ON THE VALIDITY OF THE RIEMANN HYPOTHESIS

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Abstract

In this paper, we have used the partial Euler product to examine the validity of the Riemann Hypothesis. The Dirichlet series over the Mobius function \( M(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) has been modified and represented in terms of the partial Euler product by progressively eliminating the numbers that first have a prime factor 2, then 3, then 5, .. up to the prime \( p_r \). It is shown that the series \( M(s) \) and the new series have the same region of convergence. Unlike the partial sum of \( M(s) \) that has irregular behavior, the partial sum of the new series exhibits regular behavior as \( p_r \) approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series to determine its region of convergence and to provide an answer for the validity of the Riemann Hypothesis.

1 Introduction

The Riemann zeta function \( \zeta(s) \) satisfies the following functional equation over the complex plain \([1]\)

\[
\zeta(1 - s) = 2(2\pi)^2 \cos(0.5s\pi)\Gamma(s)\zeta(s),
\]

where, \( s = \sigma + it \) is a complex variable and \( s \neq 0 \).

For \( \sigma > 1 \) (or \( \Re(s) > 1 \)), \( \zeta(s) \) can be expressed by the following series

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

or by the following product over the primes \( p_i \)’s

\[
\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right).
\]

where, \( p_1 = 2, \prod_{i=1}^{\infty} (1 - 1/p_i^s) \) is the Euler product and \( \prod_{i=1}^{r} (1 - 1/p_i^s) \) is the partial Euler product. The above series and product representations of \( \zeta(s) \) are absolutely convergent for \( \sigma > 1 \).

The region of the convergence can be extended to \( \Re(s) > 0 \) by using the alternating series \( \eta(s) \) where

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},
\]

1
and
\[ \zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s). \]  

(5)

One may notice that the term \(1 - 2^{1-s}\) is zero at \(s = 1\). This zero cancels the simple pole that \(\zeta(s)\) has at \(s = 1\) enabling the extension (or analog continuation) of the zeta function series representation over the critical strip \(0 < \Re(s) < 1\).

It is well known that all the non-trivial zeros of \(\zeta(s)\) are located in the critical strip \(0 < \Re(s) < 1\). Riemann stated that all the non-trivial zeros were very probably located on the critical line \(\Re(s) = 0.5\) [2]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function.

The Mobius function \(\mu(n)\) is defined as follows:
\[
\mu(n) = 1, \text{ if } n = 1.
\]
\[
\mu(n) = (-1)^k, \text{ if } n = \prod_{i=1}^{k} p_i, \text{ } p_i\text{'s are distinct primes.}
\]
\[
\mu(n) = 0, \text{ if } p^2 | n \text{ for some } p.
\]

The Dirichlet series \(M(s)\) with the Mobius function is defined as
\[
M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.
\]

(6)

This series is absolutely convergent to \(1/\zeta(s)\) for \(\Re(s) > 1\) and conditionally convergent to \(1/\zeta(s)\) for \(\Re(s) = 1\). The Riemann hypothesis is equivalent to the statement that \(M(s)\) is conditionally convergent to \(1/\zeta(s)\) for \(\Re(s) > 0.5\).

Gonek, Hughes and Keating [3] have done an extensive research into establishing a relationship between \(\zeta(s)\) and its partial Euler product for \(\Re(s) < 1\). Gonek stated ”Analytic number theorists believe that an eventual proof of the Riemann Hypothesis must use both the Euler product and functional equation of the zeta-function. For there are functions with similar functional equations but no Euler product, and functions with an Euler product but no functional equation.” In sections 4 and 5, we will present a functional equation for \(\zeta(s)\) using its partial Euler product. The method is based on writing the Euler product formula as follows
\[
1/\zeta(s) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \prod_{r+1}^{\infty} \left(1 - \frac{1}{p_i^s}\right).
\]

The above equation is valid for \(\sigma > 1\). To be able to represent \(\zeta(s)\) in term of its partial Euler product for \(\sigma \leq 1\), we have to replace the term \(\prod_{r}^{\infty} (1 - 1/p_i^s)\) with an equivalent one that allows the analytic continuation for the representation of \(\zeta(s)\) for \(\sigma \leq 1\). Thus, the new term, that we need to introduce to replace \(\prod_{r}^{\infty} (1 - 1/p_i^s)\), must have a zero that cancels the pole that \(\zeta(s)\) has at \(s = 1\). In the section 4, we will use the complex analysis to compute this new term. In section 5, we then use the new representation to compute the sum \(\sum_{i=1}^{r} p_i^\sigma\) for \(\sigma < 1\). This sum is then used to examine the validity of the Riemann Hypothesis.

In this paper, we claim the the Riemann Hypothesis is invalid. We support our claim by proving that the series \(M(\sigma)\) is divergent for \(\sigma < 1\). We achieved this results by introducing a method to represent the Dirichlet series \(M(s)\) (defined by Equation (6)) in terms of the partial
Euler product. This task is achieved (sections 2) by first eliminating the numbers that have the prime factor 2 to generate the series \( M(s, 2) \). For the series \( M(s, 2) \), we then eliminate the numbers with the prime factor 3 to generate the series \( M(s, 3) \), and so on, up to the prime number \( p_r \). In essence, we have applied the sieving technique to modify the series \( M(s) \) to include only the numbers with prime factors greater than \( p_r \). In section 3, we have shown that the series \( M(s) \) and the new series \( M(s, p_r) \) have the same region of convergence.

So far, the efforts to use the series \( M(s) \) to examine the validity of the Riemann Hypothesis have failed due to the irregular behavior of the partial sum of the series \( M(s) \). In section 6, we have shown that the partial sum of the new series \( M(s, p_r) \) exhibits regular behavior as \( p_r \) approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series and consequently determine its region of convergence. With this analysis, we have shown that non-trivial zeros can be found arbitrary close to the line \( s = 1 \).

2 Applying the Sieving Method to the Dirichlet Series \( M(s) \).

The Dirichlet series \( M(s) \) with the Mobius function is defined as

\[
M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},
\]

where \( \mu(n) \) is the Mobius function. Thus,

\[
M(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s} \ldots.
\]

It should be pointed out that our definition of \( M(s) \) is different from \( M(x) \) that is commonly defined in the literature as \( M(x) = \sum_{n \leq x} \mu(n) \).

Now, we introduce the series \( M(s, 2) \) by eliminating all the numbers that have a prime factor 2. Thus, \( M(s, 2) \) can be written as

\[
M(s, 2) = 1 - \frac{1}{3^s} - \frac{1}{5^s} + \frac{0}{7^s} - \frac{1}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s} \ldots.
\]

The analysis in this paper heavily relay on the testing the convergence of a series by comparing it with another conditionally convergent series. Therefore, rearrangement and permutation of the terms may have a significant impact on the region of convergence of both series. Therefore, it essential to have the same index for both series \( M(s) \) and \( M(s, 2) \) referring to the same term. Hence, the the above series can be re-written as

\[
M(s, 2) = 1 + \frac{0}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s} \ldots,
\]

or

\[
M(s, 2) = \sum_{n=1}^{\infty} \frac{\mu(n, 2)}{n^s},
\]

where

\[
\mu(n, 2) = \mu(n), \text{ if } n \text{ is an odd number},
\]
\( \mu(n, 2) = 0, \) if \( n \) is an even number.

The above series \( M(s, 2) \) can be further modified by eliminating all the numbers that have a prime factor 3 to get the series \( M(s, 3) \) where

\[
M(s, 3) = 1 - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{23^s} + \frac{0}{25^s},
\]

or more conveniently

\[
M(s, 3) = 1 + \frac{0}{2^s} - \frac{0}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s},
\]

and so on.

Let \( I(p_r) \) represent, in ascending order, the integers with distinct prime factors that belong to the set \( \{ p_i : p_i > p_r \} \). Let \( \{ 1, I(p_r) \} \) be the set of 1 and \( I(p_r) \) (for example, \( \{ 1, I(2) \} \) is the set of square free odd numbers), then we define the series \( M(s, p_r) \) as

\[
M(s, p_r) = \sum_{n=1}^{\infty} \frac{\mu(n, p_r)}{n^s},
\]

where

\[
\mu(n, p_r) = \mu(n), \text{ if } n \in \{ 1, I(p_r) \},
\]

otherwise, \( \mu(n, p_r) = 0. \)

It can be easily shown that \( M(s, p_r) \) converges absolutely for \( \Re(s) > 1 \) for every prime number \( p_r \). Furthermore, it can be shown that, for \( \Re(s) > 1 \), \( M(s, p_r) \) satisfies the following equation

\[
M(s) = M(s, p_r) \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^s} \right).
\]

Since

\[
M(s) = \prod_{i=1}^{\infty} \left( 1 - \frac{1}{p_i^s} \right),
\]

then we conclude that, for \( \Re(s) > 1 \), \( M(s, p_r) \) approaches 1 as \( p_r \) approaches infinity.

3 Convergence of the series \( M(s, p_r) \) within the strip \( 0.5 < \Re(s) \leq 1 \).

In this section, we will deal with the question of the conditional convergence of the series \( M(s, p_r) \) over the strip \( 0.5 < \Re(s) \leq 1 \). This task can be achieved by examining the convergence of the series \( M(s, p_r) \) along the real axis (or along the line \( 0.5 < \sigma \leq 1 \)). Theorems 1 and 2 establishes the relationship between the conditional convergence of the two series \( M(s) \) and \( M(s, p_r) \) for \( 0.5 < \sigma \leq 1 \).

