Integrability of Maxwell's Equations

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Abstract

This article adds to [4] some nearer explanations. It is shown again that Maxwell's equations are integrable, but I am doing it without quaternions, which is simpler, albeit more superficial. It suffices however to take insight into the very nature of classical electrodynamics and path integration in quantum electrodynamics.

1 Introduction

In its covariant form, Maxwell's equations are given as

$$\Box A^{\mu}(x^0, \mathbf{x}) = j^{\mu}(x^0, \mathbf{x}), \qquad (0 \leq \mu \leq 3),$$

along with two other conditions: the first is **charge conservation**,

$$\partial j^0/\partial x^0 + \nabla \cdot \mathbf{j} = 0,$$

and the second equation is the socalled **Lorentz gauge**,

$$\partial A^0/\partial x^0 + \nabla \cdot \mathbf{A} = 0$$

see: [3, II-18-6].

Both, Lorentz gauge and charge conservation are intimatedly related, and according to Poincaré's Lemma, they both state their local integrability on single connected regions in space time, on which they are continuously differentiable (see [1, Sec. 2-12 to 2-13]): Let $\Omega \subset \mathbb{R}^4$ be single connected set contained in an open set $U \subset \mathbb{R}^4$, and let $f_0, ..., f_3$ be continuously differentiable function on U into either $\mathbb{R}$ or $\mathbb{C}$. Then the following statements are equivalent:

1. $\partial_0 f_0(x) + \cdots + \partial_3 f_3(x) = 0 \quad \forall x \in \Omega$
2. For any two paths γ and γ' in Ω joining $a, b \in \Omega$: $\int f \cdot d\gamma = \int f \cdot d\gamma'$.

*To the generous bookseller from Herman Kershaw, who once, at the book fair's last day in Frankfurt, ceded H. Cartan's book Differential Forms [1] to me, then a poor student, after having passed some checks that I really cared for it.

2 Problem Statement

So, obviously, one would like to condense the 4-vector Maxwell equation into a single scalar equation $\Box L(x) = Q(x)$, where $L(x) = \int_a^x A \cdot d\gamma$ and $Q(x) = \int_b^x j \cdot d\gamma'$. However, there are two concerns as to this:

3 Concern 1

The first problem is that the domain of definition of j or A may not be simply connected, like for instance a circle in two or more spatial dimensions: In case of a circle one might end with two and more different solutions, and in general, according to to the number of holes and turns of loops.

In order to see that this concern can be overvcome, let me note that for a continuously differentiable function F on $U \subset \mathbb{R}^4$ and $a, x \in U$ the integral $\int_a^x \partial F \cdot d\gamma$ is invariant w.r.t. diffeomorphisms $\Psi : U \to U$ (i.e. bijective mappings which are differentiable along with their inverse): This is so, because ∂F transforms contravariantly under Ψ, whereas $d\gamma$ transforms covariantly, so that the Jacobi matrix cancels against its inverse.

With this, let me explore the situation of two space and one time coordinate first: Given charges, confined to a circle in the xy-plane, and let the time axis be perpendicular to it, then there are two classes of closed paths: One to the side of the circle, and one intersecting the circle. Those to the side contract homeomorphically to a point, so yield a zero result. So, what about the loops which intersect the circle one or more times?

The point now is that with the time perpendicular to the circle in the xy-plane, the particles will not go in a circle, but in a spiral around the time axis, and the originally thought circle is being torn into a spiral. This spiral can be stretched homeomorphically into a line along the time axis. That leaves us with loops encircling the time axis one or several times. Now I observe that I can lift these loops further up the time axis without affecting the values of path integral along those loops. And, if the particles haven't stayed for eternity in that place - which may be assumed not to have happened, then, lifting the loops further towards $t \to \infty$, no charges will then be present and a potential vector field will likewise converge to zero. So, by the principle that all physical observable quantities are to be confined to a bounded region in spacetime, the path integrals along all loops (in the assumed 3-dimensional spacetime) give a zero value.

