A pedagogical derivation of the Navier-Stokes equation

Armando V.D.B. Assis
Departamento de Física,
Universidade Federal de Santa Catarina,
88040-900 Florianópolis, SC, Brazil

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This brief paper is part of my research on the origins of turbulence. Since the derivations of the Navier-Stokes equation are frequently cumbersome, I would like to provide this pedagogical derivation (I hope), discussing the properties of the continuum fluids under a heuristical approach.

FROM THE NEWTON SECOND LAW

Consider a *continuum* fluid such that its very small parts are large enough to contain a large number of atomic and/or molecular components, within a tridimensional spatial region in which the Newton first law holds. Any velocity, say $\vec{v}$, will satisfy $|\vec{v}\,| << c$, where c is the speed of light in the empty space.

Newton's second law applied to a small and closed [1] subsystem σ of our continuum fluid, with boundary $\partial\,(\delta V)$ and volume δV gives:

$$\dot{\vec{v}}_{cm(\sigma)} \int_{\delta V} \rho(\vec{r},t)dV = \int_{\delta V} \rho(\vec{r},t)\dot{\vec{v}}_{cm(\sigma)}\, dV = \int_{\sigma} d\vec{F}_{ext} = \int_{\delta V} \rho(\vec{r},t)\vec{f}(\vec{r},t)dV + \oint_{\partial(\delta V)} \mathbf{T}\cdot\hat{n}dS, \tag{1}$$

where [2] $\dot{\vec{v}}_{cm(\sigma)}$ is the acceleration of σ's center of mass, $\rho(\vec{r},t)$ is the fluid density scalar field; $\vec{F}_{ext}$ is the resultant of the external forces acting on σ, being these forces of two distinct kinds, viz.: a) from a source external to the fluid, providing an acceleration field $\vec{f}(\vec{r},t)$ (e.g., gravitational), and b) from the fluid external to σ but acting on $\partial\,(\delta V)$ and being given by a tensor $\mathbf{T}$.

Now, we define the thermodynamic pressure. The thermodynamic pressure is characterized by its normal effect on a given surface, on a given boundary. In fact, by definition, we consider the normal effect of a given surface within the fluid on an element of fluid colliding towards this surface at some of its points, and the element of fluid turns out to gain an amount $\delta\mathcal{N}\,\hat{n}$ of linear momentum from the surface, where $\hat{n}$ is the unitary normal vector exterior to the surface at the colliding point, with $\delta\mathcal{N} > 0$. From the Newton third law, the surface gains $-\delta\mathcal{N}\,\hat{n}$ of linear momentum from the colliding fluid element during δt, the collision time interval. Any extra effect on the boundary, normal and/or tangent to the boundary surface, related to the flux of linear momentum through the boundary, that generates deformation, is not related to the thermodynamic pressure by definition [3]. The thermodynamic pressure turns out to be the remaining effect due to the fluid when the fluid is ideal (without deformation effects in spite of a choosen coordinate system) and when the fluid is static, that generates flux of linear momentum through the boundary of control volumes within the fluid. Thus, the thermodynamic pressure tensor, the pressure tensor for short, must be diagonal, since the elements outside the diagonal encapsulate tangent effects (remember the components outside the diagonal have different indices). One must have this normal effect due to the pressure in spite of a particular choosen surface within the fluid. This implies the transformed pressure tensor with components $p_{ij}^{'}$, viz.:

$$p_{ij}^{'} = \sum_{k}\sum_{l} a_{ik}a_{jl}p_{kl}, \tag{2}$$

with $(k,l) \in \{1,2,3\} \times \{1,2,3\}$, must remain diagonal $\forall$ tridimensional coordinate transformations at an arbitrary point within the fluid, where the transformation elements a_{ij} are given by the dot product between the orthonormal basis vectors, $\hat{e}_i^{'} \cdot \hat{e}_j$, where the basis $B^{'} = \{\hat{e}_k^{'}\}$, with $k \in \{1,2,3\}$, is the basis for the transformed second rank tridimensional cartesian pressure tensor $\mathbf{P}^{'} = \hat{e}_i^{'} \cdot \mathbf{P}^{'} \cdot \hat{e}_j^{'}$ at an arbitrary point, and $B = \{\hat{e}_k\}$, with $k \in \{1,2,3\}$, is the basis for the original second rank tridimensional cartesian pressure tensor $\mathbf{P} = \hat{e}_i \cdot \mathbf{P} \cdot \hat{e}_j$ at the same arbitrary point. Hence, with $\mathbf{P} = (p_{kl})$ diagonal, viz., $p_{kl} \equiv p_{kl}\delta_{kl}$, where δ_{kl} is the Kronecker delta ($\delta_{kl} = 1$ if $k = l$, or $\delta_{kl} = 0$ if $k \neq l$), the eq. (2) reads:

$$p'_{ij} = \sum_k \sum_l a_{ik} a_{jl} p_{kl} \delta_{kl} = \sum_k a_{ik} a_{jk} p_{kk} = \sum_k a_{ik} a_{kj} p_{kk} = \sum_k \hat{e}'_i \cdot \hat{e}_k \hat{e}_k \cdot \hat{e}'_j p_{kk} = \hat{e}'_i \cdot \left(\sum_k p_{kk} \hat{e}_k \hat{e}_k \right) \cdot \hat{e}'_j, \tag{3}$$

since the transformation element $a_{jk} = \hat{e}'_j \cdot \hat{e}_k = \hat{e}_k \cdot \hat{e}'_j = a_{kj}$ is symmetric in virtue of the commutativity of the dot product between vectors. The identity tensor $\mathbf{1}$ may be written in terms of the canonical dyadic tensor, i.e., $\mathbf{1} = \sum_k \hat{e}_k \hat{e}_k$. Also, in virtue of the adopted orthonormality for the basis B (the same for B'), we have got $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$. The tensor between brackets in the right-hand side of the eq. (3) is diagonal, but the scalar result emerging, namely p'_{ij}, will be the components of a diagonal tensor only if the components of the tensor within the brackets in the right-hand side of the eq. (3) do not depend on k, viz., $p_{kk} = -p$. The minus sign is adopted since the linear momentum transfer on an adopted control boundary within the fluid, in virtue of the fluid external to the boundary, is opposite to the local exterior unitary normal vector at the collision point on this boundary, as previously discussed (the boundary surface gains $-\delta\mathcal{N}\,\hat{n}$ of linear momentum from the colliding fluid element within δt, the entire collision time interval). Hence, under this physical constraint provided by the normal effect due to the pressure in spite of a control boundary orientation within the fluid, the eq. (3) turns out to give:

$$p'_{ij} = \hat{e}'_i \cdot \left(\sum_k p_{kk} \hat{e}_k \hat{e}_k \right) \cdot \hat{e}'_j = (-p)\, \hat{e}'_i \cdot \left(\sum_k \hat{e}_k \hat{e}_k \right) \cdot \hat{e}'_j = (-p)\, \hat{e}'_i \cdot \mathbf{1} \cdot \hat{e}'_j = (-p)\, \hat{e}'_i \cdot \hat{e}'_j = -p\delta_{ij}. \tag{4}$$

One says, in virtue of the eq. (4), the pressure is a scalar field, invariant under transformation of coordinates, a point property. We showed the orientation of the basis locally used to write the pressure tensor is irrelevant. This latter result means the pressure is locally isotropic.

