Combined Gravitational Action (IV): *Exploitation*[1]

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Abstract: In previous papers related to the concept of Combined Gravitational Action (CGA), we have established the theoretical foundations of CGA as an alternative gravity theory that has already allowed us to resolve some unexpected and challenging problems both inside and outside the Solar System. These include the anomalous deceleration of Pioneer 10, the observed secular increase of the Astronomical Unit, the apsidal motion anomaly of eclipsing binary star systems, and the study of CGA-effects in non-compact and compact stellar objects. All of this has been achieved without fully utilizing the CGA-formalism. Therefore, the main purpose of this paper is to thoroughly explore the CGA-equations to investigate, among other things, CGA-spin-orbit coupling precession and the application of CGA to Large-Scale Structures in order to address the issue of galactic rotation curves.

Keywords: combined gravitational action; combined gravitational potential energy; Newton's law of gravitation; Kepler's third law; solar system; eclipsing binary stars; binary pulsars; dark matter

1. Introduction

Before the advent of the CGA as an alternative gravity theory, it was always stressed that the study of compact stellar objects exclusively belonged to the GRT-domain because their strong compactness is enough to bend local space-time in such a way that observable GRT-effects should occur. However, as we shall see, the CGA is also able to investigate, predict, and explain the same type of compact stellar objects, all within the context of usual Euclidean geometry and the Galilean relativity principle. This reflects the tangible fact that the propagation of the gravitational field and the action of gravitational force are both independent of the topology of space-time.

But why should the CGA arrive at the same results as GRT or even better in some cases? Because if we take the concept of the curvature of space-time apart, we find that, contrary to Newton's gravity theory, the CGA and GRT take into full consideration the relative motion of the test-body and the speed of light in local vacuum, which in CGA plays the role of a specific kinematical parameter of normalization and in GRT is considered as the speed of gravity propagation. The main consequence of the CGA-formalism [1,2,3,4] is the dynamic gravitational field (DGF), which is, in reality, an induced field. It is more precisely a sort of gravitational induction due to the relative motion of a material body in the vicinity of the principal gravitational source. Furthermore, in the present work, we will show in sections 5 and 6 that the existence of dynamic gravitational acceleration at the galactic scale should be attributed to the gravitodynamical evidence of dark matter (DM) itself, and the characteristic acceleration introduced by Milgrom as a universal constant in his theory of Modified Newtonian Dynamics (MOND) is, in fact, a special case of Eq.(32) (*see* [3]). Consequently, MOND as an alternative theory to the DM 'hypothesis' becomes, by means of the CGA, an additional support for DM!

[1] This paper is dedicated to the memory of Prof. Thomas C. Van Flandern, 26 June 1940 – 9 January 2009

2. Generalization of Kepler's third law and its Consequences

In this section, we will generalize Kepler's third law, meaning we will extend the said law to the context of CGA for a binary system – a system of two massive bodies gravitationally linked – by using the Eq.(27) in Ref.[3], which is, as we know from the preceding papers [1,2], a generalization of Newton's gravitational law. Consequently, this generalization allows us to define the following physical quantities: (i) the CGA-orbital angular velocity; (ii) the CGA-orbital angular momentum J; (iii) the CGA-angular rate of the spin-orbit coupling precession.

2.1. The generalized Kepler's third law

Let us consider two spherical massive bodies A and B with masses m_A and m_B moving in orbits of radii a_A and a_B around a common center of mass C defined by

$$m_A a_A = m_B a_B \ . \tag{1}$$

Therefore, the ratio of the two masses is

$$\frac{m_A}{m_B} = \frac{a_B}{a_A} . \tag{2}$$

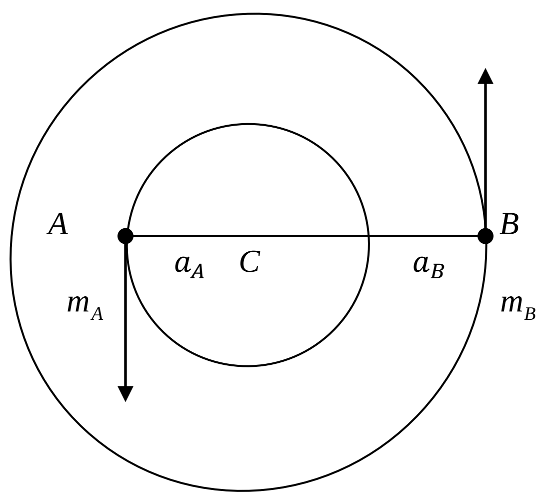

Figure 1. Two spherical massive bodies *A* and *B* moving in almost circular orbits of radii a_A and a_B about their common center of mass *C*.

Further, according to Eq.(27) in Ref.[3], the combined gravitational attraction (resultant force) between the two orbiting spherical massive bodies is given by:

$$\mathbf{F} = -\frac{k}{r^3}\left[1 + \frac{GM}{c^2 r}\right]\mathbf{r}, \quad k = GMm, \tag{3}$$

which can also be written as

$$m\frac{v_{\text{orb}}^2}{r^2}\mathbf{r} = -\frac{k}{r^3}\left[1 + \frac{GM}{c^2 r}\right]\mathbf{r}, \quad v_{\text{orb}} = \Omega r = 2\pi p^{-1} r. \tag{4}$$

From (3) and (4), we get, after some algebraic manipulation, the expected generalized Kepler's third law

$$\frac{r^3}{p^2} = \frac{GM}{4\pi^2}\left[1 + \frac{GM}{c^2 r}\right]. \qquad (5)$$

In the case of the binary system (A, B) gravitationally linked, M is generally considered as the total mass of the system.

2.2. CGA-orbital angular velocity

It follows from the generalized Kepler's third law (5) that, in the context of CGA, the expression of CGA-orbital angular velocity Ω should be different from the usual expression defined in the framework of Newton's gravity theory. Thus, according to (5), we have

$$\Omega \equiv \Omega(r) = \sqrt{\frac{GM}{r^3}}\left[1 + \frac{GM}{c^2 r}\right]^{1/2} . \qquad (6)$$

The CGA-orbital angular velocity (6) may be reduced to the classical expression $\sqrt{GM/r^3}$ for the case $(GM/c^2 r) \ll 1$. As we can see, there is no singularity in expression (6), unlike in general relativity theory where the expression of orbital angular velocity contains a singularity. Moreover, it follows from (6) that the CGA-orbital velocity, $v_{\text{orb}} \equiv v_{\text{orb}}(r) = \Omega r$, takes the explicit form

$$v_{\text{orb}} \equiv v_{\text{orb}}(r) = \sqrt{\frac{GM}{r}}\left[1 + \frac{GM}{c^2 r}\right]^{1/2}. \qquad (7)$$

Once again, when the term $(GM/c^2 r)$ is significantly less than unity, the CGA-orbital velocity (7) reduces to the usual one. This is why we have already used the usual expression $\sqrt{GM/r}$ in [3] when deriving Eq.(27), in which M plays the role of the central mass and m the test-body mass. In general, such an approach does not affect the results that can be obtained from the use of the new expressions (5, 6, 7) when M in this case is the total mass $(M = m_A + m_B)$ of the binary system (A, B) since we are exclusively dealing with massive bodies (two-body problem) in orbital motion. For instance, the Earth-Moon system is characterized by the value of $(GM/c^2 r) = 1.168 \times 10^{-11}$.

Let us return to the CGA-orbital angular velocity and, for simplicity, consider temporarily the massive body A playing the role of the main gravitational source and B as a test-body moving around A at the orbital velocity v_{orb}. Under such conditions (Fig.1) and by taking into account the expression of the combined gravitational attraction $\mathbf{F}$, the two massive bodies as a binary system (A, B) must have the same orbital angular velocity vector $\mathbf{\Omega}$ defined by

$$\mathbf{\Omega} = \boldsymbol{\omega} + \Delta\boldsymbol{\omega} , \qquad (8)$$

around their common center of mass C. Hence, using the expression of $\mathbf{F}$ and equating it to the centrifugal force, and after performing some algebraic calculations and neglecting the infinitesimal quantity, $2\|\boldsymbol{\omega}\| \cdot \|\Delta\boldsymbol{\omega}\| \cos\theta$, resulting from the scalar product of $\boldsymbol{\omega}$ and $\Delta\boldsymbol{\omega}$, we get

$$\frac{m_A m_B}{r^2}\left[1+\frac{GM}{c^2 r}\right]=m_A(\boldsymbol{\omega}^2+\Delta\boldsymbol{\omega}^2)a_A=m_B(\boldsymbol{\omega}^2+\Delta\boldsymbol{\omega}^2)a_B\,. \tag{9}$$

Using the relation (2), we obtain $r=a_A+a_B=a_A(1+m_A m_B^{-1})$, thus (9) becomes after substitution and simplifications

$$\frac{GM}{r^3}\left[1+\frac{GM}{c^2 r}\right]=(\boldsymbol{\omega}^2+\Delta\boldsymbol{\omega}^2)\,, \tag{10}$$

Eq.(10) is also true for an elliptical orbit, in which case r becomes the semi-major axis of the orbit of one massive body relative to the other, which is at the focus of the ellipse. Now, using Eq.(10), we can easily determine the magnitude of $\Delta\boldsymbol{\omega}$, which is of the form

$$\Delta\omega=\frac{GM}{c\,r^2}=\frac{Gm_A(1+q)}{c\,r^2}\,, \tag{11}$$

where $q=m_B/m_A$ is the mass ratio of the binary system $(A,\,B)$.

