is the charge distribution, $\boldsymbol{\rho}(q)$ is the current density.
This only treats the real part of the full equation. The full equation runs:

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[1] http://en.wikipedia.org/wiki/Divergence_theorem

$$s(t,\boldsymbol{q}) = \nabla\rho(t,\boldsymbol{q}) = s_0(t,\boldsymbol{q}) + \boldsymbol{s}(t,\boldsymbol{q}) \tag{9}$$

$$= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{\nabla},\boldsymbol{\rho}(t,\boldsymbol{q})\rangle \pm \nabla_0\boldsymbol{\rho}(t,\boldsymbol{q}) + \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \pm \left(\pm\boldsymbol{\nabla}\times\boldsymbol{\rho}(t,\boldsymbol{q})\right)$$

$$= \nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{v}(t,\boldsymbol{q}),\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \mp \langle\boldsymbol{\nabla},\boldsymbol{v}(t,\boldsymbol{q})\rangle\,\rho_0(t,\boldsymbol{q})$$

$$\pm\nabla_0\boldsymbol{v}(t,\boldsymbol{q}) + \nabla_0\rho_0(t,\boldsymbol{q}) + \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})$$

$$\pm\left(\pm(\rho_0(t,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(t,\boldsymbol{q}) - \boldsymbol{v}(t,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\right)$$

$$s_0(t,\boldsymbol{q}) = 2\nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{v}(q),\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \mp \langle\boldsymbol{\nabla},\boldsymbol{v}(t,\boldsymbol{q})\rangle\,\rho_0(t,\boldsymbol{q}) \tag{10}$$

$$\boldsymbol{s}(t,\boldsymbol{q}) = \pm\nabla_0\boldsymbol{v}(t,\boldsymbol{q}) \pm \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \tag{11}$$

$$\pm\left(\pm(\rho_0(t,\boldsymbol{q})\,\boldsymbol{\nabla}\times\boldsymbol{v}(t,\boldsymbol{q}) - \boldsymbol{v}(t,\boldsymbol{q})\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}))\right)$$

The red sign selection indicates a change of handedness by changing the sign of one of the imaginary base vectors. (Conjugation also causes a switch of handedness). If temporarily no creation and no annihilation occur, then these equations reduce to equations of motion.

$$\nabla_0\rho_0(t,\boldsymbol{q}) \pm \nabla_0\boldsymbol{\rho}(t,\boldsymbol{q}) = \pm\langle\boldsymbol{\nabla},\boldsymbol{\rho}(t,\boldsymbol{q})\rangle - \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \left(\pm\boldsymbol{\nabla}\times\boldsymbol{\rho}(t,\boldsymbol{q})\right) \tag{12}$$

$$\nabla_0\rho(t,\boldsymbol{q}) = \langle\boldsymbol{\nabla},\boldsymbol{\rho}(t,\boldsymbol{q})\rangle - \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \boldsymbol{\nabla}\times\boldsymbol{\rho}(t,\boldsymbol{q}) \tag{13}$$

$$\nabla_0\rho_0(t,\boldsymbol{q}) = \mp\langle\boldsymbol{\nabla},\boldsymbol{\rho}(t,\boldsymbol{q})\rangle \tag{14}$$

$$\nabla_0\boldsymbol{\rho}(t,\boldsymbol{q}) = \mp\,\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \boldsymbol{\nabla}\times\boldsymbol{\rho}(t,\boldsymbol{q}) \tag{15}$$

The field $\rho(t,\boldsymbol{q})$ can be split in a (relative) stationary background $\rho_b(\boldsymbol{q})$ and the moving private field $\rho_p(t,\boldsymbol{q})$.
If $\boldsymbol{v}(t,\boldsymbol{q})$ is a constant then

$$s_0(t,\boldsymbol{q}) = 2\nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{v},\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \tag{16}$$

$$\boldsymbol{s}(t,\boldsymbol{q}) = \pm\,\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \left(\pm\boldsymbol{v}\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\right) \tag{17}$$

$$s(t,\boldsymbol{q}) = 2\nabla_0\rho_0(t,\boldsymbol{q}) \mp \langle\boldsymbol{v},\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\rangle \pm \boldsymbol{\nabla}\rho_0(t,\boldsymbol{q}) \mp \left(\pm\boldsymbol{v}\times\boldsymbol{\nabla}\rho_0(t,\boldsymbol{q})\right) \tag{18}$$

The continuity equation has a direct relation to a corresponding conservation law[2]. The conserved quantity is $\rho_0(t,\boldsymbol{q})$ or its integral

$$Charge = \int_V \rho_0\,dV$$

[2] http://en.wikipedia.org/wiki/Conservation_law

Noether's theorem[3] provides the relation between conserved quantities, differentiable symmetries and the Lagrangian[4].