We will write the tensor $\mathbf{T}$ in the eq. (1) in terms of two distinct tensors that generates two distinct kinds of effects on a piece of fluid: a) normal effects that do not distort the infinitesimal elements of fluid dV (see [3]), provided by the locally isotropic pressure tensor $\mathbf{P}$ whose elements are given by the eq. (4); b) *normal* effects that distort the infinitesimal elements of fluid dV, called *strain*; and *tangent* effects that distort the infinitesimal elements of fluid dV, called *shear*. Strain, as a normal effect, seems to require a diagonal tensor, since, as pointed out before in our previous discussion regarding pressure, the elements outside the diagonal encapsulates tangent effects. We will see this is the case for strain. Furthermore, it is important and instructive to point out that the strain deformation is related to each layer of fluid such that it does not require a relative motion between a pair of layers in contact to be defined. The pure strain is a measure of the proximity between points within a same layer of fluid. The strain is normal to a given layer of fluid, but its effect is to generate a deformation in a perpendicular layer such that points within this perpendicular layer turn out to increase their mutual proximities. Imagining an infinitesimal die of fluid with rectangular faces $\mathcal{F}_1$ to $\mathcal{F}_6$, being $\mathcal{F}_1$ the opposite to $\mathcal{F}_6$ face, the normal strain on $\mathcal{F}_1$ and the normal strain on $\mathcal{F}_6$, e.g., would augment the proximities of the points pertaining to the faces $\mathcal{F}_2$ to $\mathcal{F}_5$. The entire effect of strain on our infinitesimal die of fluid follows this reasoning for the two remaining pairs of opposite faces. Differently, the shear requires a pair of layers under tangent contact and relative motion. It follows that for our infinitesimal die of fluid the shear on this die is due to the fluid externally surrounding this die and its respective tangent to each die face relative motions. The shear is analogous, in relation to its dissipative effect, to the friction between solid surfaces, but with a fundamental difference: there is not static shear, as the static friction coefficient in solid mechanics, viz.: for fluids, once one has not got relative motion between fluid layers, one has not got shear effects. The shear tensor has as its distortion effect the inclination between adjacent faces. Given this heuristical remarks, we write the tensor $\mathbf{T}$:

$$\mathbf{T} = \mathbf{P} + \mathbf{\Gamma} = -p\mathbf{1} + \mathbf{\Gamma}, \tag{5}$$

where $\mathbf{\Gamma}$ is the tensor that encapsulates the effects of strain and shear.

Physically, the distortion effects related to strain and shear are related to relative motions, viz.: for strain, the relative motion between adjacent points contained in a same layer of fluid; for shear, the relative motion between adjacent points contained in adjacent layers, these adjacent layers under mutual tangent contact and relative motion. Infinitesimally, we model these relative motion effects as being given by first order effects, viz., we turn out to be interested in the first order variation of the velocity of the points contained in a same layer of fluid, for strain, and in the first order variation of the velocity of the points cointained in parallelly adjacent layers of fluid under mutual contact and relative movement, for shear. Hence, given a small piece of layer within a fluid, let $\hat{e}_i$ be the unitary vector normal to this small piece of layer, and let $\hat{e}_j$ be a unitary vector tangent to this small piece of layer; with one more, analogously to $\hat{e}_j$, unitary vector tangent to this small piece of layer, $\hat{e}_k$, perpendicular to $\hat{e}_j$ and dextrogyre, one has a local basis for a small piece of layer within the fluid. E.g., for strain, in the $\hat{e}_j$ direction, we are interested in how quickly two points, say A and B, contained in the same piece of fluid layer, these points separated by a small amount of displacement δx_j in the $\hat{e}_j$ direction, become mutually distant (or mutually close) in this direction. The component velocity of the point A in the $\hat{e}_j$ direction is, say, $v_j\,(A)$. The component velocity of the point B in the $\hat{e}_j$ direction is, say, $v_j\,(B)$. The relative velocity between A and B in the $\hat{e}_j$ direction measures how quickly A and B become mutually distant. But to measure the deformation, one should compare how this mutual instantaneous separation rate in $\hat{e}_j$ direction increases (or decreases) in relation to the mutual separation δx_j in this direction. Related to the strain in the $\hat{e}_j$ direction, we are interested in the quantity:

$$\frac{1}{\delta x_j}\left[v_j\,(B)-v_j\,(A)\right]=\frac{1}{\delta x_j}\left[\left(v_j\,(A)+\frac{\partial v_j\,(A)}{\partial x_j}\delta x_j\right)-v_j\,(A)\right]=\frac{\partial v_j\,(A)}{\partial x_j}=\left.\frac{\partial v_j}{\partial x_j}\right|_A. \tag{6}$$

Now, e.g., for shear, in the $\hat{e}_j$ direction, we are interested in how quickly two points, say A and C, contained in two parallelly adjacent pieces of fluid layer under relative tangent motion, these points separated by a small amount of normal displacement δx_i in the $\hat{e}_i$ direction, become mutually distant (or mutually close) in the $\hat{e}_j$ direction. The component velocity of the point A in the $\hat{e}_j$ direction is, as before, $v_j\,(A)$. The component velocity of the point C in the $\hat{e}_j$ direction is, say, $v_j\,(C)$. The relative velocity between A and C in the $\hat{e}_j$ direction measures how quickly A and C become mutually distant. But to measure the deformation, one should compare how this mutual instantaneous separation rate in $\hat{e}_j$ direction increases (or decreases) in relation to the perpendicularly mutual separation δx_i in the $\hat{e}_i$ direction. Related to the shear in the $\hat{e}_j$ direction, we are interested in the quantity:

$$\frac{1}{\delta x_i}\left[v_j\,(C)-v_j\,(A)\right]=\frac{1}{\delta x_i}\left[\left(v_j\,(A)+\frac{\partial v_j\,(A)}{\partial x_i}\delta x_i\right)-v_j\,(A)\right]=\frac{\partial v_j\,(A)}{\partial x_i}=\left.\frac{\partial v_j}{\partial x_i}\right|_A. \tag{7}$$

One refers to the shear as the angular deformation, since, for a small time interval δt, the relative tangent displacement between A and C, between their mutually tangent layers:

$$\left[v_j\,(C)-v_j\,(A)\right]\delta t=\frac{\partial v_j\,(A)}{\partial x_i}\delta x_i\delta t, \tag{8}$$

divided by the height δx_i between them:

$$\frac{1}{\delta x_i}\left[v_j\,(C)-v_j\,(A)\right]\delta t=\frac{\partial v_j\,(A)}{\partial x_i}\delta t \tag{9}$$

gives the infinitesimally small angular distrortion, say $\tan(\delta\theta)$, of the face initially paralell to $\hat{e}_i$ and $\hat{e}_k$ at A:

$$\tan(\delta\theta)=\delta\theta=\frac{\partial v_j\,(A)}{\partial x_i}\delta t, \tag{10}$$

such that the shear time rate $\delta\theta/\delta t$:

$$\frac{\delta\theta}{\delta t}=\frac{\partial v_j\,(A)}{\partial x_i}=\left.\frac{\partial v_j}{\partial x_i}\right|_A, \tag{11}$$

provides this angular distortion per unit of time at A, exactly the eq. (7). Similar reasoning gives the strain and shear effects related to all the tridimensional directions, from which one turns out to realize that the strain and shear first order effects, simply the strain and shear effects, since we are neglecting higher order terms in our model, are all encapsulated in the *gradient velocity dyad* tensor:

$$\mathbf{\Lambda}\equiv\vec{\nabla}\vec{v}\Rightarrow\Lambda_{ij}=\left(\vec{\nabla}\vec{v}\right)_{ij}=\frac{\partial v_j}{\partial x_i}. \tag{12}$$

Now we investigate the simmetry of the tensor $\mathbf{\Lambda}$. Since $\mathbf{\Lambda}$ is a tridimensional rank two tensor, we decompose this tensor in its symmetric and skew-symmetric parts, $\mathbf{\Lambda}^{+}$ and $\mathbf{\Lambda}^{-}$, respectively:

$$\mathbf{\Lambda}^{+}=\frac{1}{2}\left(\mathbf{\Lambda}+\mathbf{\Lambda}^{t}\right),\quad \mathbf{\Lambda}^{-}=\frac{1}{2}\left(\mathbf{\Lambda}-\mathbf{\Lambda}^{t}\right),\qquad (13)$$

where $\mathbf{\Lambda}^{t}$ is the transposed tensor of $\mathbf{\Lambda}$, with elements $\Lambda_{ij}^{t}=\Lambda_{ji}$, and:

$$\mathbf{\Lambda}=\mathbf{\Lambda}^{+}+\mathbf{\Lambda}^{-}.\qquad (14)$$

Since $\Lambda_{ij}^{-}=-\Lambda_{ji}^{-}$, for the components of $\mathbf{\Lambda}^{-}$, in virtue of its skew-symmetry, $\Lambda_{ii}^{-}=0$, and $\mathbf{\Lambda}^{-}$ has got three degrees of freedom. Hence, the deformation effect of the skew-symmetric part of $\mathbf{\Lambda}$, $\mathbf{\Lambda}^{-}$, on two elements of fluid [4], say at the points E_1 and E_2, these points separated by a mutual small displacement vector $\vec{s}$ from E_1 to E_2, may be written as:

$$\left(\mathbf{\Lambda}^{-}\right)\cdot\vec{s}=\vec{\omega}\times\vec{s},\qquad (15)$$