**Theorem 1** For \( s = \sigma + i0 \), where \( 0.5 < \sigma \leq 1 \) and for every prime number \( p_r \), the series \( M(\sigma) \) converges conditionally if and only if the series \( M(\sigma, p_r) \) converges conditionally. Furthermore, \( M(\sigma) \) and \( M(\sigma, p_r) \) are related as follows

\[
M(\sigma) = M(\sigma, p_r) \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^s} \right).
\]
The proof of Theorem 1 is outlined in Appendix 1.

**Theorem 2** For \( s = \sigma + it \), where \( 0.5 < \sigma \leq 1 \) and for every prime number \( p_r \), the series \( M(s) \) converges conditionally if and only if the series \( M(s, p_r) \) converges conditionally. Furthermore, \( M(s) \) and \( M(s, p_r) \) are related as follows

\[
M(s) = M(s, p_r) \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^s} \right). \tag{11}
\]

The proof of the first part of Theorem 2 follows from the fact that \( M(s, p_r) \) is a Dirichlet series and consequently this series is conditionally convergent if and only if the series \( M(\sigma, p_r) \) is conditionally convergent.

The second part of the theorem can be proved by first defining \( M(s, p_r; N_1, N_2) \) as the sum

\[
M(s, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{\mu(n, p_r)}{n^s}. \tag{12}
\]

Then, we have

\[
M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N). \tag{13}
\]

If both series \( M(s, p_{r-1}) \) and \( M(s, p_r) \) are convergent, then as \( N \) approaches infinity, we obtain

\[
M(s, p_{r-1}) = M(s, p_r) \left( 1 - \frac{1}{p_r^s} \right).
\]

By repeating this process \( r - 1 \) times, we then obtain

\[
M(s) = M(s, p_r) \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^s} \right). \tag{11}
\]

Note that if we multiply both sides of the above equation by \( \prod_{i=1}^{r} (1 + p_i^{-s}) \)

\[
M(s, p_r) = \frac{1}{\zeta(s) \prod_{i=1}^{r} (1 - p_i^{-2s})} \prod_{i=1}^{r} \left( 1 + \frac{1}{p_i^s} \right). \tag{11}
\]

As \( p_r \) approaches infinity, we then have

\[
M(s, p_r) = \frac{\zeta(2s)}{\zeta(s)} \prod_{i=1}^{r} \left( 1 + \frac{1}{p_i^s} \right). \tag{11}
\]

It should be pointed out that the sieving method applied to the Dirichlet series with Mo-\( \text{bious function can be also applied to the Dirichlet series with Lioville function. The Dirichlet}
\)

\[
L(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{14}
\]

as
where
\[ \lambda(n) = 1, \text{ if } n = 1, \]
\[ \lambda(n) = 1, \text{ if } n \text{ has an even number of prime factors including multiplicities}, \]
\[ \lambda(n) = -1, \text{ if } n \text{ has an odd number of prime factors including multiplicities}. \]

Following the same process, we define the series \( L(s, p_r) \) as
\[ L(s, p_r) = \sum_{n=1}^{\infty} \frac{\lambda(n, p_r)}{n^s}, \tag{15} \]
where
\[ \lambda(n, p_r) = \lambda(n), \text{ if } n \in \{1, I(p_r)\}, \]
otherwise, \( \lambda(n, p_r) = 0. \)

It can be easily shown that \( L(s, p_r) \) converges absolutely for \( \Re(s) > 1 \) for every prime number \( p_r \). Furthermore, it can be also shown that, for \( \Re(s) > 1 \), \( L(s, p_r) \) satisfies the following equation
\[ L(s, p_r) = L(s) \prod_{i=1}^{r} \left( 1 + \frac{1}{p_i^s} \right). \]

It is well known in the literature that, on RH, we have
\[ \sum_{n \leq x} \lambda(n) = O(x^{1/2+\epsilon}), \]
where \( \epsilon \) is an arbitrary small number.

Using the above equation and following similar steps to those used for Theorems (1) and (2), we may obtain the following theorem.

**Theorem 3** For \( s = \sigma + it \), where \( 0.5 < \sigma \leq 1 \) and for every prime number \( p_r \), the series \( L(s) \) converges conditionally if and only if the series \( L(s, p_r) \) converges conditionally. Furthermore, \( L(s) \) and \( L(s, p_r) \) are related as follows
\[ L(s, p_r) = L(s) \prod_{i=1}^{r} \left( 1 + \frac{1}{p_i^s} \right). \tag{16} \]

4 Functional representation of \( \zeta(s) \) using its partial Euler product.

Theorem 1 of the previous section provides a relationship between \( \zeta(s) = 1/M(s) \) and the partial Euler product \( \prod_{i=1}^{r}(1 - 1/p_i^s) \). In this section and the following one, we will derive a functional representation for \( \zeta(s) \) using its partial Euler product. In this section, we will use the prime counting function to compute this functional representation and in the following section we will use the von Mangoldt function to achieve the same task. This functional representation is then used to compute the sum \( \sum_{i=1}^{r} p_i^\sigma \) for \( \sigma < 1 \). In section, 6 we will use this sum to show that the series \( M(\sigma, p_r) \) is diverges for \( \sigma < 1 \).
We will start this task by first writing \( \zeta(s) \) for \( \sigma > 1 \) as follows

\[
\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right) \prod_{r+1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \tag{17}
\]

For \( \sigma > 0.5 \), we have

\[
\log \prod_{i=r+1}^{r+2} \left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r+1}^{r+2} \log \left(1 - \frac{1}{p_i^s}\right),
\]
or

\[
\log \prod_{i=r+1}^{r+2} \left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r+1}^{r+2} \left( -\frac{1}{p_i^s} - \frac{1}{2p_i^{2s}} - \frac{1}{3p_i^{3s}} - \cdots \right).
\]

Let \( \delta \) be defined as the sum

\[
\delta = \sum_{i=r+1}^{r+2} \left( -\frac{1}{2p_i^{2s}} - \frac{1}{3p_i^{3s}} - \frac{1}{4p_i^{4s}} \cdots \right). \tag{18}
\]

Thus,

\[
\log \prod_{i=r+1}^{r+2} \left(1 - \frac{1}{p_i^s}\right) = -\sum_{i=r+1}^{r+2} \frac{1}{p_i^s} + \delta. \tag{19}
\]

Since \( |\delta| < \sum_{n=p_{r+1}}^{\infty} \left( \frac{1}{2n^{2s}} + \frac{1}{3n^{3s}} + \frac{1}{4n^{4s}} \cdots \right) \), thus \( \delta = O(p_{r+1}^{1-2\sigma}/(2\sigma - 1)) \). Furthermore, if \( 2\sigma - 1 \) is a fixed positive number, then \( \delta = O(p_{r+1}^{1-2\sigma}) \). It should be pointed out that for \( \sigma = 0.5 \) and \( t \neq 0 \), \( \delta \) is convergent to a finite number by the virtue of the Prime Number Theorem.

Using the Prime Number Theorem (PNT) with a suitable constant \( a > 0 \), the number of primes less than \( x \) is given by [4, page 43]

\[
\pi(x) = \text{Li}(x) + O \left( xe^{-a\sqrt{\log x}} \right), \tag{20}
\]
or

\[
\pi(x) = \text{Li}(x) + O \left( x/(\log x)^k \right), \tag{21}
\]
where \( \text{Li}(x) \) is the Logarithmic Integral of \( x \) and \( k \) is a number greater than zero.

Using Stieltjes integral [5], we may write the sum \( \sum_{i=r+1}^{r+2} \frac{1}{p_i^s} \) for \( \sigma > 1 \) as follows

\[
\sum_{i=r+1}^{r+2} \frac{1}{p_i^s} = \int_{x=p_{r+1}}^{x=p_{r+2}} \frac{d\pi(x)}{x^\sigma}. \tag{22}
\]

Using Equation (21) for the representation of \( \pi(x) \), we may then write the integral in Equation (22) as [5, Theorem 2, page 57]

\[
\sum_{i=r+1}^{r+2} \frac{1}{p_i^s} = \int_{x=p_{r+1}}^{x=p_{r+2}} \frac{1}{x^\sigma} \frac{1}{\log x} dx + O \left( \frac{1}{(\log p_{r+1})^k} \right), \tag{23}
\]
where \( k \) is a number greater than zero. Therefore,

\[
\sum_{i=r+1}^{r+2} \frac{1}{p_i^s} = \int_{x=p_{r+1}}^{\infty} \frac{1}{x^\sigma} \frac{1}{\log x} dx - \int_{x=p_{r+2}}^{\infty} \frac{1}{x^\sigma} \frac{1}{\log x} dx + O \left( \frac{1}{(\log p_{r+1})^k} \right). \tag{24}
\]
Recalling that the Exponential Integral $E_1(r)$ is given by

$$E_1(r) = \int_r^\infty \frac{e^{-u}}{u} \, du,$$

and using the substitutions $u = (\sigma - 1) \log p_r$, $du = (\sigma - 1) dx/x$ and $x^\sigma/x = e^u$, then for $\sigma > 1$, we may write Equation (24) as

$$\sum_{i=1}^{r^2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r1}) - E_1((\sigma - 1) \log p_{r2}) + O\left(\frac{1}{(\log p_{r1})^k}\right). \quad (25)$$

Combining Equations (19) and (25) and noting that, for $\sigma > 1$, $E_1((\sigma - 1) \log p_{r2})$ approaches zero as $p_{r2}$ approaches infinity, we may write Equation (17) for $\sigma > 1$ as

$$-\log \zeta(\sigma) = \sum_{i=1}^r \log \left(1 - \frac{1}{p_i^\sigma}\right) - \sum_{i=r+1}^\infty \frac{1}{p_i^\sigma} + \delta,$$

or

$$\log \zeta(\sigma) + \sum_{i=1}^r \log \left(1 - \frac{1}{p_i^\sigma}\right) - E_1((\sigma - 1) \log p_{r+1}) = \epsilon,$$

where $\epsilon = O(1/(\log p_{r1})^k)$ is an arbitrarily small number attained by setting $p_r$ sufficiently large. Therefore,

$$\zeta(\sigma) \prod_{i=1}^r \left(1 - \frac{1}{p_i^\sigma}\right) \exp(-E_1((\sigma - 1) \log p_{r+1})) = 1 + \epsilon. \quad (26)$$

As $p_r$ approaches infinity, $\epsilon$ approaches zero. Hence, the right side of the above equation approaches $1$ as $p_r$ approaches infinity.