Another interpretation of ρ

The PAD $\rho(t, \boldsymbol{q})$ can be used to define a charge probability density and probability current density. The Dirac equation appears to be a special form of continuity equation.

The Dirac equation runs

$$\nabla_0 \psi + \boldsymbol{\nabla \alpha} \psi = m \beta \psi \tag{1}$$

$\boldsymbol{\alpha}$ and β represent the matrix form of the sign selections of quaternions.

We keep the sign selections of $(t, \boldsymbol{q})$ and their derivatives fixed. Thus $\boldsymbol{\alpha}$ and β only influence the **spinor** ψ.

$$\alpha_1 = \begin{bmatrix} 0 & \boldsymbol{i} \\ -\boldsymbol{i} & 0 \end{bmatrix} \tag{2}$$

$$\alpha_2 = \begin{bmatrix} 0 & \boldsymbol{j} \\ -\boldsymbol{j} & 0 \end{bmatrix} \tag{3}$$

$$\alpha_3 = \begin{bmatrix} 0 & \boldsymbol{k} \\ -\boldsymbol{k} & 0 \end{bmatrix} \tag{4}$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

There exist also a relation between $\alpha_1, \alpha_2, \alpha_3$ and the Pauli[5] matrices $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{6}$$

$$1 \longmapsto I, \qquad \boldsymbol{i} \longmapsto \sigma_1, \qquad \boldsymbol{j} \longmapsto \sigma_2, \qquad \boldsymbol{k} \longmapsto \sigma_3 \tag{7}$$

This combination is usually represented in the form of gamma matrices[6]. These matrices are not used in this paper. They are used when a complex Hilbert space must handle quaternionic behavior.

Transferring the matrix form of the Dirac equation into quaternionic format delivers two quaternionic fields ψ_R and ψ_L that couple two equations of motion.

$$\nabla_0 \psi_R + \boldsymbol{\nabla} \psi_R = m \psi_L \tag{8}$$

[3] http://en.wikipedia.org/wiki/Noether's_theorem
[4] http://en.wikipedia.org/wiki/Lagrangian#Lagrangians_in_quantum_field_theory
[5] http://en.wikipedia.org/wiki/Pauli_matrices
[6] http://en.wikipedia.org/wiki/Gamma_matrices

$$\nabla_0\psi_L - \boldsymbol{\nabla}\psi_L = m\psi_R \tag{9}$$

The mass term m couples ψ_L and ψ_R. The fact $\mathrm{m} = 0$ decouples ψ_L and ψ_R.

$$\psi_R = \psi_L^* = \psi_0 + \boldsymbol{\psi} \tag{10}$$

Thus the fields are each other's quaternionic conjugate.
Reformulating the quaternionic equations gives

$$\nabla_0(\psi_0 + \psi) + \nabla(\psi_0 + \psi) = \mathrm{m}(\psi_0 - \psi) \tag{11}$$

$$\nabla_0(\psi_0 - \psi) - \nabla(\psi_0 - \psi) = \mathrm{m}(\psi_0 + \psi) \tag{12}$$

Summing the equations gives via

$$\nabla\psi = \nabla \times \psi - \langle\nabla, \psi\rangle \tag{13}$$

The result

$$\nabla_0\psi_0 - \langle\nabla, \psi\rangle = \mathrm{m}\ \psi_0 \tag{14}$$

$$\nabla \times \psi = 0 \tag{15}$$

$$\nabla_0\psi + \nabla\psi_0 = -\mathrm{m}\ \psi \tag{16}$$

Compare this with the continuity equations

$$\nabla_0\rho_0(t, \boldsymbol{q}) - \langle\boldsymbol{\nabla}, \boldsymbol{\rho}(t, \boldsymbol{q})\rangle = s_0(t, \boldsymbol{q}) \tag{17}$$