[5] since the right-hand of eq. (15) equivalently carries the three degrees of freedom of $\mathbf{\Lambda}^{-}$ within $\vec{\omega}$, where $\vec{\omega}$ is a vector to be determined. Physically, eq. (15) translates the meaning of a deformation effect due to $\mathbf{\Lambda}^{-}$ that generates rotation of E_2 about E_1 with a rotation axis parallell to $\vec{\omega}$ passing through E_1. From eqs. (12) and (13), with $\vec{s}=\sum_j s_j\hat{e}_j$ and $\vec{\omega}=\sum_k \omega_k\hat{e}_k$, eq. (15) reads:

$$\sum_i\left[\left(\mathbf{\Lambda}^{-}\right)\cdot\vec{s}\right]_i\hat{e}_i=\sum_i\sum_j\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial v_i}{\partial x_j}\right)s_j\hat{e}_i=\left(\sum_k\omega_k\hat{e}_k\right)\times\left(\sum_j s_j\hat{e}_j\right)=\sum_k\sum_j\omega_k s_j\left(\hat{e}_k\times\hat{e}_j\right).\qquad (16)$$

Using the Levi-Civita alternating symbol, $\epsilon_{ijk}=1$ for ijk-indexes' even permutations [of $(1,2,3)$], $\epsilon_{ijk}=-1$ for odd permutations or $\epsilon_{ijk}=0$ for any repeated index, one easily reaches $\hat{e}_k\times\hat{e}_j=\sum_i\epsilon_{kji}\hat{e}_i$, rewrites the eq. (16), and also reaches:

$$\sum_i\sum_j\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial v_i}{\partial x_j}\right)s_j\hat{e}_i=\sum_i\sum_j\left(\sum_k\epsilon_{kji}\omega_k\right)s_j\hat{e}_i\Rightarrow\sum_i\left\{\sum_j\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial v_i}{\partial x_j}\right)-\left(\sum_k\epsilon_{kji}\omega_k\right)\right]s_j\right\}\hat{e}_i=\vec{0}.\qquad (17)$$

Since the basis vector $\hat{e}_i$ are linearly independent, $\forall\ i\in\{1,2,3\}$, we necessarily have got:

$$\sum_j\left[\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial v_i}{\partial x_j}\right)-\left(\sum_k\epsilon_{kji}\omega_k\right)\right]s_j=0.\qquad (18)$$

Also, since $\vec{s}$ is arbitrary, the equality in eq. (18) will hold in any case *iff*:

$$\sum_k\epsilon_{kji}\omega_k=\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial v_i}{\partial x_j}\right).\qquad (19)$$

For $i=j$, the eq. (19) is trivial, i.e., $0=0$ in spite of any $\vec{w}$. This means that the diagonal elements of $\mathbf{\Lambda}^{-}$ are not relevant for $\vec{\omega}$ and the vice versa. Hence, we are interested in $i\neq j$. The summation in the left-hand side of the eq. (19) gives for any fixed pair $(j,i)\in[\{1,2,3\}\times\{1,2,3\}-\{(1,1),(2,2),(3,3)\}]$:

$$\sum_k\epsilon_{kji}\omega_k=\epsilon_{iji}\omega_i+\epsilon_{jji}\omega_j+\epsilon_{kji}\omega_k=\epsilon_{kji}\omega_k=\pm\omega_k^{k\neq j\neq i},\qquad (20)$$

where the $(+)$ sign remains if kji is an even permutation [of $(1,2,3)$], and the $(-)$ sign remains if kji is an odd permutation. But, in a case of odd permutation, the minus sign reverts the subtraction within the brackets in the right-hand side of the eq. (19), and the situation remains the same as in the respective case of even permutation. Hence, the eq. (19) simply gives:

$$\omega_k\left(k\neq j\neq i\text{ even}\right)=\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i}-\frac{\partial v_i}{\partial x_j}\right).\qquad (21)$$

From eq. (21), we easily conclude that:

$$\vec{\omega}=\frac{1}{2}\vec{\nabla}\times\vec{v}.\qquad (22)$$

The eq. (22) is important to verify that the skew-symmetric part, $\mathbf{\Lambda}^-$, of the gradient velocity dyad tensor, $\mathbf{\Lambda}$, has not dynamical meaning within the fluid mechanics. In fact, consider a rigid rotation of a portion of fluid, e.g., a cylindrical recipient filled with a viscous continuum fluid and rotating with constant angular velocity $\dot{\theta}$ about its simmetry axis in a region with gravitational acceleration field $\vec{g} = -g\hat{u}$, where $g \approx 9.8\, ms^{-2}$, being $\hat{u}$ the unitary vector along the simmetry axis of the cylindrical recipient. Once the stationary regime is reached, any pair of points pertaining to the fluid will preserve the mutual distance between its points. Thus, there will not exist distortion, hence, no viscous effects will be present. But, in this case, a point with cylindrical coordinates (r, θ, z) will have instantaneous velocity $\vec{v} = r\dot{\theta}\hat{e}_\theta$, giving, in virtue of the eq. (22):

$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{v} = \frac{1}{2r}\hat{e}_z \frac{\partial}{\partial r}\left(r^2\dot{\theta}\right) = \dot{\theta}\hat{e}_z. \tag{23}$$

From which, since $\vec{s} = r\hat{e}_r$ in the eq. (15), we have got for that equation:

$$\left(\mathbf{\Lambda}^-\right) \cdot \vec{s} = r\dot{\theta}\left(\hat{e}_z \times \hat{e}_r\right) = r\dot{\theta}\hat{e}_\theta = \vec{v} \neq \vec{0}. \tag{24}$$

The eq. (24) states the distortion effects, viscous ones, due to $\mathbf{\Lambda}^-$ generate the rotation velocity $\vec{v}$ of a pointlike fluid element E_2 about E_1 (here, E_1 pertains to the cylinder axis of symmetry) [see the reasoning leading to the eq. (15) [6]] or, equivalently, that the rotation velocity $\vec{v}$ of a pointlike fluid element E_2 about E_1 is related to distortion effects, viscous ones, due to $\mathbf{\Lambda}^-$. But, since this rotation is rigid, as discussed before, distortion effects are absent, leading to the conclusion that $\mathbf{\Lambda}^-$ must be neglected from the distortion reasoning, from the viscous effects, viz., distortion effects must be encapsulated solely within the symmetric part of the gradient velocity dyad tensor, since the hypothesis that there will be distortion effects, with physical sense in any situation, related to the skew-symmetric part of $\mathbf{\Lambda}$ leads to a contradiction in the counter-example of rigid rotation.