2.3. CGA-orbital angular momentum

In Newton's gravity theory, the CGA-orbital angular momentum, L, of the binary system $(A,\,B)$ and the CGA-orbital angular velocity, Ω , are connected by

$$L\equiv L(r)=\mu\Omega r^2, \tag{12}$$

where $\mu=m_A m_B/M$ is the reduced mass of the binary system $(A,\,B)$.The expression (12) allows us to affirm in the context of CGA that the equality between the mutual combined gravitational attraction, $\mathbf{F}_{\mathrm{G}}$, and the combined centrifugal force

$$\mathbf{F}_{\mathrm{C}}=\frac{L^2}{\mu}\cdot\frac{\mathbf{r}}{r^4}\,, \tag{13}$$

ensures the orbital gravitational stability of the binary system $(A,\,B)$. Hence, such stability occurs according to equation

$$\mathbf{F}_{\mathrm{G}}-\mathbf{F}_{\mathrm{C}}=0\,, \tag{14}$$

which is actually an extension of d'Alembert's principle to gravitational physics in the context of CGA. Now, let us focus our attention on expression (13), which is called the 'combined' centrifugal force. It is actually a combination of two forces: the static centrifugal force $\mathbf{F}_{\mathrm{SC}}$ and the dynamic centrifugal force $\mathbf{F}_{\mathrm{DC}}$. To be certain of this combination, it is best to rewrite (13) in the following explicit form

$$\mathbf{F}_{\mathrm{C}}=\mu\left(\frac{GM}{r^2}\right)\frac{\mathbf{r}}{r}+\mu\left[\frac{GM}{cr}\right]^2\frac{\mathbf{r}}{r^2}\,. \tag{15}$$

Therefore, the two components of $\mathbf{F}_{\mathrm{C}}$ are of the form:

$$\mathbf{F}_{\mathrm{SC}}=\mu\left(\frac{GM}{r^2}\right)\frac{\mathbf{r}}{r}\,,\quad \mathbf{F}_{\mathrm{DC}}=\mu\left[\frac{GM}{cr}\right]^2\frac{\mathbf{r}}{r^2}\,. \tag{16}$$

That is why $\mathbf{F}_{\mathrm{C}}$ is called 'combined' centrifugal force. The extra-component force $\mathbf{F}_{\mathrm{DC}}$ is induced by the motion of the binary system (A, B) relative to the center of mass. Hence, the existence of $\mathbf{F}_{\mathrm{DC}}$ itself as an extra-component force implies that the mentioned orbital gravitational stability should not be considered an absolute fact, since $\mathbf{F}_{\mathrm{DC}}$ causes, among other effects, some very small secular gravitational perturbations which are, on average, reflected in the precession of the elliptical orbit, as we will see below.

2.4. CGA-spin-orbit coupling precession

In this subsection, we will study another post-Keplerian parameter, namely, the CGA-spin-orbit coupling precession rate, Ψ_{CGA} , under the effect of CGA-spin-orbit coupling, which is originally caused by the couple $(\boldsymbol{\Lambda}, \mathbf{F}_{\mathrm{D}})$ (*see* [3]), and generally occurs in one component of the system (*A*, *B*) when the spin angular momentum vector $\mathbf{S}$ of that component is misaligned with the orbital angular momentum vector **L**. The coupling of spin and orbital angular momenta causes the spin vector $\mathbf{S}$ to precess around **L** with the angular precession rate, Ψ_{CGA}, proportional to $\Delta\omega$, or equivalently

$$\Psi_{\mathrm{CGA}}\ (\mathrm{rad/s}) = K\Delta\omega\ . \tag{17}$$

Where K is a coefficient of proportionality to be determined later . In the framework of GRT, the spin-orbit coupling precession is called 'geodetic precession' or 'De Sitter precession', and is physically attributed to the curvature of space-time. However, for the CGA, this phenomenon is a pure consequence of the action of the couple $(\boldsymbol{\Lambda}, \mathbf{F}_{\mathrm{D}})$ and that is why it is legitimately called 'CGA-spin-orbit coupling precession' or simply 'CGA-orbital precession' as we will see because as Einstein himself argued in 1912, "*The gravitation acts more strongly on a moving body than on the same body in case it is at rest*." But Einstein's claim was stated in 1912, that is to say, before the publication of the final version of GRT in 1915 in which, as we know, the very realistic concept of the gravitational force is abandoned and replaced by the concept of the curvature of space-time. At the same time, Einstein claimed that GRT may be reduced to Newtonian gravity theory for low velocities and weak gravitational fields! We now return to the coefficient of proportionality, K, contained in the relation (17). As we are exclusively dealing with an orbital motion and for the purpose of our investigation, it seems more natural and very convenient to define it as a function of the form: $K \equiv K(r,q) = \frac{3}{2}(1+q/6)(1+q)^{-1}c^{-1}v_{\mathrm{orb}}(r)$, where $q = m_B/m_A$ is the mass ratio of the binary system (A, B) and $v_{\mathrm{orb}}(r) = \Omega r$ this is CGA-orbital velocity (7). We have the explicit expression:

$$K \equiv K(r,q) = \frac{3(1+q/6)}{2(1+q)}\frac{\Omega r}{c}\,. \tag{18}$$

By substituting expressions (11) and (18) into (17), we obtain

$$\Psi_{\mathrm{CGA}} = \frac{3\pi G(1+q/6)m_A}{c^2 r\, p}\,. \tag{19}$$

For an exact elliptical orbit, we have $r = a(1-e^2)$, where a is the semi-major axis and e is the orbital eccentricity. Hence, after substitution, we find, the expected expression of CGA-spin-orbit coupling precession

$$\Psi_{\text{CGA}} = \frac{3\pi G(1+q/6)m_A}{c^2 a(1-e^2)p}. \quad (20)$$

Eq.(20) exclusively concerns the massive body B of mass m_B when acting as a test-body orbiting the main gravitational source A of mass m_A. However, when $q=1$, *i.e.*, $m_A = m_B$ in such a case, the two massive bodies have the same CGA-orbital precession rate because, as it was known [3], when the two massive bodies should mutually play the role of the main gravitational source. For the case of the test-body, the causal origin of B's orbital precession rate (20) is, of course, the couple of the dynamic gravitational field-forces $(\mathbf{\Lambda}, \mathbf{F}_{\text{D}})$ induced by B during its motion in A's gravitational field. Moreover, as repeatedly stated in this paper, the dynamically induced $\mathbf{\Lambda}$ and $\mathbf{F}_{\text{D}}$ have a significant gravitational influence on the evolution and behavior of the massive bodies.

Curiously, Lorentz had already arrived at a conclusion very comparable to that of Einstein, but more than a decade before him. In his influential work entitled 'Considerations on Gravitation' published in 1900, Lorentz wrote, "*Every theory of gravitation has to deal with the problem of the influence exerted on this force by the motion of the heavenly bodies*." [5]. Lorentz's claim clearly reinforces the fact that $\mathbf{\Lambda}$ and $\mathbf{F}_{\text{D}}$ are induced by the motion of the massive test-body B in the gravitational field of the central body A. In [2], we have already calculated the angular deflection of starlight as a direct consequence of $(\mathbf{\Lambda}, \mathbf{F}_{\text{D}})$, so it is evident that the gravitational redshift and gravitational lensing should also be caused by the same $(\mathbf{\Lambda}, \mathbf{F}_{\text{D}})$.This serves as a counterexample to the concept of the curvature of space-time as an interpretation of gravitation. Let us return to Eq.(20), which is also applicable to eclipsing binary star systems and binary pulsars, as we will see later on. We will begin by applying it to the Earth-Moon system as a whole. First, we will investigate the secular precession of the Moon's orbit under the effect of CGA-spin-orbit coupling caused by the dynamic gravitational field-force $(\mathbf{\Lambda}, \mathbf{F}_{\text{D}})$ induced by the Moon during its motion in the immediate local Earth's gravitational field (g-field). Secondly, we will investigate the same phenomenon for the Earth-Moon system in the Sun's g-field.