The same argument now applies to 4 dimensional spacetime: all 3-dimensional holes in space tear up along the time axis and can be stretched out towards $t \to \infty$ and $t \to -\infty$ resp.. And, since there both fields and charges are supposed to vanish, all integrals along closed paths again give zero.

4 Concern 2

The 2nd concern is that apart from the ranges of definition of j and A, the path integration must not be taken w.r.t. the Euclidean metrics, but w.r.t. the Minkowsi metrics, i.e.: the path integration has to be restricted to regions in which the Lorentz matrix is invertible. Now, with one time and two space coordinates, spacetime is divided into 3 disjoint such regions, separated by the forward and backward light cones. In 4 dimensions, however, in the space-like region $\{(x^0, \mathbf{x}) \mid |x^0|^2 < \|\mathbf{x}\|^2\}$ the Lorentz metrics is twice degenerate, because the 4-dimensional Euclidean unit ball cuts into an upper and lower hemisphere: the Lorentz metrics simply does not distinct between corresponding points of either halves. The upper and lower hemispheres can be associated with the positive and negative sign of the determinant, or equivalently, a positive or negative sign of parity.

5 Conclusion

So, in 3 dimensions of space-time we end up with 3 connected regions in which Maxwell's equations can be integrated to one scalar equation, each. And in 4 dimensions of space-time, there are four regions in which Maxwell's equations can be integrated. The excluded region in 4 dimensions, defined up to rotation in the 3-dimensional space, is the union of forward and backward light cone with $\{(x^0, \cdots, x^3) \mid x^3 = 0\}$. The set $\{(x^0, \cdots, x^3) \mid x^3 = 0\}$ could be included to either be part of the upper or lower hemisphere. But a smarter solution would be the replacement of $\mathbb{R}^3$ by the triple $(\sigma_x, \sigma_y, \sigma_z)$ of Pauli matrices in which case the 2-fold coverage $SU(2)$ of $SO(3)$ would smoothly resolve the parity flip. In [4] just that is done. Anyhow, time inversion homeomorphically maps forward and backward light cones onto oneanother, and so does space inversion with the spacelike positive and negative parity cones. The four regions therefore come from the four combinations of the two discrete symmetries that the groups $O(4)$ and $U(4)$ possess: time and space inversion.

So, where did we reach? We showed that one can rewrite Maxwell's equations into a quadrupel of wave equations of action integrals, and solving these, will give us the solution of Maxwell's equations in form of a quadruplet of actions. The general solution then will be any complex linear combination of the four component solutions up to the additions of constant complex vectors χ in $\mathbb{C}^4$. And then, we get a $U(4)$-symmetry on top. (I could have chosen real linear combinations with $O(4)$ on top, but since the fields A^μ are complex-valued, it's better to extend to the complex from scratch.) What is that symmetry group on top of the solutions good for? It's simply that the equations are invariant under that group. Above, in non-relativistic classical mechanics time and location coordinates are equivalent and part of one (Euclidean) symmetry group. We now achieved right that equivalence of space and time coordinates in Maxwell's relativistic theory, either in the sense that we can flip and rotate the 4-dimensional space-time ball such that the coordinates interchange.

However: the fact that space-like regions aren't simply connected, wheras the light cones are, reflects that time always is a special dimension: it is the only of the coordinates which the particles cannot revisit in their life: So, this introduces an inherent break of symmetry. Because of this, we should always be able to tell the time axis apart from the spatial ones.

This in mind, let me investigate what happens, if I would disregard this situation in terms of the above mentioned Pauli matrices: in this unphysical case I would encircle the Pauli matrices, and this two times, to come out to the same space-time position. And after the first turn parity (or: spin) would flip! So, it turns out that one can equivalently forbid that path by declaring parity (or fermoionic spin) to be a conserved quantity.