Similarly, for $\Re(s) > 1$, we can use the following expression for $E_1(s)$

$$E_1(s) = \int_1^\infty \frac{e^{-xs}}{x} \, dx,$$

to show that

$$\lim_{r \to \infty} \left\{ \zeta(s) \prod_{i=1}^r \left(1 - \frac{1}{p_i^\sigma}\right) \exp(-E_1((s - 1) \log p_{r+1})) \right\} = 1. \quad (27)$$

Let the function $G(s, p_r)$ be defined as

$$G(s, p_r) = \zeta(s) \prod_{i=1}^r \left(1 - \frac{1}{p_i^\sigma}\right) \exp(-E_1((s - 1) \log p_{r+1})) \quad (28)$$

where, $G(s, p_r)$ is a regular function for $\Re(s) > 1$. Referring to Equation (27), the function $G(s, p_r)$ approaches $1$ as $p_r$ approaches infinity. It should be noted that, for each $p_r$, the function $\exp(-E_1((s - 1) \log p_{r+1}))$ is an entire function, the function $\zeta(s)$ is analytic everywhere except at $s = 1$ and the function $\prod_{i=1}^r (1 - 1/p_i)$ is analytic for $\Re(s) > 0$. Thus, for any $\sigma > 1$, the function $G(s, p_r)$ can be considered as a sequence of analytic functions. Furthermore, as $p_r$ (or $r$) approaches infinity, this sequence is uniformly convergent over the half plane with $\sigma > 1 + \epsilon$ (where, $\epsilon$ is an arbitrary small number). Therefore, by the virtue of the Weiestrass
theorem, the limit is also analytic function [6] (Weiestrass theorem states that if the function sequence \( f_n \) is analytic over the region \( \Omega \) and \( f_n \) is uniformly convergent to a function \( f \), then \( f \) is also analytic on \( \Omega \) and \( f_n \) converges uniformly to \( f \) on \( \Omega \)). If we define this limit as \( G(s) \), where

\[
G(s) = \lim_{r \to \infty} G(s, p_r)
\]

then, \( G(s) \) is analytic over the half plane \( \Re(s) > 1 \) and it is equal to 1 by the virtue of Equation (27).

The Prime Number Theorem (PNT) allows us to extend the above results to the line \( s = 1 + it \). Moreover, we will show that if RH is valid, then for the strip \( s = \sigma + it \) where, \( 0.5 < \sigma < 1 \), the above results will also be valid with the limit of \( G(s, p_r) \) is 1 as \( p_r \) approaches infinity.

We will start this task by showing that although both \( \zeta(s) \) and \( E_1((s - 1) \log p_{r+1}) \) have a singularity at \( s = 1 \), the product \( G(s, p_r) \) has a removable singularity at \( s = 1 \) for every \( p_r \). This can be shown by first expanding \( \zeta(s) \) as a Laurent series about its singularity at \( s = 1 \)

\[
\zeta(s) = \frac{1}{s - 1} + \gamma - \gamma_1 (s - 1) + \gamma_2 \frac{(s - 1)^2}{2!} - \gamma_3 \frac{(s - 1)^3}{3!} + \ldots,
\]

where \( \gamma \) is the Euler-Mascheroni constant and \( \gamma_i \)'s are the Stieltjes constants. For \( s = 1 + \epsilon \), where \( \epsilon = \epsilon_1 + i\epsilon_2 \), \( \epsilon_1 \) and \( \epsilon_2 \) are arbitrary small numbers, the above equation can be written as

\[
\zeta(s) = \frac{1}{\epsilon} + \gamma - \gamma_1 \epsilon + \gamma_2 \frac{\epsilon^2}{2!} - \gamma_3 \frac{\epsilon^3}{3!} + \ldots
\]

Furthermore, for \( \sigma > 1 \), using the definition of the Exponential Integral, we may write \( E_1(s) \) as

\[
E_1(s) = -\gamma - \log s + s - \frac{s^2}{2!} + \frac{s^3}{3!} - \frac{s^4}{4!} + \ldots
\]

Thus, for \( s = 1 + \epsilon \), we have

\[
\exp (-E_1((s - 1) \log p_r)) = e^\gamma \epsilon \log p_r \exp \left( -\epsilon \log p_r \frac{(\epsilon \log p_r)^2}{2!} - \frac{(\epsilon \log p_r)^3}{3!} + \ldots \right).
\]

By taking the product \( \zeta(s) \exp (-E_1((s - 1) \log p_r)) \) and allowing \( \epsilon \) to approach zero, we then obtain at \( s = 1 \) (in the same sense as computing \( \sin x/x \) at \( x = 0 \))

\[
\zeta(s) \exp (-E_1((s - 1) \log p_r)) = e^\gamma \log p_r.
\]

However, it is well known that the partial Euler product at \( s = 1 \) can be written as [8]

\[
\prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) = \frac{e^{-\gamma}}{\log p_r} + O \left( \frac{1}{(\log p_r)^2} \right).
\]

Multiplying Equations (34) and (35), we may conclude that at \( s = 1 \), \( G(s, p_r) \) approaches 1 as \( p_r \) approaches infinity. Furthermore, for \( s = 1 + it \) and \( t \neq 1 \), the value of \( \exp(-E_1(it \log p_r)) \) approaches 1 as \( p_r \) approaches infinity and since

\[
\lim_{r \to \infty} \left\{ \zeta(s) \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^s} \right) \right\} = 1,
\]
Therefore, for \( s = 1 + it \), we have the following

\[
\lim_{r \to \infty} G(s, p_r) = \lim_{r \to \infty} \left\{ \zeta(s) \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^\sigma} \right) \exp \left( -E_1((s-1) \log p_{r+1}) \right) \right\} = 1.
\]

So far, we have shown that the function \( G(s, p_r) \) is uniformly convergent to 1 when \( \Re(s) > 1 \) and using PNT, \( G(s, p_r) \) is convergent to 1 for \( \Re(s) = 1 \). In the following, we will show that, assuming the validity of the Riemann Hypothesis, the function \( G(s, p_r) \) is uniformly convergent to 1 for every value of \( s \) with \( \Re(s) > 0.5 + \epsilon \), where \( \epsilon \) is an arbitrary small number. Toward this goal, we will first show that the function \( G(s, p_r) \) is convergent for any value of \( s \) on the real axis with \( \sigma > 0.5 \). This can be achieved by first writing the expressions for \( G(\sigma, p_{r1}) \) and \( G(\sigma, p_{r2}) \) (where \( r_2 \) is an arbitrary large number greater than \( r_1 \))

\[
G(\sigma, p_{r1}) = \zeta(\sigma) \exp \left( -E_1((\sigma-1) \log p_{r1+1}) \right) \prod_{i=1}^{r_1} \left( 1 - \frac{1}{p_i^\sigma} \right),
\]

\[
G(\sigma, p_{r2}) = \zeta(\sigma) \exp \left( -E_1((\sigma-1) \log p_{r2+1}) \right) \prod_{i=1}^{r_2} \left( 1 - \frac{1}{p_i^\sigma} \right),
\]

Since the function \( G(s, p_r) \) is analytic that is not equal to 0 for \( \sigma > 0.5 \), hence we can divide Equation (37) by Equation (36) and then take the logarithm to obtain

\[
\log \left( \frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})} \right) = E_1((\sigma-1) \log p_{r1+1}) - E_1((\sigma-1) \log p_{r2+1}) + \log \left( \prod_{i=r1+1}^{r2} \left( 1 - \frac{1}{p_i^\sigma} \right) \right).
\]

To compute the logarithm of the partial Euler product in Equation (38), we recall Equation (19)

\[
\log \prod_{r1+1}^{r2} \left( 1 - \frac{1}{p_i^\sigma} \right) = - \sum_{i=r1+1}^{r2} \frac{1}{p_i^\delta} + \delta,
\]

where \( \delta = O(p_r^{-2\sigma}/(2\sigma - 1)) \). Furthermore, on RH, we have

\[
\pi(x) = \text{Li}(x) + O \left( \sqrt{x} \log x \right),
\]

where \( \text{Li}(x) \) is the Logarithmic Integral of \( x \). Using Equation (39) for the representation of the prime counting function, we may then obtain (Appendix 2)

\[
\sum_{i=r1+1}^{r2} \frac{1}{p_i^\sigma} = E_1((\sigma-1) \log p_{r1+1}) - E_1((\sigma-1) \log p_{r2}) + \varepsilon,
\]

where \( \varepsilon = O \left( \frac{1}{(\sigma-0.5)^2} p_r^{-0.5-\sigma} \log p_{r1} \right) \). Hence, Equation (38) can be written as

\[
\log \left( \frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})} \right) = \varepsilon + \delta + E_1((\sigma-1) \log p_{r2}) - E_1((\sigma-1) \log p_{r2+1}).
\]

Since, for \( \sigma > 0.5 + \epsilon, \varepsilon + \delta \) and \( E_1((\sigma-1) \log p_{r2}) - E_1((\sigma-1) \log p_{r2+1}) \) can be made arbitrary small by choosing \( p_{r1} \) arbitrary large, thus the limit of \( G(\sigma, p_r) \) exists as \( p_r \) approaches infinity and it is given by

\[
G(\sigma) = \lim_{r \to \infty} G(\sigma, p_r)
\]
This proves that, on RH, \( G(\sigma, p_r) \) is convergent as \( p_r \) approaches infinity and thus \( G(\sigma) \) exists for \( \sigma > 0.5 \). In Appendix 3, we have shown that, on RH and for \( \Re(s) > 0.5 \), we have

\[
\sum_{i=1}^{r^2} \frac{1}{p_i^s} = E_1((s-1) \log p_{r+1}) - E_1((s-1) \log p_{r+2}) + \varepsilon,
\]

where \( \varepsilon = O\left(\frac{t+1}{(\sigma-0.5)^2} p_{r+1}^{-0.5-\sigma} \log p_{r+1}\right) \). Thus, we can follow the same steps and show that \( G(s, p_r) \) is convergent as \( p_r \) approaches infinity and thus \( G(s) \) exists for \( \Re(s) > 0.5 \).