2.4.1. Moon's secular CGA-orbital precession in the immediate local Earth's g-field

We have the following orbital and physical parameters of the Moon. Orbital eccentricity: $e = 0.0549$; orbital period: $P = 27.32$ days; semi-major axis: $a = 3.844\times10^8$ m; Moon's mass: $m_B = 7.3477\times10^{22}$ kg; Earth's mass: $m_A = 5.9722\times10^{24}$ kg; mass ratio: $q = m_B / m_A = 1.230317\times10^{-2}$. After substituting all these parameters into Eq.(20), we get the rate of the secular CGA-orbital precession:

$$\Psi_{\text{CGA}} = 4.674804\times10^{-17}\ \text{rad/s} = 30.444760\,\text{mas/cy}. \quad (21)$$

Where 'mas/cy' is an abbreviation for 'milliarcsecond per century'. Kinematically, the value (21) means that the Moon's orbit itself rotates around the Earth at the velocity of 56.71m/cy and the CGA-orbital precession period, *i.e*., the temporal interval for a complete rotation, 2π, is 4.26×10^9 yr which is very comparable to the age of our Solar System! Now, let us evaluate the magnitude of the couple $(\mathbf{\Lambda}, \mathbf{F}_\text{D})$ responsible for this secular orbital precession. First, we have according to Eqs.(42,43) in [3], the following expressions:

$$\Lambda = \frac{1}{a}\left[\frac{GM}{ca}\right]^2 \quad \text{and} \quad F_\text{D} = \frac{m}{a}\left[\frac{GM}{ca}\right]^2 .$$

Hence, after a direct substitution and calculation, we obtain

$$\Lambda = 3.112\times10^{-14}\ \text{m/s}^2 \quad \text{and} \quad F_\text{D} = 2.28660 \times 10^9\ \text{N}.$$

2.4.2. Earth-Moon system secular CGA-orbital precession in the Sun's g-field

Historically, De Sitter studied the so-called geodetic precession for the first time in 1916 as a consequence of GRT. He derived a formula similar to (20) and published it in his seminal article entitled "On Einstein's theory of gravitation and its astronomical consequences. Second paper." [6]. De Sitter applied his formula to the Earth-Moon system and found that the spin-orbit coupling contribution is reflected in the Earth-Moon orbital precession by $1.9194\ \text{arc}\sec/\text{cy}$. However, the most complete investigation on the subject was conducted by Brumberg *et al*. [7,8]. Their corrected value of the geodetic precession is $1.9199\ \text{arc}\sec/\text{cy}$, which is quite comparable to our value as we will see below. De Sitter attributed the causal origin to the curvature of space-time.

However, the existence of the same phenomenon in the framework of CGA with the same amount found by [7,8] is considered a counterexample to the concept of the curvature of space-time itself as an interpretation of gravitation. Due to its distance from the Sun, the Earth-Moon system can be regarded as a single body rotating in the Sun's g-field. Furthermore, since the Earth physically dominates the system under consideration, we take the orbital and physical parameters of the Earth as the average for the entire system and denote $m_B = m_\oplus + m_\text{M}$ as the total mass of the system (Earth, Moon).

Eccentricity: $e = 0.0167$; period: $P = 365.25$ days; semi-major axis: $a = \text{AU} = 149.597870 \times 10^9$ m ; Moon's mass: $m_\text{M} = 7.3477 \times 10^{22}$ kg ; Earth's mass: $m_\oplus = 5.9722 \times 10^{24}$ kg ; Sun's mass: $m_A = m_\odot = 1.9891 \times 10^{30}$ kg ; mass ratio: $q = m_B/m_A = 3.039403 \times 10^{-6}$.

By substituting all these parameters into Eq.(20), we get:

$$\Psi_\text{CGA} = 2.949535 \times 10^{-15}\ \text{rad/s} = 1.919918\ \text{arc}\sec/\text{cy} . \tag{22}$$

This is in excellent agreement with the value found by Brumberg *et al*. [7,8]. Further, according to Eq. (22), an entire precession cycle would take 6.750285×10^7 years, which is 1.47% of the total age of our Solar System.

2.4.3. CGA-spin-orbit coupling Effect in Eclipsing Binary Star Systems

Finally, we apply formula (20) to investigate the CGA-orbital precession under the effect of CGA-spin-orbit coupling caused by the couple $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$ in the well-known eclipsing binary star systems: AS Camelopardalis and DI Herculis. We also study the same phenomenon in the binary pulsars PSR B1913+16 and PSR B1534+12, as well as the double pulsar PSR J0737-3039. Since we have already studied the CGA-apsidal motion in AS Camelopardalis and DI Herculis, respectively, we can use the same orbital and stellar parameters according to [3]. For AS Cam: $e = 0.1695$; $P = 3.430$ days $a = 17.20\ R_{\odot}$; $m_A = 3.3\ m_{\odot}$; $m_B = 2.5\ m_{\odot}$ and DI Her: $e = 0.489$; $P = 10.55$ days ; $a = 43.12\ R_{\odot}$; $m_A = 5.15\ m_{\odot}$; $m_B = 4.52\ m_{\odot}$. Moreover, as the two systems are characterized by the mass ratio $q < 1$, this implies, among other things, that in each system the primary star A of mass m_A should play the role of the main gravitational source and the secondary star B of mass m_B should be the orbiting test-body. Therefore, in the two systems (AS Cam, DI Her), the effect of CGA-spin-orbit coupling should concern exclusively the secondary star, *i.e.*, we are dealing with B's CGA-orbital precession caused by the couple $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$. Hence, after substituting all the necessary parameters in Eq. (20), we obtain the following CGA-orbital precession rates for AS Cam and DI Her, respectively:

$$\Psi_{\mathrm{CGA}}^{(1)} = 1.202686\ \mathrm{deg/cy}\ , \qquad \Psi_{\mathrm{CGA}}^{(2)} = 1.169855\ \mathrm{deg/cy}\ . \tag{23}$$

2.4.4. CGA-spin-orbit coupling Effect in Binary Pulsars and Double Pulsars

Let us return to the compact stellar objects, discussed in [3] and investigate the CGA-orbital precession under the effect of GCA-spin-orbit coupling. It is expected that the CGA-rate, Ψ_{CGA}, for the binary pulsars PSR B1913+16 and the double pulsar PSR J0737-3039 should be significantly greater than that of eclipsing binary star systems. This is mainly due to the strength of the dynamic gravitational force, $\mathbf{F}_{\mathrm{D}}$, in these compact stellar objects. For example, in [3], we have already determined the values of $F_{\mathrm{D}} = 6.1476\times10^{24}$ N and $F_{\mathrm{D}} = 1.730\times10^{24}$ N for AS Cam and DI Her, respectively, as well as the values of $F_{\mathrm{D}} = 5.831340\times10^{26}$ N and $F_{\mathrm{D}} = 4.677426\times10^{27}$ N for PSR B1913+16 and PSR J0737-3039, respectively. Therefore, a simple comparison shows:

$$\frac{F_{\mathrm{D}}\,(\text{in compact binary pulsar sytem})}{F_{\mathrm{D}}\,(\text{in ordinary binary star system})} = 761$$

In the context of GRT, Weisberg and Taylor [9] found a theoretical geodetic precession rate of 1.213 deg/yr for PSR B1913+16, and Manchester *et al*. [10] predicted geodetic precessional periods of 75 years and 71 years for PSR J0737-3039A and PSR J0737-3039B, respectively, which correspond to the following geodetic precession rates: $\Psi_{\mathrm{GR}}^{(\mathrm{A})} = 4.8\ \mathrm{deg/yr}$ and $\Psi_{\mathrm{GR}}^{(\mathrm{B})} = 5.070\ \mathrm{deg/yr}$. Since we have already studied the CGA-apsidal motion in the above--mentioned pulsars, we can use the same orbital and stellar parameters as in [3] for PSR B1913+16: $e = 0.6171$; $P = 0.322997$ day ; $a = 1.950100\times10^{9}$ m ; $m_A = 1.4414\ m_{\odot}$; $m_B = 1.3867\ m_{\odot}$; and PSR J0737-3039: $e = 0.0877$;

$P = 0.102251\text{day}$; $a = 8.8\times10^{8}\,\text{m}$; $m_A = 1.338\,m_{\odot}$; $m_B = 1.249\,m_{\odot}$. Moreover, as the two compact systems are characterized by the mass ratio $q \approx 1$, this implies that in each system, the two compact neutron stars should mutually play the role of the main gravitational source. In this case, the effect of CGA-spin-orbit coupling should concern each system as a whole, i.e., we are dealing with *A* and *B*'s CGA-orbital precession caused by the couple $(\mathbf{\Lambda}, \mathbf{F}_{\text{D}})$ in each system. Therefore, after substituting all the necessary parameters in Eq.(20), we obtain the following CGA-orbital precession rates for PSR B1913+16 and PSR J0737-3039, respectively:

$$\Psi_{\text{CGA}}^{(1913)} = 1.24760 \text{ deg/yr} , \qquad \Psi_{\text{CGA}}^{(0737)} = 5.03980 \text{ deg/yr} . \tag{24}$$

As we can see repeatedly, just like with other investigated gravitational phenomena, our calculated values are in good agreement with those theoretically predicted by [9,10]. Once again, the CGA, as a post-Newtonian gravity theory, is very capable of studying and predicting both old and new gravitational phenomena inside and outside the Solar System,–in weak and strong (combined) gravitational fields.