With the skew-symmetric part of the gradient velocity dyad tensor neglected:

$$\mathbf{\Lambda} - \mathbf{\Lambda}^- = \mathbf{\Lambda}^+ \overset{\text{New}}{\equiv} \mathbf{\Lambda},\ \Lambda_{ij} = \Lambda^+_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right), \tag{25}$$

in virtue of the eqs. (12), (13) and (14). Since the distortion effects are related to strain (normal) and shear (tangent) on a *same* local piece of fluid layer, we may decompose the gradient velocity dyad tensor in two independent tensors: $\mathbf{\Lambda}_n$, related to the normal effects (strain) on a local piece of fluid layer, and $\mathbf{\Lambda}_t$, related to the tangent effects (shear) on this same local piece of fluid layer. Hence, we write down:

$$\mathbf{\Lambda} = \mathbf{\Lambda}^+ = \mathbf{\Lambda}_n + \mathbf{\Lambda}_t. \tag{26}$$

Now, consider a small die of fluid, an infinitesimally small part of the continuum fluid. Consider a distortion of this small die during the infinitesimally small elapsed time $0 < t < \delta t \to 0$ necessary to accomplish this distortion. This distortion is due to strain and shear. Considering each effect independently: a) one verifies, in virtue of the eq. (11), that the *pure shear* effects (related, only, to $\mathbf{\Lambda}_t$) turn out to imply that the diagonal ii-elements of $\mathbf{\Lambda}_t$ may necessarily not contain only vanishing terms of the form $\partial v_i/\partial x_i = 0$, since, within an entire process of small distortion, the tangential effect of shear on a small face along a direction $\hat{e}_j$ parallel to this, say, ij-face of the infinitesimal die, due to the orthogonal velocity variation $\partial v_j/\partial x_i$ [see eq. (7)] along $\hat{e}_i$, the exterior to the ij-face normal unitary vector, turns out to be inclined in relation to the original ij-face at $t = 0$, viz., the pure shear distortion may depend on $\partial v_i/\partial x_i \neq 0$; b) differently, *pure strain* effects (related, only, to $\mathbf{\Lambda}_n$) remains normal during the entire infinitesimal distortion, since there is not relative rotation between elementary die faces due to pure strain distortions, viz., the pure strain effects are eminently normal, being these effects in analogy with the thermodynamic pressure effects.

Since the effects of strain are purely normal, as discussed above, the tensor $\mathbf{\Lambda}_n$ cannot have elements outside its diagonal. Following exactly the same reasoning that led from the eq. (2) to the eq. (4) (there, also related to conservation of the diagonal form of the thermodynamic pressure tensor in spite of the local basis used to represent the pressure tensor, in virtue of the normal property of the thermodynamic pressure), since the situation for $\mathbf{\Lambda}_n$ is exactly the same in relation to the conservation of its diagonal form despite of the local basis used to represent it, we necessarily have got for the $\mathbf{\Lambda}_n$ elements:

$$\left(\mathbf{\Lambda}_n\right)_{ij} = \lambda_n \delta_{ij}, \tag{27}$$

where λ_n is the same common element for all the diagonal elements of $\mathbf{\Lambda}_n$. There is the very well known result from the theory of symmetric tensors that establishes: there will exist an orthogonal basis, called principal basis, defining three axes, the principal axes, such that a symmetric tensor becomes diagonal, viz., a symmetric tensor represented in the principal basis is diagonal. The principal axes related to a symmetric tensor $\mathbf{\Lambda}$ are obtained by the solution of the *secular equation* from:

$$\left(\mathbf{\Lambda}^+ - \lambda^+\mathbf{1}\right) \cdot \vec{a}_+ = \vec{0}, \tag{28}$$

where $\vec{a}_+$ is an eigenvector of the new gradient velocity dyad tensor $\mathbf{\Lambda}$ [$= \mathbf{\Lambda}^+$, in our case, see eq. (25)], with λ^+ being the respective eigenvalue. Since the eq. (28) is homogeneous, with the trivial solution $\vec{a}_+ = \vec{0}$, the indeterminacy of the system of linear and homogeneous equations obtained from the eq. (28) requires:

$$\mathbf{det}\left(\mathbf{\Lambda}^+ - \lambda^+\mathbf{1}\right) = 0, \tag{29}$$

the secular equation that gives the eigenvalues λ^+ of $\mathbf{\Lambda} = \mathbf{\Lambda}^+$ related to their respective eigenvectors $\vec{a}_+$. Physically, both the tensors, $\mathbf{P}$ and $\mathbf{\Lambda}_n$, must be 3-fold degenerated, since any axis may be a principal axis, since these tensors are invariant ones, implying that these tensors have got, each, identical eigenvalues. From eq. (27), we easily verify the 3-fold degenerated eigenvalue of the normal effects related to pure strain within the gradient velocity dyad tensor is λ_n. Furthermore, since $\mathbf{\Lambda} = \mathbf{\Lambda}^+$ turns out to be diagonal in its principal basis $\{\vec{a}_+\}$, and since $\mathbf{\Lambda}_n$ is also diagonal despite of a basis [see the eq. (27)], the tensor $\mathbf{\Lambda}_t$ related to pure shear effects within the gradient velocity dyad tensor, given by the eq. (26):

$$\mathbf{\Lambda}_t = \mathbf{\Lambda} - \mathbf{\Lambda}_n = \mathbf{\Lambda}^+ - \mathbf{\Lambda}_n, \tag{30}$$

turns out to be diagonal in the $\mathbf{\Lambda}$'s (=$\mathbf{\Lambda}^+$'s) principal basis $\{\vec{a}_+\}$. Also, since the trace of any tensor is invariant despite of a basis used to represent it, viz., in fact, for the trace of, say, $\mathbf{\Lambda}_t$, $\mathbf{Tr}\left(\mathbf{\Lambda}_t\right)$:

$$\mathbf{Tr}\left(\mathbf{\Lambda}_t\right) = \sum_i \left(\mathbf{\Lambda}_t\right)_{ii} = \sum_i \left(\sum_j \sum_l a^{'}_{ij} a^{'}_{il} \left(\mathbf{\Lambda}_t\right)^{'}_{jl}\right) = \sum_j \sum_l \left(\sum_i a^{'}_{ij} a^{'}_{il}\right) \left(\mathbf{\Lambda}_t\right)^{'}_{jl} = \sum_j \sum_l \left(\sum_i \hat{e}_i \cdot \hat{e}^{'}_j \hat{e}_i \cdot \hat{e}^{'}_l\right) \left(\mathbf{\Lambda}_t\right)^{'}_{jl} =$$

$$= \sum_j \sum_l \left(\sum_i \hat{e}^{'}_j \cdot \hat{e}_i \hat{e}_i \cdot \hat{e}^{'}_l\right) \left(\mathbf{\Lambda}_t\right)^{'}_{jl} = \sum_j \sum_l \left[\hat{e}^{'}_j \cdot \left(\sum_i \hat{e}_i \hat{e}_i\right) \cdot \hat{e}^{'}_l\right] \left(\mathbf{\Lambda}_t\right)^{'}_{jl} = \sum_j \sum_l \left[\hat{e}^{'}_j \cdot \mathbf{1} \cdot \hat{e}^{'}_l\right] \left(\mathbf{\Lambda}_t\right)^{'}_{jl} =$$

$$= \sum_j \sum_l \left(\hat{e}^{'}_j \cdot \hat{e}^{'}_l\right) \left(\mathbf{\Lambda}_t\right)^{'}_{jl} = \sum_j \sum_l \delta_{jl} \left(\mathbf{\Lambda}_t\right)^{'}_{jl} = \sum_j \left(\mathbf{\Lambda}_t\right)^{'}_{jj},$$

one easily verifies the invariance of trace:

$$\mathbf{Tr}\left(\mathbf{\Lambda}_t\right) = \sum_i \left(\mathbf{\Lambda}_t\right)_{ii} = \sum_i \left(\mathbf{\Lambda}_t\right)^{'}_{ii}, \tag{31}$$

and we have got, from the eqs. (25), (27) and (30), that:

$$\mathbf{Tr}\left(\mathbf{\Lambda}_t\right) = \mathbf{Tr}\left(\mathbf{\Lambda}^+ - \mathbf{\Lambda}_n\right) = \sum_i \left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_i} + \frac{\partial v_i}{\partial x_i}\right) - \lambda_n \delta_{ii}\right] = \sum_i \frac{\partial v_i}{\partial x_i} - \lambda_n \sum_i \delta_{ii} = \vec{\nabla} \cdot \vec{v} - 3\lambda_n, \tag{32}$$

i.e., the pure strain distortion effects given by the components λ_n of $\mathbf{\Lambda}_n$ would depend on the pure shear distortion effects given by $\mathbf{\Lambda}_t$ via the following function:

$$\lambda_n \overset{!}{=} \frac{1}{3}\vec{\nabla} \cdot \vec{v} - \frac{1}{3}\mathbf{Tr}\left(\mathbf{\Lambda}_t\right), \tag{33}$$

in any local base. But it is a contradiction. The unique way by which pure shear effects do not influence pure strain effects in spite of local coordinates for any point within the fluid under arbitrary physical situations is, in virtue of the invariant relation given by the eq. (33), obtained by:

$$\mathbf{Tr}\left(\mathbf{\Lambda}_t\right) = 0. \tag{34}$$

Even with $\mathbf{Tr}\left(\mathbf{\Lambda}_t\right)$ = constant, pure shear effects via $\mathbf{Tr}\left(\mathbf{\Lambda}_t\right)$ would be exerting influence on pure strain effects, on the components λ_n of the tensor encapsulating pure strain effects given by the eq. (27), albeit constant, still a contradiction. But such a hypothetically constant effect, covariantly, is easily rejected when one recall the case of rigid rotation previously discussed, since, with no strain and no shear, such constant should be void, hence zero. Thus, we are led to conclude that:

$$\lambda_n = \frac{1}{3}\vec{\nabla} \cdot \vec{v}, \tag{35}$$

from which, we rewrite the eq. (27):

$$\left(\mathbf{\Lambda}_n\right)_{ij} = \frac{1}{3}\vec{\nabla} \cdot \vec{v}\,\delta_{ij} \Rightarrow \mathbf{\Lambda}_n = \frac{1}{3}\vec{\nabla} \cdot \vec{v}\,\mathbf{1}. \tag{36}$$