It should be noted that, while the function sequence \( G(s, p_r) \) is not uniformly convergent when the region of convergence is extended all the way to the line \( \sigma = 0.5 \), it is however uniformly convergent for any strip with \( \sigma > 0.5 + \varepsilon \), where \( \varepsilon \) is an arbitrary small number. This follows from the fact that \( \varepsilon \) (or, the \( O \) term) is bounded for any \( \sigma > 0.5 + \varepsilon \). Since \( G(s, p_r) \) is analytic for \( \Re(s) > 0 \) and it is uniformly convergent for \( \Re(s) > 0.5 + \varepsilon \), thus \( G(s) \) is analytic for the half right complex plain with \( \Re(s) > 0.5 + \varepsilon \) (Weiestrass theorem [6]). Since we have shown that \( G(s) = 1 \) for \( \Re(s) \geq 1 \), thus on RH, \( G(s) = 1 \) for \( \Re(s) > 0.5 + \varepsilon \). Hence, we have the following theorem

**Theorem 4** For \( s = \sigma + it \) and \( \sigma > 0.5 \), the following holds if RH is valid

\[
\lim_{r \to \infty} \left\{ \zeta(s) \prod_{i=1}^{r^2} \left( 1 - \frac{1}{p_i^s} \right) \exp\left(-E_1((s-1) \log p_{r+1})\right) \right\} = 1. \tag{41}
\]

\[
\lim_{r \to \infty} \{ M(s, p_r) \exp(E_1((s-1) \log p_{r+1})) \} = 1. \tag{42}
\]

It should be pointed out that Theorem 4 can be generalized to the case where there are no non-trivial zeros for values of \( s \) with \( \Re(s) > a \) (where, \( a > 0.5 \)). For this case, Equation (41) is valid for every \( s \) with \( \Re(s) > a \) and \( \varepsilon \) in Appendix 3 is given by \( O\left(\frac{t+1}{(\sigma-a)^2} p_{r+1}^{-a-\sigma} \log p_{r+1}\right) \).

Equation (41) of Theorem 4 can be written as follows

\[
\log \zeta(s) + \log \prod_{i=1}^{r^2} \left( 1 - \frac{1}{p_i^s} \right) - E_1 ((s-1) \log p_{r+2+1}) = 0,
\]

where the equality of both sides is attained as \( r_2 \) (or \( p_{r+2} \)) approaches infinity. It should be pointed out that both functions \( \log \zeta(s) \) and \( E_1((s-1) \log p_{r+2+1}) \) have a branch cut along the real axis where \( 0.5 \leq \sigma < 1 \), while the difference (i.e. \( \log \zeta(s) - E_1((s-1) \log p_{r+2+1}) \)) does not have a branch cut. For \( r < r_2 \), the above equation can be then written as

\[
\log \zeta(s) = E_1((s-1) \log p_{r+2+1}) - \sum_{i=1}^{r} \log \left( 1 - \frac{1}{p_i^s} \right) - \sum_{i=r+1}^{r^2} \log \left( 1 - \frac{1}{p_i^s} \right),
\]

Since, on RH and for \( \Re(s) > 0.5 \), (refer to Appendix 3)

\[
- \sum_{i=r+1}^{r^2} \log \left( 1 - \frac{1}{p_i^s} \right) = \sum_{i=r+1}^{r^2} \frac{1}{p_i^s} + \delta = E_1 ((s-1) \log p_{r+1}) - E_1 ((s-1) \log p_{r+2}) + \varepsilon + \delta
\]
where \( \varepsilon = O \left( \frac{t+1}{(\sigma-0.5)^2} p_r^{0.5-\sigma} \log p_r \right) \) and \( \delta = O(p_r^{1-2\sigma}/(1-2\sigma)) \), therefore

\[
\log \zeta(s) = - \sum_{i=1}^r \log \left( 1 - \frac{1}{p_i^{\sigma}} \right) + E_1 ((s - 1) \log p_{r+1}) + O \left( \frac{t+1}{(\sigma-0.5)^2} p_r^{0.5-\sigma} \log p_r \right). \tag{43}
\]

Equation (43) represents well the singularity of \( \log \zeta(s) \) at \( s = 1 \) and it allows analytic continuation for values of \( s \) with \( \Re(s) < 1 \). This analytic continuation should extend all the way to the non-trivial zeros with the highest value of \( \sigma \). Unfortunately, Equation (43) poorly represents \( \zeta(s) \) in the vicinity of the non-trivial zeros as the \( O \) term grows much faster than the growth of \( \log \zeta(s) \) in the vicinity of the simple non-trivial zeros. In the next section, we will use the von Mangoldt function to provide a better representation for \( \log \zeta(s) \) in the vicinity of the non-trivial zeros.

5 Partial Euler product functional representation of \( \zeta(s) \) using von Mangoldt function.

The derivation of Equation (43) was based on computing the sum \( \sum_{i=r+1}^{r^2} 1/p_i^s \) (Appendix 3) as follows

\[
\sum_{i=r+1}^{r^2} \frac{1}{p_i^s} = \int_{p_{r+1}}^{p_{r^2}} \frac{d\pi(x)}{x^s} = \int_{p_{r+1}}^{p_{r^2}} \frac{1}{x^s \log x} dx + \int_{p_{r+1}}^{p_{r^2}} \frac{1}{x^{s \delta}} \log x \ dx.
\]

The above can be also computed using the von Mangoldt function \( \Lambda(n) \) (where \( \Lambda(n) = \log p \), if \( n = p^k \) for some prime \( p \) and integer \( k \geq 1 \), otherwise, \( \Lambda(n) = 0 \)) to obtain

\[
\sum_{i=r+1}^{r^2} \frac{1}{p_i^s} = \sum_{n=r+1}^{r^2} \frac{1}{n^s \log n} \Lambda(n) + \Delta, \tag{44}
\]

where \( \Delta \) is added to eliminate the contribution by the terms of the form \( m^{-s} \), where \( m = p^k \) and \( 2 \leq k < |\log_2 p_{r^2}| + 1 \). In other words, \( \Delta \) is given by

\[
\Delta = \sum_{p_i = \lceil \sqrt[p_r]{p_{r^2}} \rceil}^{\lceil \sqrt[p_r]{p_{r^2}} \rceil} \frac{1}{2p_i^{2s}} + \sum_{p_i = \lceil \sqrt[p_r]{p_{r^2}} \rceil}^{\lceil \sqrt[p_r]{p_{r^2}} \rceil} \frac{1}{3p_i^{3s}} + \ldots + \sum_{p_i = \lceil \sqrt[p_r]{p_{r^2}} \rceil}^{\lceil \sqrt[p_r]{p_{r^2}} \rceil} \frac{1}{L p_i^{xs}}, \tag{45}
\]

where \( L = |\log_2 p_{r^2}| + 1 \) and \( |x| \) is the integer value of \( x \). The order of \( \Delta \) is determined by the order of the first term \( \sum_{p_i = \lceil \sqrt[p_r]{p_{r^2}} \rceil}^{\lceil \sqrt[p_r]{p_{r^2}} \rceil} 0.5/p_i^{2s} \). Thus, the order of \( \Delta \) can be computed (in the same way the order of \( \delta \) was computed) to obtain \( \Delta = O((\sqrt[p_r]{p_{r^2}})^{1-2\sigma}/(2\sigma-1)) = O(p_r^{0.5-\sigma}/(2\sigma-1)) \).

Furthermore, if \( 2\sigma - 1 \) is a fixed positive number, then \( \Delta = O(p_r^{0.5-\sigma}) \). It should be pointed out that for \( \sigma = 0.5 \) and \( t \neq 0 \), \( \Delta \) is convergent to a finite number by the virtue of PNT.