3. Consequence of the Potential Equations

Let's focus on Eqs.(5,6,7) from reference [3]. Eq.(5) represents the Laplace equation in radial coordinates for the potential outside the gravitational source *A* with mass *M* and test-body *B* with mass *m*. Eqs.(6) and (7), respectively:

$$\frac{\partial^2 U}{\partial v^2} - \frac{1}{v}\frac{\partial U}{\partial v} = 0 , \qquad \frac{\partial^2 U}{\partial r \partial v} + \frac{1}{r}\frac{\partial U}{\partial v} = 0 .$$

These two equations share a common fundamental solution $U \equiv U(r, v)$ under appropriate boundary conditions. This solution corresponds to the velocity-dependent combined gravitational potential energy function (Eq.3 in [3]):

$$U \equiv U(r, v) = -\frac{k}{r}\left(1 + \frac{v^2}{w^2}\right) ,$$

and w is a specific kinematical parameter with the dimensions of a constant velocity defined in [3] by

$$w = \begin{cases} c \ \text{ if } B \text{ is in relative motion inside the vicinity of } A \\ v_{\text{esc}} = \sqrt{2GM/R} \ \text{ if } B \text{ is in relative motion outside the vicinity of } A \end{cases} ,$$

where $k = GMm$, $r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ is the relative distance between the gravitational source *A* and the moving test-body *B*, $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is the velocity of the test-body relative to the reference frame of *A*, c is the speed of light in vacuum[2], and v_{esc} is the escape velocity at the surface of the gravitational source *A*. However, despite their different expressions, the two equations mentioned above define the same new physical quantity in the context of CGA, namely, the

[2] In paper [3], we used the symbol c_0 for the speed of light in vacuum, as both symbols c_0 and c are interchangeable.

gravitational momentum. This new concept is derived from the following fact: we have from the above equations

$$v\frac{\partial^2 U}{\partial v^2} = \frac{\partial U}{\partial v}\,, \qquad -r\frac{\partial^2 U}{\partial r \partial v} = \frac{\partial U}{\partial v}\,.$$

Further, we have

$$-\frac{\partial U}{\partial v} = \frac{2kv}{w^2 r} = \epsilon(mv)\,, \tag{25}$$

where $\epsilon \equiv \epsilon(r) = (2GM/w^2 r)$. Since ϵ is a dimensionless physical quantity and (mv) is the magnitude of the ordinary linear momentum vector, $\mathbf{p} = m\mathbf{v}$, thus Eq.(25) may be written as

$$p_{\mathrm{G}} \equiv p_{\mathrm{G}}(v, r) = \epsilon p\,, \tag{26}$$

where $p = mv$ and p_{G} is the magnitude of the *gravitational momentum vector* $\mathbf{p}_{\mathrm{G}}$, which is defined below as follows

$$\mathbf{p}_{\mathrm{G}}\colon \begin{cases} p_{\mathrm{G}}^{(1)} = \epsilon m v_x \\ p_{\mathrm{G}}^{(2)} = \epsilon m v_y \\ p_{\mathrm{G}}^{(3)} = \epsilon m v_z \end{cases}. \tag{27}$$

The quantity ϵ plays the role of the variable coefficient of proportionality between the magnitude of the gravitational momentum p_{G} and the magnitude of the ordinary linear momentum p. Therefore, the existence of the gravitational momentum as an additional physical quantity should enhance the principal momentum of the moving test-body. Note that the rate of change of the gravitational momentum vector (27) should be defined as the (partial) derivative of $\mathbf{P}_{\mathrm{G}}$ with respect to time, that is:

$$\frac{\partial \mathbf{p}_{\mathrm{G}}}{\partial t} = \epsilon\frac{d\mathbf{p}}{dt}\,, \tag{28}$$

or equivalently

$$m^{-1}\frac{\partial \mathbf{p}_{\mathrm{G}}}{\partial t} = \epsilon\frac{d\mathbf{v}}{dt}\,. \tag{29}$$

Remark: since $\epsilon \equiv \epsilon(r) = (2GM/w^2 r)$, the term on the right-hand side of Eq.(29) perfectly coincides with the second term on the right-hand side of Eqs.(15,18), (*see* [3]). Consequently, Eqs.(28) and (29) play the role of additional perturbing force and acceleration, respectively. Returning to Eq.(27) and considering the case when the test-body *B* orbits the gravitational source *A* at an average distance r inside *A*'s vicinity, the variable coefficient of proportionality, ϵ, becomes $(2GM/c^2 r)$. With the help of (26), the magnitude of the gravitational angular momentum of the orbiting test-body should be of the form

$$\ell_{\mathrm{G}} = p_{\mathrm{G}} r\,, \tag{30}$$

or explicitly

$$\ell_{\mathrm{G}} = \left(\frac{2GM}{c^2}\right)p\,. \tag{31}$$

It follows from the above equations that any material body in a state of orbital motion in a combined gravitational field is characterized by a gravitational angular momentum vector whose magnitude is given by Eq.(30) or (31). Moreover, since the quantity $(2GM/c^2 r)$ is called the ‘Schwarzschild radius’ in the framework of GRT, we arrive at the following operational definition: '*The magnitude of the gravitational angular momentum for an orbiting test-body inside the vicinity of the gravitational source is the scalar product of the Schwarzschild radius and the magnitude of the ordinary linear momentum*'. From (31), we arrive at the following result:

$$\frac{\ell_{\mathrm{G}}}{p} = \left(\frac{2GM}{C^2}\right). \tag{32}$$

4. CGA-Binet's orbital Equation

Through the present work, we have seen that the CGA, as an alternative post-Newtonian gravity theory, is very capable of predicting some old and new gravitational phenomena. For example, in the second paper [2], we have derived two important formulas: one for the perihelion advance of Mercury and the other for the angular deflection of starlight. Indeed, the two formulas have been deduced from the CGA-Binet's orbital equation, which has *exactly* the same physico-mathematical structure as the general relativistic Binet's orbital equation developed in the framework of curved space-time and Schwarzschild metric [11,12,13]. The fact seemed like a pure coincidence at first sight, but when one analyzes the paper [3] with a fully open mind, he/she will find that despite the concept of curved space-time, there is a certain *compatibility* of CGA with GRT reflected by Eq.(25) deduced from Eq.(24), which itself is an expression of the gravitational force derived by Ridgely [3] in the context of GRT. From Eq.(27) in [3], which is identical to Eq.(25) in the same paper, we can easily deduce the basic result of CGA, namely, the dynamic gravitational field-force $(\mathbf{\Lambda}, \mathbf{F}_{\mathrm{D}})$. Therefore, based on this, we can logically assert that the concept of curved space-time is merely a mathematical artifact, and the existence of such compatibility signifies, among other things, that CGA is a counterexample to GRT. Furthermore, to clarify this compatibility and make it more understandable and locatable, we will derive the aforementioned CGA-Binet's orbital equation once again as follows. Let us consider the test-body *B* of mass m orbiting the gravitational source *A* of mass *M*. Therefore, *B* evolves in the combined gravitational field. Moreover, assuming that the orbital motion takes place in the polar plane $(0, \mathbf{e}_r, \mathbf{e}_\phi)$ inside the vicinity of *A*, we can replace the orbital velocity v with $r\dot{\phi}$ $(\dot{\phi} = d\phi/dt)$ and the kinematical parameter w with c in the CGA-potential energy function, as discussed in Section 3, we find:

$$U = -\frac{k}{r}\left(1 + \frac{r^2\dot{\phi}^2}{c^2}\right). \tag{33}$$

Moreover, since the test-body *B* is in orbital motion, we obtain the following relations regarding ordinary angular momentum

$$h = mr^2\dot{\phi}, \tag{34}$$

from which we get

$$r\dot{\phi} = \frac{h}{mr}. \tag{35}$$

After substituting (35) into (33), we have

$$U = -\frac{k}{r}\left(1 + \frac{h^2}{m^2c^2r^2}\right). \tag{36}$$

We can now write the force due to the combined gravitational potential energy

$$F = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right) - \frac{\partial U}{\partial r} = -\frac{k}{r^2}\left(1 + \frac{3h^2}{m^2c^2r^2}\right) . \tag{37}$$