Now, we write the tensor that encapsulates the effects of strain and and shear, the tensor $\mathbf{\Gamma}$ in the eq. (5), as a combination of these purely independent effects:

$$\mathbf{\Gamma} = \mathbf{\Xi}_t \mathbf{\Lambda}_t + \mathbf{\Xi}_n \mathbf{\Lambda}_n, \tag{37}$$

where: $\mathbf{\Xi}_t$ is a tensor that does not depend on the velocity field, encapsulating the viscous properties of the continuum fluid purely related to the shear distortion properties of the continuum fluid; and $\mathbf{\Xi}_n$ is a tensor that does not depend on the velocity field, encapsulating the viscous properties of the continuum fluid purely related to the strain distortion properties of the continuum fluid [7]. From the eqs. (30) and (36), the eq. (37) reads:

$$\mathbf{\Gamma} = \mathbf{\Xi}_t \left[\mathbf{\Lambda}^+ - \frac{1}{3} \left(\vec{\nabla} \cdot \vec{v} \right) \mathbf{1} \right] + \mathbf{\Xi}_n \left[\frac{1}{3} \left(\vec{\nabla} \cdot \vec{v} \right) \mathbf{1} \right], \quad (38)$$

from which, the tensor $\mathbf{T}$ in eq. (5) reads:

$$\mathbf{T} = -p\mathbf{1} + \mathbf{\Xi}_t \left[\mathbf{\Lambda}^+ - \frac{1}{3} \left(\vec{\nabla} \cdot \vec{v} \right) \mathbf{1} \right] + \mathbf{\Xi}_n \left[\frac{1}{3} \left(\vec{\nabla} \cdot \vec{v} \right) \mathbf{1} \right]. \quad (39)$$

Back to the eq. (1), with the eq. (39), the eq. (1) reads, in virtue of the Gauss theorem:

$$\int_{\delta V} \left\{ \rho(\vec{r},t)\dot{\vec{v}}_{cm(\sigma)} - \rho(\vec{r},t)\vec{f}(\vec{r},t) + \vec{\nabla} \cdot \left\{ p\mathbf{1} - \mathbf{\Xi}_t \left[\mathbf{\Lambda}^+ - \frac{1}{3} \left(\vec{\nabla} \cdot \vec{v} \right) \mathbf{1} \right] - \mathbf{\Xi}_n \left[\frac{1}{3} \left(\vec{\nabla} \cdot \vec{v} \right) \mathbf{1} \right] \right\} \right\} dV = \vec{0}. \quad (40)$$

DOES $\lim_{\delta V \to 0} \sigma = \boldsymbol{cm}(\sigma)$ NEGLECT σ'S INTERNAL DEGREES OF FREEDOM?

Firstly, do permit ourselves, the following operation on the eq. (40):

$$\lim_{\delta V \to 0} \int_{\delta V} \left\{ \rho(\vec{r},t)\dot{\vec{v}}_{cm(\sigma)} - \rho(\vec{r},t)\vec{f}(\vec{r},t) + \vec{\nabla} \cdot \left\{ p\mathbf{1} - \mathbf{\Xi}_t \left[\mathbf{\Lambda}^+ - \frac{1}{3} \left(\vec{\nabla} \cdot \vec{v} \right) \mathbf{1} \right] - \mathbf{\Xi}_n \left[\frac{1}{3} \left(\vec{\nabla} \cdot \vec{v} \right) \mathbf{1} \right] \right\} \right\} dV = \vec{0}. \quad (41)$$

What does the eq. (41) mean? It means the content between brackets, the integrand, is sufficiently continuous, as well as the the volume of integration, δV, of a closed subsystem, σ, to guarantee that the acceleration of σ's center of mass, $\dot{\vec{v}}_{cm(\sigma)}$, will be properly described by the eq. (41), even for an arbitrarily small closed system σ. Here, we begin to infer some arbitrariness:

- Under valid continuity conditions, eq. (41) describes the dynamics of the center of mass of a closed system, hence the closed system may have internal degrees of freedom, internal motion, that will not be described by eq. (41);
- The transition to the continuum does not specify the number of atoms and/or molecules within an elementary element of fluid, does not specify internal degrees of freedom related to these atoms and/or molecules, a fact that will turn out to be hidden, leading to a description of the continuum via the movement of σ's center of mass.

Mechanically, there is not any restriction related to the description of a system by its center of mass. But, unfortunately, a problem arises if this system turns out to be discrete under some conditions, being the theory that describes it constructed in terms of a continuum. This theory grows upon the ground of high statistics regarding the number of particles. But, this intrinsical statistical nature behind the continuum is actually classical. The statistical results from the continuum are not the same ones one would have quantum mechanically for a vanishingly small number of particles. Effects related to a low number of particles that deviate from the expected quantum mechanical ones turn out to put a limit on the predictive capacity for a theory under the continuum hypothesis. Such theory with a continuum substratum admits one is dealing with a large number of particles even within a small scale, viz., the vanishing differentials must preserve a large number of molecules. One should take the responsability of investigation, as a physicist, concerned with the physical validity of continuum-based arguments within small systems that actually contain, for each of its small domains, few discrete elements. Later, will be concerned with situations in which the fluid turns out to present intense distortion over a large part of its assumed continuum, that may generate divergent predictions with identical initial situations, since, in quantum mechanics, two identical sets of measures at a given initial instant do not necesarily present the same set of measures subsequently; we will turn back to these issues related to neglected internal degrees of freedom. We will see there exist restrictive consequences that arise from the taken limit in the eq. (41), with these restrictive consequences leading to the origins of the turbulence. Turbulence will become incompatible , and an expression for predicting the critical Reynolds number, characterizing a fully developed turbulent flow surrounding boundaries within a fluid., arises. Here, we will deal with the continuum, exploring some heuristics of limitation throughout our analysis. We will be back to the discrete situations that become important over the large scales within domains of a material fluid, with important acquired heuristics from the continuum case.