Since the Chebyshev function \( \psi(x) \) is given by the following sum

\[
\psi(x) = \sum_{n=1}^{x} \Lambda(n)
\]

therefore, using the Stieltjes integral, one may write the sum of Equation (44) as the following integral

\[
\sum_{i=r+1}^{r^2} \frac{1}{p_i^s} = \int_{p_{r+1}}^{p_{r^2}} \frac{1}{x^s \log x} d\psi(x) + \Delta, \tag{46}
\]
where $\psi(x)$ is also given by [1]

$$
\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} + \sum_n \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \tag{47}
$$

It should be pointed out that the first term $x$ in Equation (47) is attributed to the pole of $\zeta(s)$ at $s = 1$, the sum over $\rho$ (or non-trivial zeros) is attributed to the non-trivial zeros in the critical strip and the sum over $n$ is attributed to the trivial zeros. Hence, Equation (46) can be written as

$$
\sum_{i=r_1}^{r_2} \frac{1}{p_i^s} = \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx - \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} d\left(\sum_{\rho} \frac{x^\rho}{\rho}\right) + \Delta \tag{48}
$$

where the contribution by the last two terms of Equation (47) is negligible compared with the term $\Delta$. In Appendix (3), we have shown that

$$
\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx = E_1((s - 1) \log p_{r_1}) - E_1((s - 1) \log p_{r_2}). \tag{49}
$$

For the integral with the sum over $\rho$, we first compute the integral over the $\rho$'s with $|\Im(\rho)| < T$. Thus, we have

$$
\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx \left(\sum_{|\Im(\rho)| < T} \frac{x^\rho}{\rho}\right) = \sum_{|\Im(\rho)| < T} \left(\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} d\left(\frac{x^\rho}{\rho}\right)\right). \tag{50}
$$

For the above integral, for each $\rho$, $|x^\rho/\rho|$ is a continuous function and bounded over the range $p_{r_1} \leq x \leq p_{r_2}$, therefore the interchange between the differentiation and summation is justified (alternatively, one may integrate by parts to get the same results, where the sum becomes the integrand and the differentiation is applied to the term $1/(x^s \log x)$ instead of the sum). Furthermore, for each $\rho$, $\Re(s)$ is higher than $\Re(\rho)$, therefore $\int_{p_{r_1}}^{p_{r_2}} |x^{\rho - 1}/(x^s \log x)| dx$ is convergent as $p_{r_2}$ approaches infinity. Hence, the interchange between the integral and the sum is justified. Therefore, Equation (50) can be written as

$$
\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} d\left(\sum_{|\Im(\rho)| < T} \frac{x^\rho}{\rho}\right) = \sum_{|\Im(\rho)| < T} \left(E_1((s - \rho) \log p_{r_1}) - E_1((s - \rho) \log p_{r_2})\right). \tag{51}
$$

In Appendix 4, we have shown that the sum on the right side of (51) is convergent as $T$ approaches infinity. Thus,

$$
\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} d\left(\sum_{\rho} \frac{x^\rho}{\rho}\right) = \sum_{\rho} \left(E_1((s - \rho) \log p_{r_1}) - E_1((s - \rho) \log p_{r_2})\right). \tag{52}
$$

Consequently,\n
$$
\sum_{i=r_1}^{r_2} \frac{1}{p_i^s} = E_1((s-1) \log p_{r_1}) - E_1((s-1) \log p_{r_2}) - \sum_{\rho} \left(E_1((s - \rho) \log p_{r_1}) - E_1((s - \rho) \log p_{r_2})\right) + \Delta, \tag{53}
$$

where $\Delta = O(p_{r_1}^{0.5-\sigma})$. If the function $J(s, p_{r_1}, p_{r_2})$ is defined as follows

$$
J(s, p_{r_1}, p_{r_2}) = \sum_{i=r_1}^{r_2} \frac{1}{p_i^s} - E_1((s - 1) \log p_{r_1}) + E_1((s - 1) \log p_{r_2}), \tag{54}
$$
then
\[ J(s, p_{r1}, p_{r2}) = \sum_{\rho} (E_1((s - \rho) \log p_{r1}) - E_1((s - \rho) \log p_{r2})) + \Delta. \] (55)

We notice that the function \( J(s, p_{r1}, p_{r2}) \) is analytic for every \( p_{r1}, p_{r2} \) and \( s \). This follows from the fact that although the functions \( E_1((s - 1) \log p_{r1}) \) and \( E_1((s - 1) \log p_{r2}) \) have a branch cut on the negative real axis, the difference does not have a branch cut. Moreover, although the functions \( E_1((s - 1) \log p_{r1}) \) and \( E_1((s - 1) \log p_{r2}) \) have a singularity at \( s = 1 \), the difference has a removable singularity at \( s = 1 \). This follows from the fact that as \( s \) approaches 1, the difference can be written as
\[ E_1((s - 1) \log p_{r1}) - E_1((s - 1) \log p_{r2}) = -\log ((1 - s) \log p_{r1}) - \gamma + \log ((1 - s) \log p_{r2}) + \gamma \]
or,
\[ E_1((s - 1) \log p_{r1}) - E_1((s - 1) \log p_{r2}) = -\log \log p_{r1} + \log \log p_{r2} \] (56)
Therefore, the function \( J(s, p_{r1}, p_{r2}) \) is analytic for every \( p_{r1}, p_{r2} \) and \( s \).

Referring to Appendix (4), we notice that for every \( s \) with \( \Re(s) > \max \Re(\rho) \), the term \( \sum_{\rho} (E_1((s - \rho) \log p_{r1}) - E_1((s - \rho) \log p_{r2})) \) approaches zero as \( p_{r1} \) approaches infinity. Thus, for \( \Re(s) > \max \Re(\rho) \), we have
\[ \sum_{i=1}^{r_2} \frac{1}{p_i^s} = E_1((s - 1) \log p_{r2+1}) - E_1((s - 1) \log p_{r1}) + O(p_{r1}^{-\sigma + \max \Re(\rho)}). \] (57)

To compute \( \log \zeta(s) \) using Equation (47), we recall Equation (41) of Theorem 1. Thus, for every \( s \) with \( \Re(s) > \max \Re(\rho) \), we have
\[ \log \zeta(s) = E_1((s - 1) \log p_{r2+1}) - \sum_{i=1}^{r_2} \log \left(1 - \frac{1}{p_i^s}\right), \]
where the equality of both sides is attained as \( p_{r2} \) approaches infinity. Alternatively,
\[ \log \zeta(s) = E_1((s - 1) \log p_{r2+1}) - \sum_{i=1}^{r} \log \left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r+1}^{r_2} \log \left(1 - \frac{1}{p_i^s}\right). \]
Hence,
\[ \log \zeta(s) = E_1((s - 1) \log p_{r2+1}) - \sum_{i=1}^{r} \log \left(1 - \frac{1}{p_i^s}\right) + \sum_{i=r+1}^{r_2} \frac{1}{p_i^s} + \delta. \]
Consequently, using Equations (46), (48), (49) and (52) (and noting that when \( \Re(s - \rho) > 0 \) for every \( \rho \), the sum \( \sum_{\rho} E_1((s - \rho) \log p_{r2}) \) approaches zero as \( p_{r2} \) approaches infinity), we have the following theorem

**Theorem 5** If \( \Re(s - \rho) > 0 \) for every non-trivial zero \( \rho \), then
\[ \log \zeta(s) = -\sum_{i=1}^{r} \log \left(1 - \frac{1}{p_i^s}\right) + E_1((s - 1) \log p_{r+1}) - \sum_{\rho} E_1((s - \rho) \log p_{r+1}) + O\left(p_{r}^{0.5 - \sigma}\right). \] (58)
where \( \sigma = \Re(s) \) and the \( O \) term is given by \( \delta + \Delta \).
The differentiation of \( \log \zeta(s) \) or \( \zeta'(s)/\zeta(s) \) has been extensively used in the analysis of the Riemann zeta function. Using Equation (58), we may obtain a functional representation of \( \zeta'(s)/\zeta(s) \) in terms of the partial Euler product of \( \zeta(s) \).

**Theorem 6** If \( \Re(s - \rho) > 0 \) for every non-trivial zero \( \rho \), then

\[
\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \left( \log \prod_{i=1}^{r-1} \left( 1 - \frac{1}{p_i^s} \right) \right) - \frac{p_r^{-s}(s-1)}{s-1} + \sum_{\rho} \frac{p_r^{-s}(s-\rho)}{s-\rho} + O\left(p_r^{0.5-\sigma}\right). \tag{59}
\]

where \( \sigma = \Re(s) \) and the \( O \) term is given by \( d(\delta + \Delta)/ds \).

Although Theorems (4), (5) and (6) provide a functional representation for \( \zeta(s) \) in terms of it partial Euler product, our attempts to prove or disprove the Riemann hypothesis using these representations in conjunction with other properties (such as the growth of \( \zeta(1+iT) \) with \( T \)) have failed. However, the sum \( \sum_{p_r \leq p \leq p_r + \delta} 1/p^\sigma \) for \( \sigma < 1 \) (that was computed using these theorems) has been successfully used to examine the convergence of the series \( M(\sigma) \) for \( \sigma < 1 \) as described in the next section.

### 6 The convergence of the series \( M(\sigma, p_r) \) and \( M(\sigma) \) for \( \sigma \leq 1 \).

In this section, we will first provide an estimate for the partial sum \( M(1, p_r; 1, p_r^a) \) as \( a \) approaches infinity. This estimate will be computed by using Equation (57) and noting that \( M(1, p_r) \) equals zero for every \( p_r \). Therefore for every \( p_r \), \( M(1, p_r; 1, p_r^a) \) approaches zero as \( a \) approaches infinity. We also notice that, for every \( p_r \) and \( N \), we have (see Appendix (5)),

\[
|M(1, p_r; 1, N)| = \left| \sum_{n=1}^{N} \frac{\mu(n, p_r)}{n} \right| \leq 2
\]

The estimation of the partial sum \( M(1, p_r; 1, p_r^a) \) as a function of \( a \) will then be used to establish a relationship between \( M(1, p_r; 1, p_r^a) \) and \( M(\sigma, p_r; 1, p_r^a) \). This relationship is then used to show that \( M(\sigma, p_r) \) and \( M(\sigma) \) diverge for \( \sigma < 1 \). We will describe the details of our method in the following three steps.