We have also for a central force

$$\mathbf{F} = \mathrm{f}(r)\mathbf{e}_r , \tag{38}$$

and according to Newton's second law

$$\mathbf{F} = m\boldsymbol{a} , \tag{39}$$

or more explicitly

$$\mathbf{F} = \mathrm{f}(r)\mathbf{e}_r = m\left(\ddot{r} - r\dot{\phi}^2\right)\mathbf{e}_r + m\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\mathbf{e}_\phi . \tag{40}$$

Since we are dealing with elliptical orbits, therefore, by taking into account Kepler's second law, we obtain from (40) the following differential equations relative to the directions $\mathbf{e}_r$ and $\mathbf{e}_\phi$:

$$\frac{F}{m} = \frac{\mathrm{f}(r)}{m} = \left(\ddot{r} - r\dot{\phi}^2\right) , \tag{41}$$

$$m\left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right) = \frac{m}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0 . \tag{42}$$

That is

$$r^2\dot{\phi} = \kappa = \text{constant} . \tag{43}$$

Putting

$$r = \frac{1}{u} . \tag{44}$$

Thus

$$\dot{\phi} = \frac{\kappa}{r^2} = \kappa u^2. \tag{45}$$

By differentiating relation (44), with respect to time, we get

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\left(\frac{du}{d\phi}\right)\frac{d\phi}{dt} ,$$

thus

$$\dot{r} = -\frac{1}{u^2}\dot{\phi}\left(\frac{du}{d\phi}\right) . \tag{46}$$

By substituting (45) into (46), we obtain

$$\dot{r} = -\kappa\left(\frac{du}{d\phi}\right) . \tag{47}$$

From (49), the second time derivative is

$$\ddot{r} = -\kappa\frac{d}{dt}\left(\frac{du}{d\phi}\right) = -\kappa\frac{d}{dt}\left(\frac{du}{d\phi}\right)\frac{d\phi}{d\phi} = -\frac{d}{d\phi}\left(\frac{du}{d\phi}\right)\frac{d\phi}{dt} ,$$

that is

$$\ddot{r} = -\kappa\left(\frac{d^2u}{d\phi^2}\right)\dot{\phi} . \tag{48}$$

Taking account of (45), Eq.(48) becomes

$$\ddot{r} = -\kappa^2\left(\frac{d^2u}{d\phi^2}\right)u^2. \tag{49}$$

Again, by substituting (44), (45) and (49) in (41), we get

$$\frac{F}{m} = \frac{\mathrm{f}(r)}{m} = -\kappa^2 u^2 \frac{d^2u}{d\phi^2} - \kappa^2 u^3 \ . \tag{50}$$

Since $k = GMm$, thus from (37), (44) and (50), we find

$$\frac{d^2u}{d\phi^2} + u - \left[\frac{3GMh^2}{m^2c^2\kappa^2}\right]u^2 = \frac{GM}{\kappa^2} \ . \tag{51}$$

By taking into account relations (35) and (45), the quantity included in square brackets, on the left-hand side of Eq.(51), becomes

$$\left[\frac{3GMh^2}{m^2c^2\kappa^2}\right] = \left[\frac{3GM}{c^2}\right] . \tag{52}$$

Finally, after substituting into (51), we obtain the expected CGA-Binet's orbital equation

$$\frac{d^2u}{d\phi^2} + u - \left[\frac{3GM}{c^2}\right]u^2 = \frac{GM}{\kappa^2} \ . \tag{53}$$

Eq.(53) has the exact physico-mathematical structure of the general relativistic Binet's orbital equation developed in the context of curved space-time and the Schwarzschild metric [11,12,13].

Eq.(53) is, in fact, a generalization of the classical Binet's orbital equation $d^2u/d^2\phi + u = GM/\kappa^2$. Therefore, the explicit presence of an extra small term, $(3GM/c^2)u^2$, in Eq.(53) represents the CGA-correction when the test-body evolves in the combined gravitational field. From Eq.(53), we can easily deduce the well-known formula, $\Delta\varphi = [6\pi GM/c^2a(1-e^2)]$, for the perihelion advance of a planet evolving in the combined gravitational field around a star of mass M. Here, the planet's mass is, of course, neglected compared to the star's mass.

Now, if we want to study what happens to a light ray propagating in a (combined) gravitational field of a star. We first suppose that a photon has an infinitesimally small mass, m_γ. Secondly, since $\kappa = r^2\dot{\phi} = h/m$, if we take into account the infinitesimally small mass of the photon, we obtain, for the case when $m = m_\gamma$, $(GM/\kappa^2) \approx 0$. Therefore, we can neglect the term on the RHS of Eq.(53), which reduces the equation to:

$$\frac{d^2u}{d\phi^2} + u - \left[\frac{3GM}{c^2}\right]u^2 = 0 \ . \tag{53'}$$

This equation represents trajectory of light. From Eq.(53'), we can deduce the formula $\Delta\theta = 4GM/c^2R$ for the angular deflection of a light ray grazing the limb of the star, where R is the star's radius. It is worthwhile to note that the classical formula, $\delta = \frac{1}{2}\Delta\theta$, is just half of the CGA-formula. The classical formula is derived in the framework of Newton's gravity theory, in particular, by using $\mathbf{f} = -GMmr^{-3}\mathbf{r}$ and other considerations. However, the CGA-formula is essentially derived from $\mathbf{F} = -GMmr^{-3}[1 + v^2/c^2]\mathbf{r}$. For the case of a massive photon, we find $\mathbf{F} = -2GMmr^{-3}\mathbf{r}$. It is precisely this extra factor of 2 that gives the correct angle of deflection of starlight by the Sun $\Delta\theta = \frac{4GM_\odot}{c^2R_\odot} = 1.75''$.

5. Application of CGA *to* Large-Scale Structures

Preamble: After applying the CGA – as a post-Newtonian gravity theory– to the Solar System, eclipsing binary star systems, and binary pulsars, we now aim to apply it to galactic scales. We are interested in the 'hypothetical' dark matter (DM) without delving in to all the details, as there is a wealth of research articles and books that provide authoritative coverage of the literature on the subject and derivations of the most important results. In this section, we will show that the presence of dynamic gravitational acceleration, as defined by Eq.(32) in [3], on galactic scales should not immediately be linked to potential evidence of DM and the characteristic acceleration.

$$a_0 \approx 2 \times 10^{-10}\ \mathrm{ms}^{-2}\ , \tag{54}$$

introduced by Milgrom as a universal constant [14,15,16] in his theory of Modified Newtonian Dynamics (MOND) [17,18,19,20], which is actually a special case of Eq.(32). Consequently, MOND, as an alternative theory to the DM 'hypothesis', became embodied in CGA.

Brief History: In most cases, the Newtonian gravitational inverse-square law and its well-known relativistic generalization have passed several critical tests on very different spatial and temporal scales. However, the first incongruity seems to show up only on galactic scales with the observed discrepancy between the gravitodynamical mass and the directly observable luminous mass. To resolve that discrepancy, two obvious explanations have been proposed: (i) either large quantities of invisible DM dominate the gravitodynamics of large systems [21,22,23,24]; (ii) or gravity itself is not correctly described by Newtonian theory on every scale [14,25,26,27].

5.1. Possible evidence *for* Dark Matter

Let us begin with the fundamental question: Does the observed distribution of luminous matter truly align with the rotation curve without the need for additional DM? To the best of our knowledge, no galaxies are known to exhibit an extended rotation curve that can be explained solely by the presence of luminous matter without invoking DM. The primary evidence for significant amounts of matter in the Universe that are not linked to luminous components-comes from various observations. Several observations, typically interpreted as supporting the existence of (cold) DM [28,29,30,31,32,33], include:

1- The rotation curves of galaxies compared with their luminous matter distribution;
2- The gas content of clusters compared with velocity, x-ray or lensing mass estimates;
3- The normalization of galaxy clustering compared with microwave anisotropies;
4- The shape of the large-scale galaxy correlations;
5- Cosmic flows and redshift space distortions;
6- The amplitude of weak lensing by large scale structure.

5.2. MOND

As already mentioned, to avoid the need for DM, the best known suggested modification to Newtonian gravity theory [14,15,16,17] is usually referred to as MOND. The basic idea of MOND is that there exists a fundamental acceleration (54) *below* which the real acceleration is larger than the Newtonian one $\|\boldsymbol{a}_{\mathrm{N}}\| = a_{\mathrm{N}}$. This is essentially formulated by the real observed acceleration, a, through the relation

$$\boldsymbol{a}_{\mathrm{N}} = \mu(a/a_0)\boldsymbol{a} \,, \tag{55}$$

where $\mu(x)$ is an interpolating function with limits

$$\mu(x) = \begin{cases} x \text{ if } x \ll 1 \\ 1 \text{ if } x \gg 1 \end{cases} . \tag{56}$$

MOND should reach its *regime* only when $\|\boldsymbol{a}\| = a \ll a_0$, and in this limit

$$a = \sqrt{a_0 \frac{F}{m}} \,, \tag{57}$$

which is often given as the fundamental equation of MOND. Furthermore, Milgrom [14,15,16] suggested the following expression

$$\mu(x) = \frac{x}{\sqrt{1+x^2}} \,. \tag{58}$$

Phenomenologically speaking, MOND works well for the observed phenomena at the level of galaxies and clusters of galaxies almost exactly like the DM paradigm. For example, MOND has reproduced the (flat) galactic rotation curves and explained the Tully-Fisher law ($v^4 \propto L$) without, of course, invoking the DM hypothesis. However, advocates of the DM paradigm claim that MOND works well for the cited phenomena because the arguments from that of $\mu(x)$ and the value of a_0 have been rigged to obtain these remarkable results.