One may easily prescribe important properties related to the viscosity tensors in the eq. (37). One may be tempted to consider the viscous effects related to a given physical situation, e.g., the rigid rotation previously discussed, as being very different from another situation, e.g., a fluid surrounding a falling sphere. There are two characteristics, properties, of viscous fluids related to the viscosity per se [note one has two tensors in each term of the right-hand side, related to viscosity, of the eq. (37)]. Putting more heuristically, under physical grounds, the viscosity has got a mathematical property that characterizes this very physical property emerging from the response the fluid gives to efforts applied by external constraints. The external efforts are recognized as being the causes of distortion on the fluid. The fluid response is a function of the efforts. Being this fluid response an intrinsical property of the fluid, it must be independent of any specific flow regime. This response would exist previously within the fluid properties despite of a subsequent application of external efforts. Since different efforts cause different flows, velocity fields and positions seem to be necessary to define viscosity, since different velocity and positions seem to be related to different distortions. In the rigid rotation case, there is not distortion, but one may have viscosity if the fluid is viscous, viz., if the fluid will present distortion under another physical situation. The descriptive approach to define viscosity in terms of distortions is void in a rigid rotation. Hence:

- In the descriptive approach, the properties that are geometrically measured prior to define viscosity via macroscopic geometrical responses, inferrred from distortions, in terms of variations, are encapsulated within the tensors $\mathbf{\Lambda}_t$ and $\mathbf{\Lambda}_n$ in the eq. (37), depending on a posteriori observations to be observed⇒inferred⇒experimentally defined;

- The intrinsical properties of the fluid, previously contained within fluid's material properties, are encapsulated within the tensors $\mathbf{\Xi}_t$ and $\mathbf{\Xi}_n$. This level of property, intrinsical, turns out to be related to the average response a continuum fluid presents in the macroscopical level from its microscopical content. This characterizes thermodynamical properties of the continuum system of particles, a macroscopical response arising from its intrinsical properties. E.g., for homogeneous and isotropic fluids, there is not a recognition, macroscopically, among differences of spatial location, and the fluid behaves exactly in the same manner as if it was located at another position in space, since, internally, the fluid has got a microscopical behaviour that homogeneously emerges on average. In this case, external constraints are not providing an average bias in any specific direction, although a large bunch of the fluid may present, the bunch as a whole, very different kinematical aspects over different regions of space. Thermodynamical properties, related to the $\mathbf{\Xi}$-tensors, turn out to make sense under our primordial assumption the fluid actually has got high statistics even for its elementary portions. Under the continuum assumption, the intrinsical properties of the fluid, thermodynamical, turn out to be sufficiently described on average, as the spirit of the thermodynamics permeates the description. If one wants to obtain these properties from the microscopical information directly, the apparata of the statistical mechanics turn out to be the necessary arsenal. In any case, viz., in any level of description of a *continuum fluid*, the high statistics is intrinsically assumed and, by construction, the connection between the properties of both the $\mathbf{\Xi}$-tensors and of both the $\mathbf{\Lambda}$-tensors in the eq. (37) is exterior. This is an ad hoc property within our construction of the theory, and these tensors turn out to be complementary ab initio.

Since physical properties are defined by experiments that assert values to these properties, one cannot separate, under an instrumentalist point of view, these two characteristics, pointed above, as independent instances of the viscosity, since they are correlated as an exterior property. They subsists correlated.

- Now, one may define the intrinsical properties of the fluid as being intrinsically homogeneous and isotropic (nature presents a variety of fluid materials that manifest in this way even under extreme conditions, e.g., liquids), being its properties purely thermodynamical, and with no dependence on positions and velocities. Under this interest, from our previous comments on homogeneity and isotropy, we conclude the $\mathbf{\Xi}$-tensors must be diagonal with 3-fold degenerated eigenvalues:

$$\mathbf{\Xi}_n = \xi_n \mathbf{1}; \quad \mathbf{\Xi}_t = \xi_t \mathbf{1}. \tag{42}$$

The eq. (41) turns out to read:

$$\lim_{\delta V\to 0}\int_{\delta V}\left\{\rho(\vec{r},t)\dot{\vec{v}}_{cm(\sigma)}-\rho(\vec{r},t)\vec{f}(\vec{r},t)+\vec{\nabla}\cdot\left\{p\mathbf{1}-\xi_t\left[\mathbf{\Lambda}^{+}-\frac{1}{3}\left(\vec{\nabla}\cdot\vec{v}\right)\mathbf{1}\right]-\xi_n\left[\frac{1}{3}\left(\vec{\nabla}\cdot\vec{v}\right)\mathbf{1}\right]\right\}\right\}dV=\vec{0}. \tag{43}$$

Since the position of δV is arbitrary, one should infer the limit in the eq. (43) remains valid at each local domain of the fluid. Under the continuum hypothesis σ shrinks into a fluid material point, $\delta V\to 0$, and $\vec{r}_{cm(\sigma)}\to\vec{r}$. If σ is sufficiently small and the fields as well as the fluid is a well behaved continuum one may state the integrand within the eq. (43) remains constant over σ. Physically, this would imply local dynamics of the fluid would not depend on the way one is partitioning the fluid to describe its local properties. This set of well behaved characteristics would justify the Newton second law holds for each small closed system σ, eq. (43), in virtue of a stronger sufficient assumption:

$$\rho(\vec{r},t)\dot{\vec{v}}-\rho(\vec{r},t)\vec{f}(\vec{r},t)+\vec{\nabla}\cdot\left\{p\mathbf{1}-\xi_t\left[\mathbf{\Lambda}^{+}-\frac{1}{3}\left(\vec{\nabla}\cdot\vec{v}\right)\mathbf{1}\right]-\xi_n\left[\frac{1}{3}\left(\vec{\nabla}\cdot\vec{v}\right)\mathbf{1}\right]\right\}=\vec{0}. \tag{44}$$

Stronger, since one is stating the Newton second law stated via the eq. (43) will hold once the fluid is well behaved enough to provide its averaged bahaviour over its small scales exactly coincides with its actual behaviour. Its is an intrinsically classical description of a fluid. This understanding of nature fails within the quantum mechanical domain. Within quantum mechanics, we know the classical kinematics may follow in its form, when the classical description is an averaged description via the Ehrenfest theorem. We will describe cases (not here) in which the continuum hypothesis lacks. The eq. (44) looses its objective meaning, quantum mechanically, to decribe fluids, in the sense the classically obtained statistical results emerging from measures on quantum mechanical systems does not describe an unique objective reality of the fluid particles before an accomplished measured, since the fluid, before the measure, is a quantum object. The very same initial conditions might provide very different possible outcomes. The individuality lacks, quantum mechanically, before an accomplished measure, and the limit process we have got taken within the fluid turns out to have no objective sense. Our descriptive method lacks on pure quantum objects, and concepts of positions to define distortions do not apply to pure quantum objects. E.g., $\vec{v}(\vec{r},t)$ for quantum packets is absolute nonsense, as Heisenberg teach us.

Quantum mechanics aside, the $\vec{v}$ turns out to be the velocity of a fluid element at the position $\vec{r}$ at the instant t, hence $\vec{v}(\vec{r},t)$. Aplying the eq. (44) to the particular case of planar flow of an uniform fluid with constant density, with the velocity field of the planar layers being parallel to the x axis with normal along $\hat{y}$, only the x-component of $\vec{v}$ will not be vanished. $\vec{\nabla}\cdot\vec{v}=0$, due to the constant $\rho(\vec{r},t)$, following from the continuity equation. The continuity equation is extremely important throughout a discussion concerned with the origins of turbulence. This is so due to the interpretation one obtains when considering the meaning of the terms of the continuity equation and its relation with the origins of the classical arising from non linear conditions on an important tensorial relation that turn out to characterize, classically, non linear effects due to the Eq. (44). This relation will be discussed in details, but not here, as well as the conditions implying non linear effects. We see the only non vanishing components of $\mathbf{\Gamma}$ [eq. (38)] are $\Gamma_{12}=\Gamma_{21}=(1/2)\,\xi_t\;\partial v_x/\partial y$. Hence, the viscous force $d\vec{F}$ acting on an infinitesimal element with surface area $d\vec{S}=dS\hat{y}$ is given by:

$$d\vec{F}=\mathbf{\Gamma}\cdot d\vec{S}=\frac{1}{2}\xi_t\frac{\partial v_x}{\partial y}dS\hat{x}. \tag{45}$$

The eq. (45) is exactly the Newton law for viscosity that defines the dynamical viscosity η, taking [8] $\xi_t=2\eta$ in the Eq. (45). In virtue of the eqs. (12) and (13) for the symmetric part of a tensor, i.e., $\mathbf{\Lambda}^{+}=(1/2)\left[\nabla\vec{v}+(\nabla\vec{v})^t\right]$, and $\xi_t=2\eta$, with the eq. (42), the eq. (38) reads:

$$\mathbf{\Gamma}=\eta\left[\nabla\vec{v}+(\nabla\vec{v})^t-\frac{2}{3}\left(\vec{\nabla}\cdot\vec{v}\right)\mathbf{1}\right]+\xi_n\left(\vec{\nabla}\cdot\vec{v}\right)\mathbf{1}. \tag{46}$$