- In the first step, we will show that, for every \( a \) and as \( p_r \) approaches infinity, the partial sum \( M(1, p_r; 1, p_r^a) \) is a function of only \( a \) (independent of \( p_r \)).

  Toward this end, we define the function \( f(a, p_r) \) as

  \[
f(a, p_r) = M(1, p_r; 1, p_r^a) = \sum_{n=1}^{p_r^a} \frac{\mu(n, p_r)}{n}.
\]

We will then show that, for every \( a \) and as \( p_r \) approaches infinity, the function \( f(a, p_r) \) approaches a deterministic function \( F(a) \). In other words; if we plot \( M(1, p_r; 1, N) \) (where \( N = p_r^a \)) as a function of \( a = \log N/\log p_r \), then for each value of \( a \) and as \( p_r \) approaches infinity, \( f(a, p_r) \) approaches a unique value \( F(a) \). This result can be achieved by first dividing the prime numbers that are in the range \( p_r < x \leq p_r^2 \) into \( N \) sections. The first section comprises of all the prime numbers that are in the range \( p_r < x \leq p_r^{1+\delta} \) (where, \( \delta << 1 \) and
it is given by $\delta = 1/(\log p_r)^\alpha$, $\alpha > 1$ and $(\log p_r)^\alpha << p_r$. The second section comprises of all the prime numbers that are in the range in the range $p_r^{1+\delta} < x \leq p_r^{1+2\delta}$ and so on (where the $j$-th section comprises of all the prime numbers that are in the range $p_r^{1+(j-1)\delta} < x \leq p_r^{1+j\delta}$). Hence,

$$N\delta = 1. \quad (60)$$

The process of dividing the prime numbers into sections continues for primes greater than $p_r^2$. Thus, the total number of sections $L$ over the range $p_r < x \leq p_r^a$ is given by $(a-1)N$.

If we define $K_i$ as the sum of the reciprocals of the prime numbers in section $j$ (where $i = j + N$), then by Mertens' Theorem, $K_i$ is given by

$$K_i = \log \log p_r{(i+1)\delta} - \log p_r^{i\delta} + \frac{O(1/\log p_r)}{i},$$

where $1 \leq i\delta \leq a$. Hence, for sufficiently large $p_r$, we then have

$$K_i = \frac{1}{i} + \frac{1}{i}O(1/\log p_r) + O \left(1/i^2\right), \quad (61)$$

where $O(1/\log p_r)$ can be made arbitrary small by selecting $p_r$ arbitrary large. Therefore, we may consider that each $K_i$ is comprised of two terms. The first one is a deterministic or regular term defined as $G_i$ and it is given by $1/i$. The second one is an irregular term defined as $R_i$ and it is the remaining part of $K_i$ (i.e. the irregular term $R_i$ is given by $K_i - D_i$). Hence,

$$K_i = G_i + R_i, \quad (62)$$

where

$$G_i = \frac{1}{i}, \quad (63)$$

and

$$R_i = K_i - G_i = \frac{1}{i}O(1/\log p_r) + O \left(1/i^2\right). \quad (64)$$

Notice that although $O \left(1/i^2\right)$ behaves regularly, we have lumped it with the irregular term $R_i$ due to its negligible effect on the partial sum $M(1, p_r; 1, p_r^a)$ as $p_r$ (and consequently $N$) approaches infinity.

Next, we will device an algorithm to construct a series that is equivalent to the series $M(1, p_r; 1, p_r^a)$ from these $(a-1)N$ sections (that are comprised of the prime numbers with their associated values of $K_i$’s) and the products of $K_i$’s (with the appropriate signs). This series starts with the number 1. Then, instead of subtracting the terms $1/p_r, 1/p_{r+1}, ...$, we subtract the values of $K_i$’s for the first $N$ sections. These sections are ordered based on the value of the largest member within each section. It can be easily shown that the value of $M(1, p_r; 1, p_r^2)$ constructed by this method is given $1 - \log 2$ plus a factor that is determined by the sum of $N$ terms of the form $(1/i)O(1/\log p_r) + O(1/i^2)$ and this factor (as mentioned earlier) can be made arbitrary small by selecting $p_r$ arbitrary large. In other words;

$$M(1, p_r; 1, p_r^2) = 1 - \sum_{i=N}^{2N} K_i,$$
and

\[
\lim_{p_r, N \to \infty} M(1, p_r; 1, p_r^2) = 1 - \lim_{N \to \infty} \frac{1}{i} = 1 - \lim_{N \to \infty} \frac{1}{N + j}.
\]

Thus

\[
\lim_{p_r, N \to \infty} M(1, p_r; 1, p_r^2) = 1 - \int_0^1 \frac{1}{1 + x} \, dx = 1 - \log 2
\]

The previous results can be also attained using Stieltjes integral, For \(1 < a < 2\), we have

\[
M(1, p_r; 1, p_r^a) = 1 - \sum_{p_r \leq p_i < p_r^a} \frac{1}{p_i}.
\]

If we define \(M_1(1, p_r; 1, p_r^a)\) as

\[
M_1(1, p_r; p_r, p_r^a) = \sum_{p_r \leq p_i < p_r^a} \frac{1}{P_i},
\]

then,

\[
M(1, p_r; 1, p_r^a) = 1 - M_1(1, p_r; p_r, p_r^a) = 1 - \int_{p_r}^{p_r^a} \frac{d\pi(x)}{x} = 1 - \int_{1}^{a} \frac{d\pi(p_r^y)}{p_r^y}.
\]

On RH, we have

\[
d\pi(p_r^y) = d\text{Li}(p_r^y) + dO(\sqrt{p_r^y} \log(p_r^y)),
\]

or

\[
d\pi(p_r^y) = \frac{1}{\log(p_r^y)} \frac{\log p_r}{p_r^y} \, dy + dO(\sqrt{p_r^y} \log(p_r^y)) = \frac{dy}{yp_r^y} + dO(\sqrt{p_r^y} \log(p_r^y)).
\]

Hence, for \(1 < a < 2\), we have

\[
M(1, p_r; 1, p_r^a) = 1 - \int_1^a \frac{dy}{y} + \int_1^a \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^y} = 1 - \log(a) + O(g_1(p_r, a)),
\]

where

\[
O(g_1(p_r, a)) = \int_1^a \frac{dO(\sqrt{p_r^y} \log(p_r^y))}{p_r^y}.
\]

As \(p_r\) approaches infinity, \(O(g_1(p_r, a))\) approaches zero. Consequently,

\[
\lim_{p_r, N \to \infty} M(1, p_r; 1, p_r^a) = 1 - \log a.
\]

The terms of the series \(M(1, p_r; 1, p_r^a)\) in the range \(p_r \leq x < p_r^3\) are either a reciprocal of a prime or a reciprocal of the product of two primes. To reconstruct these terms, we start with 1 and subtract the sum of \(K_i\)'s for the sections of primes in the range \(p_r \leq x < p_r^3\) and then add to it the sum of the terms that are the product of \(K_{i1}\)'s and \(K_{i2}\)'s for any two sections of the prime numbers (where the product of any member of the one section with any member of the second section is less than \(p_r^3\)). Hence
$$M(1, p_r; 1, p_r^3) = 1 - \sum_{i=N}^{3N} K_i + \frac{1}{2} \sum_{i=N}^{2N} \left(K_{3N-i} \sum_{j=N}^i K_j\right)$$

where the factor of 1/2 was added to the last term since each term of the form 1/(p_1 p_2) is repeated twice. As $p_r$ (and consequently $N$) approach infinity, we have

$$\lim_{p_r, N \to \infty} M(1, p_r; 1, p_r^3) = 1 - \lim_{N \to \infty} \sum_{i=N}^{3N} \frac{1}{i} + \frac{1}{2} \lim_{N \to \infty} \sum_{i=N}^{2N} \left(\frac{1}{3N-i} \sum_{j=N}^i \frac{1}{j}\right).$$

The above equation can be easily computed using integration methods. Hence, as $p_r$ (and consequently $N$) approaches infinity, the partial sum $M(1, p_r; 1, p_r^a)$ for $a \leq 3$ is independent of $p_r$.

The previous results can be also attained using Stieltjes integral. For $1 < a < 3$, we have

$$M(1, p_r; 1, p_r^a) = 1 - \sum_{p_r \leq p_i \leq p_r^a} \frac{1}{p_i} + \sum_{p_r \leq p_i < p_r^a \leq p_r} \frac{1}{p_i p_j},$$

If we define $M_2(1, p_r; 1, p_r^a)$ as

$$M_2(1, p_r; 1, p_r^a) = \sum_{p_r \leq p_i < p_r^a} \frac{1}{p_i p_j} = \frac{1}{2} \sum_{p_r \leq p_i < p_r} \frac{1}{p_i} M_1(1, p_r; p_r, p_r^a/p_i),$$

then

$$M_2(1, p_r; 1, p_r^a) = \frac{1}{2} \int_1^{p_r} \frac{d\pi(p_r y)}{p_r y} (\log(a - y) + O(g_1(p_r, a - y))).$$

Hence

$$M(1, p_r; 1, p_r^a) = 1 - \log(a) + O(g_1(p_r, a)) + \frac{1}{2} \int_1^{a-1} \frac{\log(a - y)}{y} dy + O(g_2(p_r, a - 1)),$$

where

$$O(g_2(p_r, a - 1)) = \int_1^{a-1} \frac{O(g_1(p_r, a - y))}{y} dy + \int_1^{a-1} \log(a - y) dO(\sqrt{p_r y} \log(p_r y)) + \int_1^{a-1} \log(a - y) dO(\sqrt{p_r y} \log(p_r y)).$$

as $p_r$ approaches infinity, $O(g_2(p_r, a - 1))$ approaches zero. Thus, for $1 \leq a < 3$, we have

$$\lim_{p_r, N \to \infty} M(1, p_r; 1, p_r^a) = 1 - \log a + \int_1^{a-1} \frac{\log(a - y)}{y} dy$$

Therefore, as $p_r$ approaches infinity, $M(1, p_r; 1, p_r^a)$ is only dependent on $a$.