5.3. MOND's conceptual difficulties

Before delving into MOND's ambiguities, we must remember that Newton's laws of motion and the law of gravitation are closely intertwined. Any simple modifications without profound intellectual reflection, can lead to incalculable consequences. Therefore, a fundamental question arises: Does MOND apply equally to decelerations as it does to accelerations, or does the motion need to involve just a change in the vector direction of acceleration to exhibit MOND effects? The prevailing interpretation suggests that all changes in velocity are subject to MOND. Consequently, a key distinction from ordinary dynamics emerges when considering a test-body moving away from a central gravitational source. In MOND, the test-body's deceleration never falls below the value of a_0, preventing it from escaping to infinity. This means that in MOND, there are no unbound orbits.

Several critical difficulties with Milgrom's original scheme, as stated, were first identified by Felten [34] soon after the introduction of MOND in 1983. One pertinent example is that, in MOND, the accelerations a given to a test-body by two or more attracting bodies acting mutually do not add linearly. However, in Newtonian gravity theory, accelerations $a_{\rm N}$ do add linearly, so their square roots cannot do so. Alternatively, this means that because acceleration is not inversely proportional to mass, momentum is not conserved for an isolated system in general. Furthermore, as the gravitational force is no longer linear, in the MONDian framework, the resultant gravitational force is not equal to the sum of partial forces. In particular, as pointed out by Felten [34], the motion of the center of mass of a body no longer obeys classical mechanics.

5.4. MOND is an adjustable phenomenological theory

MOND is exceptionally successful in explaining the shapes of galactic rotation curves because it is an *adjustable* theory. More precisely, MOND has three principal parameters that need to be *adjusted*, namely a_0, $\mu(x)$ and the stellar mass-to-luminosity ratio Υ. Originally Milgrom [14,15,16] used the forms (54) and (58); but several authors have employed different values such as $1\times10^{-10}\,\mathrm{m\,s^{-2}}$, $1.2\times10^{-10}\,\mathrm{m\,s^{-2}}$, $3.4\times10^{-10}\,\mathrm{m\,s^{-2}}$ and $3.9\times10^{-10}\,\mathrm{m\,s^{-2}}$ instead of (54). This is clearly uncomfortable for MOND since a_0 is supposed to be a universal constant. Therefore, it follows from the above values that a_0 is not strictly speaking a universal constant as speculated by Milgrom, but an *adjustable* parameter with the dimensions of acceleration defined by

$$1\times10^{-10}\ \mathrm{m s^{-2}} \le a_0 \le 4\times10^{-10}\ \mathrm{m s^{-2}}. \tag{59}$$

Regarding the interpolating function, Bekenstein [35] proposed another one $\mu(x) = \left[\sqrt{1+4x} - 1\right]\left[\sqrt{1+4x} + 1\right]^{-1}$ instead of (58), and Famaey and Binney [36] suggested $\mu(x) = x(1+x)^{-1}$ rather than (58).

5.5. Some empirical difficulties *with* MOND

Without mentioning the *adjustability*, some authors have claimed that MOND has been very successful in explaining observations of rotation curves for a variety of objects over a wide range of scales (see *e.g.*, Milgrom [37]; Bekenstein and Sanders [38]). However, a huge number of investigations have indicated difficulties in reconciling MOND with data under its main postulation that there is *no* DM, and that the critical acceleration parameter a_0 has a well-fixed value. For example, concerning the astronomical evidence for large amounts of DM, Faber and Gallagher [39] and Davis *et al*., [40] have already revealed that in and around galaxies, it comes almost exclusively from applications of Newton's second law to galaxies and clusters of galaxies. Certainly, the accelerations in these large cosmological systems are much smaller than those for which the law has been well tested in the laboratory or in the Solar System. Kent [41] pointed out that while MOND could fit his H I rotation curve data there was a factor of 5 required in the value of a_0 and also no clear evidence for the slightly falling rotation curves that MOND would still predict. Hernquist and Quinn [42] examined simulations of shell galaxies within MOND, and arrived at the conclusion that the observed value and radial distribution of shells in NGC 3923 could not be rigorously explained without a DM halo. The and White [43] found that a MONDian fit to the coma cluster requires a *higher* value of a_0 than for galaxies and also does not predict the correct temperature profile for the x-ray gas. Lake [44] identified inconsistencies between MOND and observations of a group of seven dwarf galaxies: DDO 125, IC 1613, VCC 381, NGC 3109, DDO 154, IC 3522, and NGC 3198. Furthermore, Lake and Skillman [45] found that MONDian fits to the Local Group dwarf IC 1613 would need values of a_0 at least an order of magnitude *below* the preferential values. Gerhard and Spergel [46] studied dwarf spheroidal galaxies in the Local Group and concluded that a number of the dwarfs need to contain some DM even under the MOND paradigm. This *conclusion* is exactly the main purpose of our present section as we will see. We now return to CGA with the following typical galaxy scenario.

6. Typical Galaxy Scenario

Let us consider a test-body B as a star of mass m_* in rotational motion with velocity $v(r)$ at a radial distance r sufficiently far from the galactic center of the main gravitational source A, which is, in the present *scenario*, a typical galaxy of mass $M(r)$ inside the radius r. With these considerations and by taking into account the definition of the kinematical parameter w in Section 3, Eq.(32) of the dynamic gravitational acceleration [3] becomes

$$\Lambda = \frac{GM(r)}{r^2}\frac{v^2(r)}{v_{\text{esc}}^2} . \tag{60}$$

Since $m_* \ll M(r)$, we get the following expressions for rotational and escape velocities *at* the radius r

$$v^2(r) = GM(r)r^{-1} , \tag{61}$$

$$v_{\text{esc}}^2 = 2GM(r)r^{-1} . \tag{62}$$

After substitution in (60), we obtain the very important expression for the dynamic gravitational acceleration at the galactic scale

$$\Lambda = \frac{1}{2}\frac{v^2(r)}{r} . \tag{63}$$

Furthermore, considering the fact that

$$\boldsymbol{a} = \frac{v^2(r)}{r^2}\mathbf{r} . \tag{64}$$

Eq.(64) represents the ordinary centrifugal acceleration in terms of vectors. Therefore, according to Eq.(64), we have:

$$\boldsymbol{\Lambda} = \frac{1}{2}\boldsymbol{a} . \tag{65}$$

Alternatively, we can express Eq.(65) in terms of a force vector as follows:

$$\mathbf{F}_{\text{D}} = \frac{1}{2}(m_*\boldsymbol{a}) . \tag{66}$$

As we can easily see, the second term in brackets in Eq.(66) represents the well-known Newton's second law that governs classical dynamics. Therefore, Eq.(66) can be written as $\mathbf{F}_{\text{D}} = \frac{1}{2}\mathbf{f}$. But what does Eq.(66) mean? Firstly, it means that at the galactic level, the dynamic gravitational force $\mathbf{F}_{\text{D}}$ is always equal to half of the inertial force $\mathbf{f} = m_*\boldsymbol{a}$. Secondly, since $\mathbf{F}_\mathrm{D}$ is actually an additional force, this signifies, among other things, that at large-scale structures, the force defined by Newton's second law is not really a single force in the common classical sense, but a resultant $\mathbf{F}$ of two forces $\mathbf{f}$ and $\mathbf{F}_\mathrm{D}$, that is