Thus, in terms of components, using the Einstein summation convention on two repeated indexes, the most general viscosity tensor $\mathbf{\Gamma}$, given by the eq. (38), providing the viscous force per unit area on the boundary surface of an enclosed volume of fluid, being this force exerted by the elements external to this enclosed volume of fluid, reads:

$$\Gamma_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda} \right) + \zeta\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda}, \qquad (47)$$

where η e $\zeta \equiv \xi_t$ are the thermodynamic viscous coefficients discussed before. Writing the eq. (44) in terms of components, one has:

$$\rho\dot{v}_i - \rho f_i + \frac{\partial p}{\partial x_i} - \frac{\partial \Gamma_{ik}}{\partial x_k} = \rho\dot{v}_i - \rho f_i + \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}\left[\eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda} \right) + \zeta\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda}\right] = 0 \therefore$$

$$\rho\dot{v}_i - \rho f_i + \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}\left[\eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_\lambda}{\partial x_\lambda} \right)\right] - \frac{\partial}{\partial x_i}\left(\zeta\frac{\partial v_\lambda}{\partial x_\lambda}\right) = 0. \qquad (48)$$

The coefficients of viscosity, η and ζ, are functions that depends on the thermodynamic pressure and on the temperature. Taking into consideration our previous comments on these coefficients, we may suppose the viscous properties that depend on the viscous kinematics are entirely encapsulated within the $\boldsymbol{\Lambda}$-tensors, $\boldsymbol{\Lambda}_n$ and $\boldsymbol{\Lambda}_t$ in the eq. (37), from which the intrinsic viscous properties of the fluid within the $\boldsymbol{\Xi}$-tensors, $\boldsymbol{\Xi}_n$ and $\boldsymbol{\Xi}_t$ in the eq. (37), hence η and ζ from the eqs. (42) and (47), do not depend on positions and velocities; and the eq. (48) is rewritten:

$$\rho\dot{v}_i - \rho f_i + \frac{\partial p}{\partial x_i} - \eta\frac{\partial^2 v_i}{\partial x_k^2} - \eta\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right) + \frac{2}{3}\eta\frac{\partial}{\partial x_i}\left(\frac{\partial v_\lambda}{\partial x_\lambda}\right) - \zeta\frac{\partial}{\partial x_i}\left(\frac{\partial v_\lambda}{\partial x_\lambda}\right) = 0.$$

This equation turns out to be rewritten due to the dummy indexes:

$$\rho\dot{v}_i - \rho f_i + \frac{\partial p}{\partial x_i} - \eta\frac{\partial^2 v_i}{\partial x_k^2} - \frac{1}{3}\eta\frac{\partial}{\partial x_i}\left(\frac{\partial v_\lambda}{\partial x_\lambda}\right) - \zeta\frac{\partial}{\partial x_i}\left(\frac{\partial v_\lambda}{\partial x_\lambda}\right) = \rho\dot{v}_i - \rho f_i + \frac{\partial p}{\partial x_i} - \eta\frac{\partial^2 v_i}{\partial x_k^2} - \left(\frac{1}{3}\eta + \zeta\right)\frac{\partial}{\partial x_i}\left(\frac{\partial v_\lambda}{\partial x_\lambda}\right) = 0.$$

Back to the vectorial form, one has got the so called Navier-Stokes equation under the conditions previously raised:

$$\rho(\vec{r},t)\dot{\vec{v}}(\vec{r},t) - \rho(\vec{r},t)\vec{f}(\vec{r},t) + \vec{\nabla}p(\vec{r},t) - \eta\vec{\nabla}^2\vec{v}(\vec{r},t) - \left(\frac{1}{3}\eta + \zeta\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}(\vec{r},t)\right) = \vec{0}. \qquad (49)$$

We will suppose valid the divergence-free condition, $\vec{\nabla}\cdot\vec{v} = 0$. A constant density field turns out to be a sufficient condition. But the converse may lack under less restictive conditions. Since this discussion requires the continuity equation to be stated consequently, and the continuity equation will provide an ansatz we will discuss (not here) regarding the origins of turbulence, its very instructive to discuss which consequences one obtains from the continuity equation related to fluids under absent divergence condition. The continuity equation follows from the hypothesis of mass conservation within an undeformable control volume V_0. The total mass of fluid exiting from an undeformable control volume V_0 per unit time reads:

$$\oint_{\partial V_0} \rho(\vec{r},t)\vec{v}(\vec{r},t)\cdot\hat{n}dS. \tag{50}$$

The decreasing time rate of fluid mass within this volume V_0 is given by:

$$\frac{\partial}{\partial t}\int_{V_0}\rho(\vec{r},t)dV. \tag{51}$$

The mass conservation within V_0 requires the sum of these two quantities vanishes:

$$\oint_{\partial V_0} \rho(\vec{r},t)\vec{v}(\vec{r},t)\cdot\hat{n}dS + \frac{\partial}{\partial t}\int_{V_0}\rho(\vec{r},t)dV = 0. \tag{52}$$

Applying the Gauss theorem, being V_0 undeformable, one reaches:

$$\int_{V_0}\left\{\frac{\partial\rho(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot[\rho(\vec{r},t)\vec{v}(\vec{r},t)]\right\}dV = 0. \tag{53}$$

Also, since V_0 is arbitrary:

$$\frac{\partial\rho(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{j} = \frac{\partial\rho(\vec{r},t)}{\partial t} + \rho(\vec{r},t)\vec{\nabla}\cdot\vec{v}(\vec{r},t) + \vec{v}(\vec{r},t)\cdot\vec{\nabla}\rho(\vec{r},t) = 0, \text{ with } \vec{j} = \rho(\vec{r},t)\vec{v}(\vec{r},t). \tag{54}$$

The eq. (54) is the continuity equation and $\vec{j}(\vec{r},t)$ is the mass flux density vector, giving the direction of the fluid movement at $(\vec{r},t)$ and, its magnitude, the quantity of mass per unit time per unit area flowing through the local unit area perpendicular to the velocity field at $(\vec{r},t)$. If $\rho(\vec{r},t)$ is constant $\Rightarrow \vec{\nabla}\cdot\vec{v}(\vec{r},t) = 0$ [9], which would justify a hypothesis for an absent divergence. Such hypothesis becomes reasonable for liquids, but not trivially for gases. A gas has as characteristic the tendency to occupy the entire volume of a recipient, a tendency to reach, locally, a homogeneous density, contrary to the liquids that, in virtue of the quasi-constant density for a liquid, have got a globally homogeneous density. Thus, for a given local gradient of density within a gaseous fluid, one expects this gas has got a tendency to displace its elements perpendicularly to the density gradient, to avoid an increasing local density. A gaseous fluid will have as much success to accomplish an homogeneous density as much this gas be unconstrained. Hence, the time rate of decreasing of local density will be proportional to $\vec{v}(\vec{r},t)\cdot\vec{\nabla}\rho(\vec{r},t)$. If such an analysis remains valid: for the particular case in which the rapidity by which the fluid enters an undeformable local control volume equals the rapidity by which the fluid exits this undeformable control volume $\Rightarrow \vec{\nabla}\cdot\vec{v}(\vec{r},t) = 0$, one would have, due to the continuity equation, eq. (54), the value -1 as being the proportionality constant. Hence, in virtue of the continuity equation, eq. (54):

$$\frac{\partial\rho(\vec{r},t)}{\partial t} + \vec{v}(\vec{r},t)\cdot\vec{\nabla}\rho(\vec{r},t) = 0 \Leftrightarrow \vec{\nabla}\cdot\vec{v}(\vec{r},t) = 0, \tag{55}$$

viz., such hypothetical characteristic for a gaseous fluid turns out to be equivalent to absent divergence, $\vec{\nabla}\cdot\vec{v}(\vec{r},t) = 0$, at points of fluid at which such characteristic remains valid. We will suppose the eq. (55) remains valid, under these conditions or combinations of conditions allowing absent divergence from the continuity equation. We reach the canonical form of the Navier-Stokes differential equation (but, if different assumptions appear necessary, one should turn back to these previous derivations to reach the most general form for the Navier-Stokes equation):

$$\rho\dot{\vec{v}} - \rho\vec{f} + \vec{\nabla}p - \eta\vec{\nabla}^2\vec{v} = \vec{0}. \tag{56}$$

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[1] In this paper, a closed system [or subsystem], will denote closed to mass exchange through this closed system [or subsystem] boundary, viz., the mass within the boundary is conserved. If necessary, a thermodynamically closed system will be called isolated.