This process is repeated $a - 1$ times to show that, as $p_r$ (and consequently $N$) approaches infinity, the partial sum $M(1, p_r; 1, p_r^a)$ is dependent on only $a$. Hence for every $a$, we have

$$F(a) = \lim_{p_r \to \infty} M(1, p_r; 1, p_r^a) \quad (65)$$
It should be pointed out that the series constructed by this algorithm includes both square-free terms (that form \(M(1, p_r; 1, p_r^a)\)) as well as non square-free terms. In the following, we will show that, for every \(a\) and as \(p_r\) approaches infinity, the contribution by the non square-free terms to the partial sum \(M(1, p_r; 1, p_r^a)\) approaches zero as well. Toward this end, let \(S_0\) be the sum of the terms with the factor \(1/p_r^2\). Let \(S_1\) be the sum of the remaining terms with the factor \(1/(p_r+1)^2\), \(S_2\) be the sum of the remaining terms with the factor \(1/(p_r+2)^2\), and so on. Let \(H\) be sum of all the terms associated with non square-free terms. Thus, \(H\) is given by

\[
H = \frac{1}{p_r^2} S_0 + \frac{1}{p_r+1^2} S_1 + \ldots + \frac{1}{p_r+t^2} S_L,
\]

where \(p_{r+t}\) is the largest prime that its square is less than \(p_r^a\). However,

\[
|S_0|, |S_1|, \ldots, |S_l| < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p_r^a}.
\]

Thus,

\[
|S_0|, |S_1|, \ldots, |S_l| = O(a \log p_r).
\]

Therefore

\[
H = \left(\frac{1}{p_r^2} + \frac{1}{p_r+1^2} + \ldots + \frac{1}{p_r+t^2}\right) O(a \log p_r).
\]

Hence, the contribution by the non square free terms \(H\) is given by,

\[
H = O(a \log p_r/p_r).
\]

Consequently, for every \(a\) and as \(p_r\) approaches infinity, \(H\) (or the contribution by the non-square free terms to the partial sum \(M(1, p_r; 1, p_r^a)\)) approaches zero.

- In the second step, we write the partial sum \(M(1, p_r; 1, p_r^a)\) as the sum of two components. The first is the deterministic or regular component and it is given by \(F(a)\). The second one is the irregular component \(O(M(1, p_r; 1, p_r^a))\) given by \(M(1, p_r; 1, p_r^a) - F(a)\). We will show that \(F(a)\) (as well as \(M(1, p_r; 1, p_r^a)\)) decays slower that \(1/a\) (this is the key step to disproving the Riemann Hypotheses as RH requires an exponential decay of \(F(a)\)).

Toward this end, we write the partial sum \(M(1, p_r; 1, p_r^a)\) as following sum

\[
M(1, p_r; 1, p_r^a) = 1 - \sum_{p_r \leq p_r < p_{r-a-1}} \frac{1}{p_r} M(1, p_{r+1}; 1, p_r^a/p_r) - \sum_{p_r-a-1 \leq p_r < p_{r-a}} \frac{1}{p_r}.
\]

(66)

Using Stieltjes integral, we can write the above equation as follows

\[
M(1, p_r; 1, p_r^a) = 1 - \int_1^{p_r^{a-1}} \frac{d\pi(p_r^y)}{p_r^y} M(1, p_r^y; 1, p_r^a/p_r^y) - \int_{p_r^{a-1}}^a \frac{d\pi(p_r^y)}{p_r^y},
\]

(67)

where, on RH, \(d\pi(p_r^y) = d\text{Li}(p_r^y) + dO(\sqrt{p_r^y} \log(p_r^y))\). As \(p_r\) approaches infinity, \(M(1, p_r^y; 1, p_r^{a-y})\) approaches \(F(a - y)\). Therefore, as \(p_r\) approaches infinity, we have
\[ F(a) = 1 - \int_{1}^{a} \frac{F(a-y)}{y} \, dy - \int_{a-1}^{a} \frac{dy}{y}. \] (68)

It is shown in Appendix 5 that \(|M(1, p_r; 1, p_r^a)| \leq 2\) for every \(p_r\) and \(a\). Hence \(|F(a)| \leq 2\). Consequently, \(F(a)\) approaches zero as \(a\) approaches infinity (this follows from the fact that if \(F(a)\) does not converge to zero, then the first integral of the above equation diverges as \(a\) approaches infinity which then leads to the divergence of \(F(a)\). This contradicts our earlier statement that \(|F(a)| \leq 2\). Thus, as \(a\) approaches infinity, we have

\[ \int_{1}^{a-1} \frac{F(a-y)}{y} \, dy = 1. \] (69)

Hence, we conclude that \(F(a)\) approaches zero at a rate that is no faster than the rate at which \(1/a\) approaches zero (this follows from that fact that if \(F(a) = O(1/a)\), then the above integral is given by \(O(\log(a)/a)\) which is much smaller than 1 for sufficiently large \(a\). Therefore, for some constant \(C\), we have

\[ F(a) > C/a \]

or

\[ F(a) = \Omega(1/a) \]

Thus, if we write the partial sum of \(M(1, p_r)\) as \(M(1, p_r, 1, n)\) (where \(n = p_r^n\)), then the regular component of \(M(1, p_r, 1, n)\) decays slower than \(C/\log n\).

Similarly, we can show that \(M(1, p_r; 1, p_r^a)\) decays no faster than \(1/a\). This task can be achieved by noting that irregular component of \(M(1, p_r; 1, p_r^a)\) is given by

\[
O(M(1, p_r; 1, p_r^a)) = \int_{1}^{a-1} \frac{M(1, p_r^y; 1, p_r^a-y)}{p_r^y} \, dO(\sqrt{p_r^y \log(p_r^y)}) - \int_{1}^{a-1} \frac{O(M(1, p_r^y; 1, p_r^a-y))}{yp_r^y} \, dy - \int_{a-1}^{a} \frac{dO(\sqrt{p_r^y \log(p_r^y)})}{p_r^y}. \] (70)

Therefore, if we assume that \(M(1, p_r; 1, p_r^a)\) decays at a rate faster than \(1/a\) (say \(O(h(a))\)), then \(O(M(1, p_r; 1, p_r^a))\) can be written as

\[ O(M(1, p_r; 1, p_r^a)) = O(h(a)) - F(a) \]

Later in this section, we will show that the Riemann Hypothesis requires the exponential decay for \(O(h(a))\). If we assume that \(O(h(a))\) decays exponentially with respect to \(a\), then it can be easily shown that, for sufficiently large \(p_r\) and \(a\), the first and third integrals of equation (70) are negligible compared with \(F(a)\). Hence

\[ |O(M(1, p_r; 1, p_r^a))| = |O(h(a)) - F(a)| = \left| \int_{1}^{a-1} \frac{O(M(1, p_r^y; 1, p_r^a-y))}{yp_r^y} \, dy \right| \]

or

\[ |O(h(a)) - F(a)| = \left| \int_{1}^{a-1} \frac{O(h(a-y)) - F(a-y)}{yp_r^y} \, dy \right| < \frac{1}{p_r} |F(a)| \]
Similar inconsistency is obtained if it is assumed that $M(1, p_r; 1, p_r^a)$ decays at any rate that is faster than $1/a$. Hence, both $F(a)$ and $M(1, p_r; 1, p_r^a)$ decays at a rate slower than $1/a$.

- For the third step, we will compute the partial sum $M(\sigma, p_r; 1, p_r^a)$ for $\sigma < 1$ and show that it diverges as $a$ approaches infinity.

Toward this end, we write the partial sum $M(\sigma, p_r; p_r^a, p_r^{a+\delta})$ as the following sum

$$M(\sigma, p_r; p_r^a, p_r^{a+\delta}) = - \sum_{p_r < p_i < p_r^{a+\delta-1}} \frac{1}{p_i^\sigma} M(\sigma, p_i+1; p_r^a/p_i, p_r^{a+\delta}/p_i) - \sum_{p_r^{a} < p_i < p_r^{a+\delta}} \frac{1}{p_i^\sigma}. \quad (71)$$

Using Stieltjes integral, we can write the above equation as follows

$$M(\sigma, p_r; p_r^a, p_r^{a+\delta}) = \int_1^{a+\delta-1} \frac{d\pi(p_r^y)}{p_r^y} M(\sigma, p_r^y; p_r^a/p_r^y, p_r^{a+\delta}/p_r^y) - \int_a^{a+\delta} \frac{d\pi(p_r^y)}{p_r^y},$$

or

$$M(\sigma, p_r; p_r^a, p_r^{a+\delta}) = \int_1^{a+\delta-1} \frac{d\pi(p_r^y)}{p_r^y} M(\sigma, p_r^y; p_r^a-p_r^y, p_r^{a-y+\delta}) - \int_a^{a+\delta} \frac{d\pi(p_r^y)}{p_r^y}. \quad (72)$$

However, for $\delta \log p_r << 1$, we may write

$$M(\sigma, p_r^y; p_r^a-p_r^y, p_r^{a-y+\delta}) = p_r^{(1-\sigma)(a-y)} M(1, p_r^y; p_r^{a-y}, p_r^{a-y+\delta}). \quad (73)$$

Hence

$$M(\sigma, p_r; p_r^a, p_r^{a+\delta}) = p_r^{(1-\sigma)} \int_1^{a+\delta-1} \frac{d\pi(p_r^y)}{p_r^y} M(1, p_r^y; p_r^{a-y}, p_r^{a-y+\delta}) - p_r^{(1-\sigma)} \int_a^{a+\delta} \frac{d\pi(p_r^y)}{p_r^y}. \quad (74)$$

Consequently

$$M(\sigma, p_r; p_r^a, p_r^{a+\delta}) = p_r^{(1-\sigma)a} M(1, p_r^a; p_r^{a+\delta}). \quad (75)$$

Since the partial sum $M(1, p_r; 1, p_r^a)$ decays to zero slower than the decay of $c/a$, therefore the sum $M(1, p_r; p_r^a, p_r^{a+\delta})$ decays at a rate that is much slower than $p_r^{(1-\sigma)a}$. Hence the partial sum $M(\sigma, p_r; p_r^a, p_r^{a+\delta})$ grows to infinity as $a$ approaches infinity. Consequently, the series $M(\sigma, p_r)$ and $M(\sigma)$ diverge for $\sigma < 1$. This implies that the Riemann Hypothesis is invalid and the zeros can be found arbitrary close to line $\Re(s) = 1$. 