$$\mathbf{F} = \mathbf{f} + \mathbf{F}_\mathrm{D}\,. \tag{67}$$

Consequently, if the baryonic (luminous) matter is evidently the main responsible for $\mathbf{f}$ this immediately implies that the other component $\mathbf{F}_\mathrm{D}$ is causally due to the permanent presence of some invisible matter which should be, of course, the DM. Therefore, according to Eq.(66), the DM is not strictly *inert*; on the contrary, it is gravitationally very *active* and this dynamicity is largely reflected in the manifestation of $\mathbf{F}_\mathrm{D}$ itself as an additional force. Therefore, if we take into account the universal equivalence between inertial mass and gravitational mass, we obtain from Eq.(67) the *net* force applied by DM on the *moving* ordinary matter: $\mathbf{F}_\mathrm{D} = \mathbf{F} - \mathbf{f}$. Precisely for that reason, the dynamic gravitational field-force $(\boldsymbol{\Lambda}, \mathbf{F}_\mathrm{D})$ should be *induced* by the motion of ordinary matter under the permanent gravitodynamical influence of DM, and in such a situation, Eq.(66) tells us that there is a direct *gravitodynamical link* between moving ordinary matter and DM. Accordingly, the gravitodynamical study of DM's effects on the moving ordinary matter should depend *exclusively* on the couple $(\boldsymbol{\Lambda}, \mathbf{F}_\mathrm{D})$, *i.e.*, Eqs.(65) and (66). It seems to us that the *incomprehension* of the mentioned process of DM's gravitodynamical influence has urged some researchers to conclude from this incomprehension that Newton's second law is not applicable to galactic scales. For example, in his very interesting pedagogical article entitled 'Does Dark Matter Really Exist?' published in Scientific American, August 2002, Milgrom wrote "When the acceleration is much larger than a_0, Newton's second law applies as usual: force is proportional to acceleration. But when the acceleration is small compared with a_0, Newton's second law is altered: force becomes proportional to the square of the acceleration. By this scheme, the force needed to impart a given acceleration is always smaller than Newtonian dynamics requires. To account for the observed accelerations in galaxies, MOND predicts a smaller force–hence, less gravity-producing mass–than Newtonian dynamics does (…). In this way, it can eliminate the need for dark matter." In one sense, Milgrom's claim was and still is correct since his a_0 is exceptionally comparable to the magnitude of (65), *i.e.*, $\Lambda = \frac{1}{2}a$ as we will see more explicitly. However, the exclusion of DM from *existence* is a mistake mainly caused by the above mentioned incomprehension because, here, Milgrom – as the father of MOND – has consciously or unconsciously omitted to think of the causal origin of a_0 at large-scale structures and the universality of the equivalence between inertial and gravitational mass!

As an additional clarification, let us show that *conceptually* Milgrom's law for $a \ll a_0$ is a particular case of (63). To do this, substitute Λ for Λ_0 in (63), and then multiply both sides by the quantity $2a_\mathrm{N}$ where a_N is the Newtonian acceleration in the MONDian sense. This results in $2\Lambda_0 a_\mathrm{N} = v^2(r)r^{-1}a_\mathrm{N}$. Finally, if we consider the specific situation $a^2 \approx v^2(r)r^{-1}a_\mathrm{N}$, we obtain the expected formula

$$a = \sqrt{2\Lambda_0 a_\mathrm{N}}\;. \tag{68}$$

This is remarkably very similar to Milgrom's law (57). We can also deduce an expression more general than Milgrom's interpolating function (58). In the CGA's context, we call such a function: *functional relation*. To this end, let us rewrite the vectorial Eq.(67) in terms of acceleration vector fields as follows

$$\boldsymbol{g} = \boldsymbol{a} + \boldsymbol{\Lambda} \,. \tag{69}$$

By applying the well-known definition of the scalar product of two vectors $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cos\theta$, we get

$$\boldsymbol{g}^2 = \boldsymbol{a}^2 + 2\|\boldsymbol{a}\| \cdot \|\boldsymbol{\Lambda}\| \cos\theta + \boldsymbol{\Lambda}^2 \,. \tag{70}$$

Where the angle θ is between $\boldsymbol{a}$ and $\boldsymbol{\Lambda}$, hence from (70) we get, in terms of magnitude, the equation

$$\|\boldsymbol{g}\| = \|\boldsymbol{a}\| \sqrt{1 + 2\|\boldsymbol{\Lambda}\| \cdot \|\boldsymbol{a}\|^{-1} \cos\theta + \boldsymbol{\Lambda}^2 \cdot \boldsymbol{a}^{-2}} \,. \tag{71}$$

Since according to (65), we always have $\|\boldsymbol{\Lambda}\| < \|\boldsymbol{a}\|$, thus by dividing both sides of (71) by $\|\boldsymbol{\Lambda}\|$ and substituting $x = \|\boldsymbol{\Lambda}\| \cdot \|\boldsymbol{a}\|^{-1}$ and $\eta^{-1} = \|\boldsymbol{g}\| \cdot \|\boldsymbol{\Lambda}\|^{-1}$, we obtain the expected functional relation

$$\eta \equiv \eta_\theta(x) = \frac{x}{\sqrt{1 + 2x\cos\theta + x^2}} \,, \qquad \theta \equiv (\widehat{\boldsymbol{a}, \boldsymbol{\Lambda}}) \,. \tag{72}$$

The functional relation (72) is important because it contains some physical and geometrical information about acceleration vector fields. To be precise, the modulus x defines the magnitude ratio of acceleration vector fields, and the argument θ defines the angular position of $\boldsymbol{\Lambda}$ with respect to $\boldsymbol{a}$ in the reference frame of the galaxy under consideration. Moreover, the expression of the functional relation (72) is more general than that of Milgram's interpolating function (58). Specifically, we can recover the expression (58) for the case $\theta = \pi/2$, *i.e.*, when $\boldsymbol{\Lambda} \perp \boldsymbol{a}$, and also we can recover the expression $\mu(x) = x(1+x)^{-1}$ of Famaey-Binney [37] for the case $\theta = 0$, *i.e.*, when $\boldsymbol{\Lambda} \parallel \boldsymbol{a}$. It follows from this that, in fact, the interpolating functions proposed by Milgram and Famaey-Binney are not fortuitously suggested or needed as a mathematical artifact but have a deep quantitative and qualitative role and meaning. Hence, MOND itself is *theoretically* incorporated in CGA.

6.1. Third typical galactic constant

Now, returning to our scenario. In order to make it heuristically and adequately close to reality and without specifying the shape, we attribute to our typical galaxy an average total mass $M(r) = 2 \times 10^{11}\, m_\odot$, average radius $r = 2R_0$, and the galactic (rotation) constants of $R_0 = 8.5\,\text{kpc}$ and $V_0 = 229\,\text{km s}^{-1}$. Thus our typical galaxy becomes realistically very comparable to the Milky Way.

Since the dynamic gravitational acceleration at the galactic scale (63) has a mathematical structure of a function, *i.e.*, $\Lambda \equiv \Lambda(r, v)$, therefore, in addition to the above adopted standard (rotation) constants, we have another parameter, which has the dimensions of a constant acceleration and is defined for $r = R_0$ and $v = V_0$, respectively, as follows:

$$\Lambda_0 \equiv \Lambda_0(R_0, V_0) = 1 \times 10^{-10}\ \text{ms}^{-2} \,. \tag{73}$$

According to (59), the numerical value of the third typical galactic constant Λ_0 is exactly equal to the minimal value of a_0 which is $1\times10^{-10}\ \mathrm{m\,s}^{-2}$. Moreover, we can evaluate the minimal value of the dynamic gravitational acceleration at the average radius $r = 2R_0$ with the help of average total mass $M(r)$ of the typical galaxy. By taking into account the expression (61) of rotational velocity at $r = R$, we can substitute in (63) to get:

$$\Lambda_{\min} = 4.857781 \times 10^{-11}\ \mathrm{ms}^{-2}\ . \tag{74}$$

Form (73) and (74), we obtain the following relation:

$$\Lambda_{\min} \approx \frac{1}{2}\Lambda_0\ \ . \tag{75}$$

It follows from (75) that Λ_0 may be used as an *acceleration-scale* at the galactic level, exactly like a_0 in MOND's framework. Hence, from all that, we arrive at the following result: according to CGA, MOND is the natural *sister* of DM with only a different family name! This result coincides perfectly with Milgrom's conclusion in the above-mentioned article "But it is possible that MOND follows from the dark matter paradigm in a different way. Time will tell."

6.2. Galactic rotation curves

In galaxies, particularly spiral ones, the presence of large quantities of invisible matter with a distribution different from baryonic (luminous) matter is now very well established. The primary observational evidence for the existence of DM comes from optical and 21 cm rotation curves of spiral galaxies which do not show the expected Keplerian drop-off at large radii but remain flat or even rise over their entire observed range (Faber [39]; Bosma [47]; Rubin *et al*., [48]). Theoretically, this DM is bar instabilities (Ostriker and Peebles [49]). However, the physical and chemical structure and properties of DM are still completely unknown.