[2] In this paper, the dot above a function will denote the total derivative in relation to the time. Particularly, $\dot{\vec{v}}_{cm(\sigma)} \equiv (d/dt)\,\vec{v}_{cm(\sigma)}$.

[3] As previously stated, we are under the assumption that a small piece of fluid contains a large amount of atomic and/or molecular components. Under an exclusive presence of thermodynamic pressure, a fluid may present compressibility, but such effect is not a deformation, since an increased pressure would turn out to augment the proximity between adjacent fluid elements, between distinct atomic and/or molecular components. This effect, under the continuum assumption, implies a closed elementary

fluid element, preserving the same internal mass dM, may decrease its elementary volume dV, augmenting the local density ρ. The element of volume dV would not distort, preserving its geometrical form (shape). Effects of deformation imply distortion of the infinitesimal elements of the continuum fluid and they are not related to the thermodynamic pressure by definition. Microscopically, effects of distortion, implying effects of viscosity, are related to molecular transport of linear momentum between different regions of the fluid [for liquids, deformation effects, viscous, will be mostly related to intermolecular cohesion, mostly related to dipolar and/or ionic electromagnetic interaction between electrical charges; but, in any case, mechanically, the interaction is due to linear momenta exchange, albeit the process of mechanical interaction is much more localized in liquids, at the molecular level, mostly occurring between neighbooring elements; for gases, in contrast, the freedom that intermolecular elements have got to transport linear momenta over different intermolecular regions, characterizes their relative long range capacity for the transport of molecular linear momenta], implying relative (mean) velocities between layers of fluid, with a typical layer moving with its averaged molecular velocity [here, no confusion should arise in relation to the root mean square molecular velocity, which is high in relation to the layer velocity; imagine a container enclosing a gas at a constant room temperature, with the container being in linear uniform motion with velocity $\vec{u}$ constant in a given inertial reference frame; the mean velocity of the gas molecules is the container velocity, since for a given molecule with velocity $\vec{w}+\vec{u}$ one finds another molecule with velocity $-\vec{w}+\vec{u}$, where $\vec{w}$ is the velocity of a gas molecule in relation to the container, under a homogeneity condition (if we observed the container in its referential, the mean velocity of the gas molecules would be $\vec{0}$, since for a given molecule with velocity $\vec{w}$ one finds another molecule with velocity $-\vec{w}$); but the typical average speed (root mean square) of the molecules in relation to the container is high ($\approx 517ms^{-1}$ for N_2 at $300K$); in this analogy, the velocity of the container is analogous to the velocity of the fluid layer], through which single molecules acquire extra linear momenta; such layers with their averaged molecular velocities turn out to define deformation due to the relative velocities between them [imagine two trains (layers) moving parallelly in the same direction with relative velocity; one throws a bag (molecule) from one train to the other; the faster train (layer) looses linear momentum and the slower gains...]. One may have relative velocities between points within a same layer of fluid. We will discuss deformations in detail, existing two kinds of such effects: *strain* and *shear*, as we will see.

[4] Two elements, since any vector connects two points.

[5] There is no restriction to $\vec{s}$ that requires it as an infinitesimal vector for purposes of validity of the eq. (15), viz., $\vec{s}$ is required to be arbitrary within general conditions used to obtain a general validity of the eq. (15), albeit we may be particularly interested in cases in which $\vec{s}$ connects two infinitesimally separeted pointlike elements of fluid. It will be clear in our subsequent derivation of $\vec{\omega}$ that $\vec{s}$ may be arbitrary.

[6] An important consequence arises here: since $\vec{s}$ is arbitrary, considering $\vec{s} \equiv d\vec{s}$, viz., for infinitesimal displacements between two points within the fluid, we infer the eq. (22) implies that *any fluid element turns out to present a self rotation* with angular velocity $\vec{\omega}$ given by the eq. (22) when $\vec{\omega} \neq \vec{0}$.

[7] The physics of the distortion effects was encapsulated, for both, strain and shear, within their tensors, $\mathbf{\Lambda}_n$ and $\mathbf{\Lambda}_t$ respectively, throughout our previous discussion. Within our previous analysis, the distortions were defined by their general physical characteristics, as the tangential characteristic of transference of linear momentum of shear relative to the mutual tangent motion between adjacent layers of fluid, being the normal characteristic of transference of linear momentum of strain relative to the variation of the mutual distance between points pertaining to a same layer of fluid etc. These properties did not invoke any specific behaviour related to these general definitions under specific material fluids. The intrinsic properties of a given material related to how this specific material behaves to accomplish the generally defined distortions related to strain and shear are encapsulated within the tensors of viscous properties: $\mathbf{\Xi}_t$, providing the coefficients of shear of a given material, and $\mathbf{\Xi}_n$, providing the coefficients of strain of this given material. E.g., water and glycerin, physically accomplish strain and shear, both respectivelly obeying the physics encapsulated within the tensors $\mathbf{\Lambda}_n$ and $\mathbf{\Lambda}_n$, but the properties that allow to distinguish between water or glycerin turn out to enter via their respective tensors: $\mathbf{\Xi}_t^{\mathrm{H_2O}}$ and $\mathbf{\Xi}_n^{\mathrm{H_2O}}$, for the water viscous properties, and $\mathbf{\Xi}_t^{\mathrm{C_3H_8O_3}}$ and $\mathbf{\Xi}_n^{\mathrm{C_3H_8O_3}}$, for the glycerin viscous properties. It is important to realize that a viscous fluid may behave as an ideal fluid, even with $\mathbf{\Xi}_t \neq 0\mathbf{1}$ and $\mathbf{\Xi}_n \neq 0\mathbf{1}$, provided $\mathbf{\Lambda}_t = \mathbf{\Lambda}_n = 0\mathbf{1}$. This is the case for viscous fluid experiencing rigid rotation. If the tensors $\mathbf{\Lambda}_t$ and/or $\mathbf{\Lambda}_n$ do not vanish, one turns out to infer the fluid is viscous, but the converse is not necessarily true. If the tensors $\mathbf{\Lambda}_t$ and $\mathbf{\Lambda}_n$ vanish, one cannot asseverate the fluid is not viscous. In other words, distortion is a sufficient condition for viscosity, but not the converse, i.e.: viscosity is not a sufficient condition for distortion, distortion is not a necessary condition for viscosity.

[8] Would not be necessary the minus since the tangent shear is contrary to the velocity gradient through planar layers? Firstly, we must define if we want to measure the force externally to dS, i.e., the force the externally adjacent layer of fluid exerts on the element of area $\hat{y}dS$, remembering $\hat{y}$ is *exterior* normal, of if we want to measure the force internally to dS, also said through dS, the force internally adjacent layer of fluid exerts on the element of area $\hat{y}dS$. In the first case, one takes the positive sign, since it is enough to remember the case of a newtonian viscous fluid between two parallel plates such that the superior plate is pushed to the right (along $\hat{x}$) with constant velocity in which an external agent exerts a force $d\vec{F}$ to the right, the surface $\hat{y}dS$ touching the superior plate gains an action from the superior plate that exactly equals $d\vec{F}$, being this one the force considered in the definition of a newtonian viscous fluid $d\vec{F} = \eta \partial v_x / \partial y dS \hat{x}$, i.e., the force an external agent exerts will increase with an increased velocity gtadient, from which the necessity of the positive sign. On a volume enclosed by a surface, one measures the force the external elements of fluid exert on the surface surrounding the volume, on the volume boundary, a situation that is analogous to the first case. In a second case, the exerted force will obviously be the reaction $-d\vec{F}$, viscously contrary to the movement of the plate. Since we will be

calculating external effects on surfaces with exterior normal, it must be understood the asoption of the positive sign within the Newton viscosity law, eq. (45).

[9] Of course, a non vanishing constant, once one would have a trivial identity in spite of the velocity field.