6 Outlook

The integrability of Maxwell's equations offers an interesting perspective: It suffices to path integrate the sources, $L(x) = \int j \cdot d\gamma$ (with a fixed starting point) to get the action integral of the vector field A, and the differential of this action field will give A in turn.

That alludes to what R.P. Feynman said in [2]:

> "..I was now convinced that since we had solved the problem of classical electrodynamics (and completely in accordance with my program from M.I.T., only direct interaction between particles, in a way that made fields unnecessary) that everything was definitely going to be all right. I was convinced that all I had to do was make a quantum theory analogous to the classical one and everything would be solved.."

Moreover, we touch quantum field theory by the following: Path integrating the vector field A in any of the four component regions, e.g. in the forward light cone gives a function F which has the dimension of energy by test charge. Next, an implicit additional charge factor enters from the charged sources that had been path integrated, plus we used $dx^0 = cds$, where c is the speed of light (that I tacitly set equal to 1). With e_0 being the elementary electronic charge, I can the write $e_0 F = e_0 \int A \cdot d\gamma$ in units of e_0^2/c as a dimensionless function. But $e_0^2/c = \alpha\hbar$, where α is known to be the dimensionless fine-structure constant. Add to this, sofar the test charge is a constant e_0 resting at some place $(x^0, \mathbf{x})$, but if we let $e_0 = e_0(x)$ move in spacetime, then this makes F operate on $e_0(x)$.

That all is one side of the relations. The other one is that we never left classical realms: It still holds that from $F(x)$ alone, one can exactly determine the motion of the sources $j(x)$. It therefore repeats the canon from classical gravitation: The field is the equivalent dual of the particle view: The field holds the complete observability of the particles: just by surrounding the particles and by looking at them from all sides, lets us know exactly what the particles did before, at their retarded local times. In particular, this excludes energy-momentum transfer from the particles to field bosons per se: In it, energy-momentum exchange occurs only, when the electromagnetic field reaches a charged particle target. Or, as Feynman put it equivalently: photons are edges that connect two nodes: a charged particle source with a charged particle target.

However, by surrounding the particles and looking at them, we ourselves interact with the particles at our local time, which will be seen by the particles in turn at their advanced local time, in future. That would be just Feynman's perturbative approach.

Let me make one more point as to the well-known gauge invariance of Maxwell's equations: Let F be as above the integrated action function of the vector field A in the forward light cone. Then that scalar function F should be relativistically invariant: it should be the same in all inertial systems. (And the ame holds for the sources' action function.) Now, what is ∇F? It is relativistic momentum, and $\partial_0 F$, is the relativistic energy. In particular, in order to be invariant w.r.t. moving inertial systems $x \to x'$, the difference of the vector fields A and A', $\Delta A = A - A'$ must satisfy $\partial_0(\Delta A) + \cdots + \partial_3(\Delta A) = 0$. Plus - and this is the curious situation (explained in [4]) - the charge is invariant under this Lorentz transformation, hence $\partial_0(\Delta A) = 0$, and therefore : $\partial_1(\Delta A) + \cdots + \partial_3(\Delta A) = 0$ must hold. Gauge invariance,

$$F \to F + \Psi \Rightarrow (A^0 \to A^0 + \partial^0\Psi, \;\; A^\mu \to A^j - \partial^j\Psi, \;\; (1 \leq j \leq 3)),$$

restates just this.

References

[1] Cartan, H. *Differential Forms.* Herman Kershaw, 1971.

[2] Feynman, R. P. *The Development of the Space-Time View of Quantum Electrodynamics.* http://www.nobelprize.org/nobel_prizes/physics/laureates/1965/feynman-lecture.html, 1965.

[3] Feynman, R. P., Leighton, R. B., and Sands, M. *Lectures on Physics, Vol. I-III.* Addison Wesley, 1977.

[4] Huettenbach, H. D. *Disposing Classical Field Theory.* http://vixra.org/pdf/1209.0027v1.pdf, 2012.