To exemplify the practical applicability of the formalism developed here, we will first determine the shape and behavior of the rotation curve of our typical galaxy by deriving an expression for the rotational velocity $v(r)$ as a function of the radial distance r $(r \le 2R_0)$ from (63)

$$v(r) = \sqrt{2\Lambda r}\ , \qquad r \le 2R_0\ . \tag{76}$$

It seems heuristically more convenient for the purpose of our scenario to rewrite (76) for the special case $\Lambda = \Lambda_0$ to obtain the following expected function

$$v(r) = \sqrt{2\Lambda_0 r}\ , \qquad r \le 2R_0\ . \tag{77}$$

Now, with the help of Mathematica5, we plot the function (77) for $\Lambda_0 = 1 \times 10^{-10}\ \mathrm{ms}^{-2}$. With this aim, we have conveniently converted $[0;\ 17\,\mathrm{kpc}]$ to $[0;\ 5.25\times10^{20}\ \mathrm{m}]$ and $[0;\ 300\ \mathrm{km\,s}^{-1}]$ to $[0;\ 3\times10^{5}\ \mathrm{m\,s}^{-1}]$

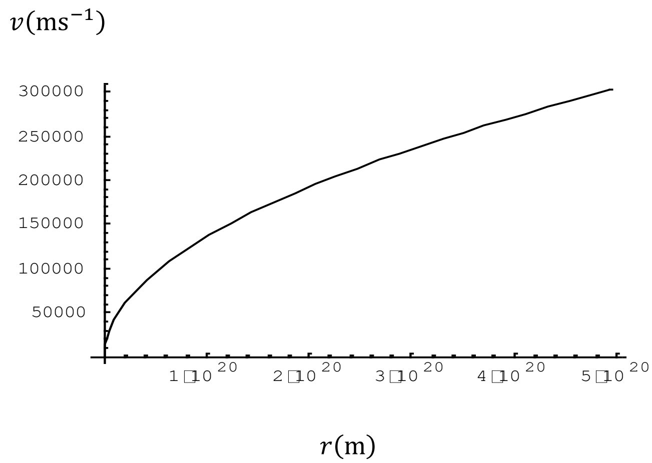

Figure 2. Rotation curve of a typical galaxy of average total mass $M(r) = 2 \times 10^{11} m_{\odot}$ and average radius $r = 17\ \mathrm{kpc}$ using the function (77) for $\Lambda_0 = 1 \times 10^{-10}\ \mathrm{m\,s^{-2}}$.

In Fig.2, we have constructed the rotation curve of our typical galaxy. This rotation curve obtained through CGA's formalism, illustrates the general shape and behavior of the majority of rotation curves of well-observed galaxies.

7. Tully-Fisher relation

Historically, Cepheid variable stars have been the primary means by which distances are measured over the local volume of space. However, beyond about $20\ \mathrm{Mpc}$, Cepheids become too faint, even for the Hubble Space Telescope, so astronomers realized that an alternative means of measuring distances was needed. Fortunately, one solution came from several astronomical observations that more conclusively show that for disc galaxies, the fourth power of the rotational velocity of stars moving around the core of the galaxy is proportional to the total luminosity of the galaxy ($v^4 \propto L$).This is well-known as the (empirical) Tully-Fisher relation [50]. Since L itself is proportional to the mass M of the galaxy, we can find $v^4 \propto M$. Similar to Milgrom's law under the expression (68), let us show that an equivalent expression to the Tully-Fisher relation may naturally occur from CGA's formalism as follows: we have from (61) $r = GM/v^2$, and after direct substitution in (63), we immediately get the desired relation.

$$v^4 = 2G\Lambda M. \tag{78}$$

This is essentially the Tully-Fisher relation expressed in a different way, which can be referred to as the 'mass-rotational velocity relation'. It is important to note that based on this law, there are two types of dependencies: implicit and explicit. Specifically, in equation (78), the quantity depends implicitly on the radial distance r, and it also explicitly depends on the mass $M \equiv M(r)$ which is contained within the radius r. Therefore, the rotational velocity is not entirely independent of the radial distance. A similar relation has already been discovered in the context of MOND for the specific case $\Lambda = \Lambda_0$.

$$v^4 = 2G\Lambda_0 M. \tag{79}$$

Since the proportionality coefficient $2G\Lambda_0$ is constant, we have $v^4 \propto M$. Accordingly, we deduce from equation (79) the two following significant relations:

$$v = (2G\Lambda_0 M)^{1/4} \ , \tag{80}$$

$$M = (2G\Lambda_0)^{-1} v^4 \ . \tag{81}$$

Now, let us illustrate graphically the double importance of the relation (79), *i.e.*, when v^4 and v are, respectively, considered as functions of the same (distributed) average total mass, M, of the typical galaxy in question. The first graph should have the same usual aspect and behavior as that defined by the original expression of the Tully-Fisher relation [50] and the second graph should have, in general, the same standard aspect and behavior as the observed rotation curves. With this aim, we have conveniently converted $[0; 2 \times 10^{11} m_\odot]$ to $[0; 4 \times 10^{41}\text{kg}]$ and $[0; 300\ \text{kms}^{-1}]$ to $[0; 3 \times 10^5\ \text{ms}^{-1}]$; and as before, with the help of Mathematica5, we plot the functions (79) and (80) for $\Lambda_0 = 1 \times 10^{-10}\ ms^{-2}$.

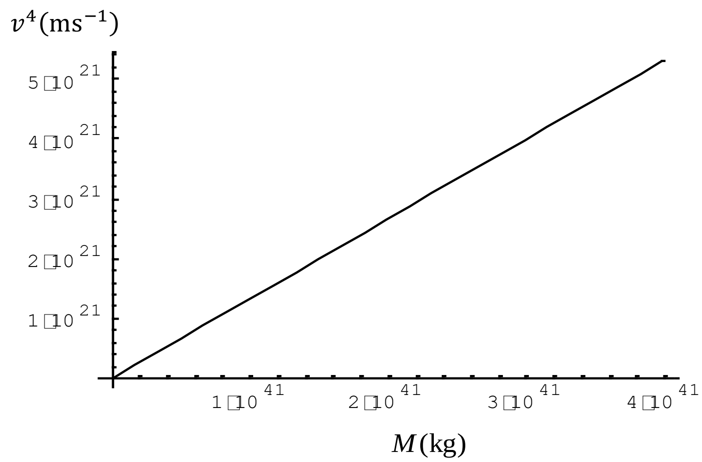

Figure 3. Fourth power of the rotational velocity, v^4, plotted versus the distributed average total mass, M, of a typical galaxy using mass-rotational velocity relation (79) for $\Lambda_0 = 1 \times 10^{-10}\ \text{m}\,\text{s}^{-2}$.

In Fig.3, we have plotted v^4 as a function of M. The graph perfectly illustrates the *correlation* between the fourth power of the rotational velocity and the distribution of the average total mass of a typical galaxy. This is in good agreement with the original empirical Tully-Fisher relation and the typical aspect and behavior of the observed curves.

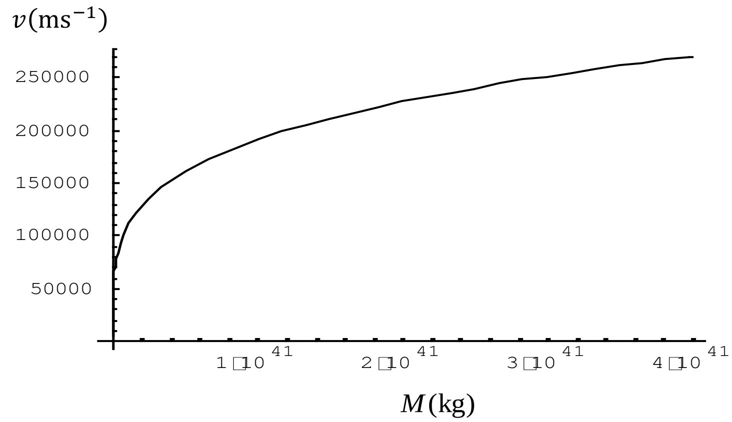

Figure 4. The rotational velocity, v, plotted versus the distributed average total mass M of a typical galaxy using relation (80) for $\Lambda_0 = 1\times10^{-10}\ \mathrm{m\,s^{-2}}$.

In Fig.4, we have plotted the rotational velocity, v, as a function of the distributed average total mass M of a typical galaxy. As we can see it, the illustrated rotation curve is in good agreement with the standard aspect and behavior of observed rotation curves. Moreover, this result has a number of interesting implications. First, according to Eqs.(79) and (80), there is an apparently universal correlation between baryonic mass and rotational velocity *through* the gravitodynamical influence of DM, which is phenomenologically reflected by the presence of the dynamic gravitational accelerations Λ_0 in Eqs.(79) and (80). Secondly, the mass-rotational velocity relation in Eqs.(79) or (80) provides the physical basis for the empirical Tully-Fisher relation that was unclear before the CGA advent.

8. Conclusion

The CGA could be regarded as an alternative gravitational model to compare with others that have already existed for a long time. As we have seen, the CGA enabled us to study and solve some old and new problems related to gravitational phenomena through a novel comprehension and interpretation of gravity itself. The famous Newton's law of gravitation was corrected and reformulated in a new more general form [1,2,3,4]. In the CGA's context, the dark matter 'hypothesis' and MOND paradigm have been finally reconciled with each other, and the empirical Tully-Fisher relation has found its physical basis.

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