And (18)

$$\boldsymbol{s}(t, \boldsymbol{q}) = \nabla_0\boldsymbol{\rho}(t, \boldsymbol{q}) + \boldsymbol{\nabla}\rho_0(t, \boldsymbol{q}) + \boldsymbol{\nabla} \times \boldsymbol{\rho}(t, \boldsymbol{q})$$

This means that

$$s_0(t, \boldsymbol{q}) \mapsto m\ \psi_0(t, \boldsymbol{q}) \tag{18}$$

$$\boldsymbol{s}(t, \boldsymbol{q}) \mapsto -\mathrm{m}\ \psi \tag{20}$$

Thus in the Dirac equation the mass term is a source term that depends on the (conjugate) field.

This equation can be extended with interactions with other fields.

$$\nabla_0\psi + \boldsymbol{\nabla}\psi = \nabla\psi = m\psi^* \tag{22}$$

$$\nabla\psi = m\psi^* - e\psi A + B \tag{23}$$

The field A is covariant with ψ. e is a coupling constant. Thus here ∇ is the covariant derivative[7]. The field B represents a source.

The following definitions specify another continuity equation:

$$\rho_{Dirac} \stackrel{\text{def}}{=} \psi\psi = \psi_0\psi_0 - \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle + 2\psi_0\boldsymbol{\psi} \tag{24}$$

$$\rho_{0_{Dirac}} = \psi_0\psi_0 - \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle \tag{25}$$

$$\boldsymbol{\rho}_{Dirac} = 2\psi_0\boldsymbol{\psi} \tag{26}$$

$$\nabla\rho_{Dirac} = 2\psi\nabla\psi = 2m\psi\psi^* = 2m\varphi \tag{27}$$

$$\nabla\psi\psi^* = \nabla\varphi = 2\psi\nabla\psi = 2m\rho_{0_{Dirac}} \tag{28}$$

The field φ is real and non-negative and represents a probability density distribution. This result defines two new continuity equations. $\rho_{0_{Dirac}}$ has a Minkowski signature.

The interpretation of φ as the probability density distribution of **presence** leads to:

$$\int_V \varphi \, dV = 1 \tag{29}$$

$$\int_V \nabla\rho_{Dirac} \, dV = 2m \tag{30}$$

The field ψ has an intrinsic spin[8]:

$$spin = \int_V \boldsymbol{\psi} \times \nabla\psi_0 \, dV \tag{31}$$

The Majorana equation

The Majorana equation[9] differs from the Dirac equation in the way that the handedness of the field ψ is changed. In the Dirac equation the mass term contains the conjugate ψ^* of the field ψ. Here all three imaginary base vectors change sign.

$$\nabla\psi = m\psi^* \tag{1}$$

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[7] See last paragraph

[8] http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf Section: Conservation of angular momentum, formula 4.70a

[9] http://en.wikipedia.org/wiki/Majorana_equation

In the Majorana equation the mass term contains the $\psi^{\otimes}$ flavor of the field ψ. In this case only one imaginary base vector changes its sign.

$$\nabla\psi = m\psi^{\otimes} \tag{2}$$

Neutrinos are supposed to obey the Majorana equation.

When the Majorana equation holds, then

$$\nabla\rho_{Dirac} = 2\psi\nabla\psi = 2m\psi\psi^{\otimes} \neq 2m\psi\psi^{*} \tag{3}$$

Due to

$$\int_V \psi\psi^{\otimes}\, dV \neq 1 \tag{4}$$

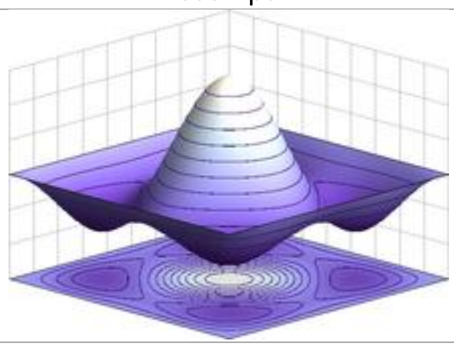
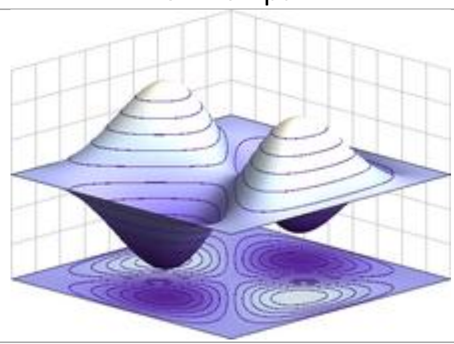
the computation of mass becomes complicated

Forbidden region

Fermions have asymmetric permutation wave functions. This fact has only significance when two or more states are considered. Let us consider the situation that the *two states are completely identical*[10] and are *nearly at the same location*. In that case the superposition of the two states is given by:

$$|\psi> = |n_1> |n_2> \pm |n_2> |n_1>$$

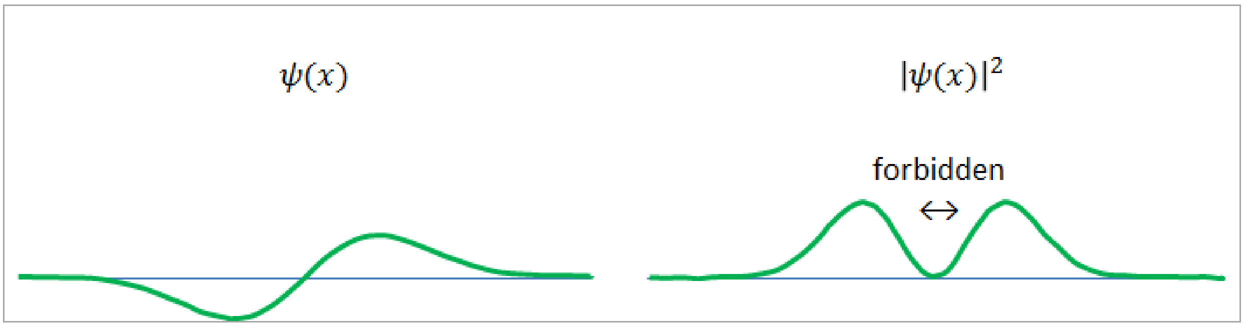
The plus sign holds for bosons and the minus sign holds for fermions. The images of the two cases are:

Boson pair	Fermion pair
	
Symmetric distribution	Asymmetric distribution

This is a two dimensional model, but it explains the general idea. Below the cut through the center of the asymmetric distribution is shown. When this is compared with the same cut of

[10] http://en.wikipedia.org/wiki/Identical_particles

the squared modulus, then it reveals a forbidden region for the asymmetric distribution. Bosons do not feature such a forbidden region.



The particles were put at the closest possible position. When fermions go to their next position, they must step over the forbidden region. Bosons do not have that restriction.

Covariant derivative

The covariant derivative D is defined as

$$Df(q) = \nabla f(q) - \boldsymbol{A}(q)\, f(q) \tag{1}$$

This is interesting with respect to a gauge transformation of the form

$$f'(q) = G(q)\, f(q) \tag{2}$$

$$G^*(q)\ G(q) = 1 \tag{3}$$

$$\nabla G(q) = \boldsymbol{H}(q) G(q) \tag{4}$$

where with a corresponding vector potential transformation

$$\boldsymbol{A}'(q) = \ \boldsymbol{A}(q) + \boldsymbol{H}(q) \tag{5}$$

$$D' = \nabla - \boldsymbol{A}(q) - \boldsymbol{H}(q) \tag{6}$$

$$D'f'(q) = \boldsymbol{H}(q)\, G(q)\, f(q) + G(q)\, \nabla f(q) \tag{7}$$

$$-\boldsymbol{A}(q) G(q) f(q) - \boldsymbol{H}(q) G(q) f(q)$$

$$= G(q)\big(\nabla f(q) - \boldsymbol{A}(q)\, f(q)\big)$$

$$D'f'(q) = G(q) D f(q) \tag{8}$$

Thus with that transformation pair not only the modulus of the function stays invariant but also the modulus of the covariant derivative stays invariant. Further

$$f'^*(q) D' f'(q) = f^*(q)\, G^*(q)\ G(q)\, D f(q) \tag{9}$$

$$= f^*(q)\, Df(q)$$

This plays a role in the Lagrangian.

References

The contents of this paper is taken from part two of the Hilbert book model:
http://www.crypts-of-physics.eu/OntheoriginofdynamicsBoek2.pdf