Appendix 1

To prove the first part of Theorem 1 (i.e. for \( s = \sigma + i0 \) and \( 0.5 < \sigma \leq 1 \), the series \( M(\sigma, p_r) \) converges conditionally if \( M(\sigma) \) converges conditionally), we first start with proving that \( M(\sigma, 2) \) is convergent if \( M(\sigma) \) is convergent. Since \( M(\sigma) \) is convergent, then for any arbitrary small number \( \delta \), there exists an integer \( N_0 \) such that for every integer \( N > N_0 \)

\[
|M(\sigma; N, \infty)| = \left| \sum_{n=N}^{\infty} \frac{\mu(n)}{n^\sigma} \right| < \delta
\]

Let the sums \( M(\sigma; 1, N), M(\sigma; N+1, 2N), M(\sigma; 2N+1, 2^2N), M(\sigma; 2^2N+1, 2^3N), ..., M(\sigma; 2^{L-1}N+1, 2^LN) \) be defined as

\[
M(\sigma; 1, N) = \sum_{n=1}^{N} \frac{\mu(n)}{n^\sigma} = A_1,
\]

\[
M(\sigma; N+1, 2N) = \sum_{n=N+1}^{2N} \frac{\mu(n)}{n^\sigma} = \delta_1,
\]

\[
M(\sigma; 2N+1, 2^2N) = \sum_{n=2N+1}^{2N} \frac{\mu(n)}{n^\sigma} = \delta_2,
\]

\[
M(\sigma; 2^2N+1, 2^3N) = \sum_{n=2^2N+1}^{2^2N} \frac{\mu(n)}{n^\sigma} = \delta_3,
\]

\[
M(\sigma; 2^{L-1}N+1, 2^LN) = \sum_{n=2^{L-1}N+1}^{2^LN} \frac{\mu(n)}{n^\sigma} = \delta_{L-1},
\]

If we define \( \delta(l) \) as the maximum of \( |\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_{L-1}|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, ..., |\delta_l + \delta_{l+1} + ... + \delta_{L-1}| \), then by the virtue of the convergence of \( M(\sigma) \),

\[
|\delta_1|, |\delta_2|, |\delta_3|, ..., |\delta_{L-1}|, |\delta_1 + \delta_2|, |\delta_1 + \delta_2 + \delta_3|, ..., |\delta_1 + \delta_2 + \delta_3 + ... + \delta_{L-1}| \leq \delta(1) \leq 2\delta.
\]

We also have

\[
|\delta_l|, |\delta_{l+1}|, |\delta_{l+2}|, ..., |\delta_{L-1}|, |\delta_l + \delta_{l+1}|, |\delta_l + \delta_{l+1} + \delta_{l+2}|, ..., |\delta_l + \delta_{l+1} + ... + \delta_{L-1}| \leq \delta(l),
\]

where by the virtue of the convergence of \( M(\sigma) \), \( \delta(l) \) approaches zero as \( l \) approaches infinity.

Furthermore, let the sums \( M(\sigma; 2; 1, N), M(\sigma; 2; N+1, 2N), M(\sigma; 2; 2N+1, 2^2N), M(\sigma; 2; 2^2N+1, 2^3N), ..., M(\sigma; 2; 2^{L-1}N+1, 2^LN) \) be defined as

\[
M(\sigma; 2; 1, N) = \sum_{n=1}^{N} \frac{\mu(n, 2)}{n^\sigma} = B_1,
\]

\[
M(\sigma; 2; N+1, 2N) = \sum_{n=N+1}^{2N} \frac{\mu(n, 2)}{n^\sigma} = \epsilon_1,
\]
\begin{align*}
M(\sigma, 2; 2N + 1, 2^2 N) &= \sum_{n=2N+1}^{2^{2N}} \frac{\mu(n, 2)}{n^\sigma} = \epsilon_2, \\
M(\sigma, 2; 2^2 N + 1, 2^3 N) &= \sum_{n=2^{2N+1}}^{2^{3N}} \frac{\mu(n, 2)}{n^\sigma} = \epsilon_3, \\
M(\sigma, 2; 2^{L-1} N + 1, 2^L N) &= \sum_{n=2^{L-1}N+1}^{2^L N} \frac{\mu(n, 2)}{n^\sigma} = \epsilon_{L-1},
\end{align*}

Since
\begin{align*}
\sum_{n=1}^{2N} \frac{\mu(n)}{n^\sigma} &= \sum_{n=1}^{2N} \frac{\mu(n, 2)}{n^\sigma} - \sum_{n=1}^{N} \frac{\mu(n, 2)}{(2n)^\sigma},
\end{align*}

thus
\begin{align*}
M(\sigma; 1, 2N) &= M(\sigma, 2; 1, 2N) - \frac{1}{2^\sigma} M(\sigma, 2; 1, N).
\end{align*}

Similarly, since
\begin{align*}
\sum_{n=2^lN+1}^{2^{l+1} N} \frac{\mu(n)}{n^\sigma} &= \sum_{n=2^lN+1}^{2^{l+1} N} \frac{\mu(n, 2)}{n^\sigma} - \sum_{n=2^{l-1}N+1}^{2^l N} \frac{\mu(n, 2)}{(2n)^\sigma},
\end{align*}

thus
\begin{align*}
M(\sigma; 2^l N + 1, 2^{l+1} N) &= M(\sigma, 2; 2^l N + 1, 2^{l+1} N) - \frac{1}{2^\sigma} M(\sigma, 2; 2^{l-1} N + 1, 2^l N).
\end{align*}

Rearranging the previous equations, we then have
\begin{align*}
A_1 + \delta_1 &= B_1 + \epsilon_1 - \frac{1}{2^\sigma} B_1, \\
\delta_2 &= \epsilon_2 - \frac{1}{2^\sigma} \epsilon_1, \\
\delta_3 &= \epsilon_3 - \frac{1}{2^\sigma} \epsilon_2, \\
\delta_{L-1} &= \epsilon_{L-1} - \frac{1}{2^\sigma} \epsilon_{L-2},
\end{align*}

where \(|\delta_1|, |\delta_2|, |\delta_3|, ..., |\delta_{L-1}|, |\delta_1 + \delta_2|, |\delta_1 + \delta_2 + \delta_3|, |\delta_1 + \delta_2 + \delta_3 + ... + \delta_{L-1}| \leq \delta(1) \leq 2\delta\) and \(\delta\) is arbitrary small. Hence
\begin{align*}
\epsilon_2 &= \frac{1}{2^\sigma} \epsilon_1 + \delta_2, \\
\epsilon_3 &= \frac{1}{2^\sigma} \epsilon_2 + \delta_3 = \frac{1}{2^2\sigma} \epsilon_1 + \frac{1}{2^\sigma} \delta_2 + \delta_3.
\end{align*}
\[
\epsilon_{L-1} = \frac{1}{2^{\sigma}} \epsilon_{L-2} + \delta_{L-1} = \frac{1}{2(L-2)^{\sigma}} \epsilon_1 + \frac{1}{2(L-3)^{\sigma}} \delta_2 + \frac{1}{2(L-4)^{\sigma}} \epsilon_3 + \ldots + \delta_{L-1}.
\]

Therefore,
\[
\epsilon_1 + \epsilon_2 + \epsilon_2 + \ldots + \epsilon_{L-1} = \left(1 + \frac{1}{2^{\sigma}} + \frac{1}{2^{2\sigma}} + \ldots + \frac{1}{2(L-1)^{\sigma}}\right) \epsilon_1 + (\delta_2 + \delta_3 + \ldots + \delta_{L-1}) + \frac{1}{2^{\sigma}}(\delta_2 + \delta_3 + \ldots + \delta_{L-2}) + \frac{1}{2^{2\sigma}}(\delta_2 + \delta_3 + \ldots + \delta_{L-3}) + \ldots + \frac{1}{2(L-3)^{\sigma}} \delta_2.
\]

Since \(|\delta_2| \leq \delta(1)|, |\delta_2 + \delta_3| \leq \delta(1)|, \ldots, |\delta_1 + \delta_2 + \delta_3 + \ldots + \delta_{L-1}| \leq \delta(1)|), hence
\[
|\delta_2 + \delta_3 + \ldots + \delta_{L-1}| + \frac{1}{2^{\sigma}} |\delta_2 + \delta_3 + \ldots + \delta_{L-2}| + \ldots + \frac{1}{2(L-3)^{\sigma}} |\delta_2| \leq \left|\delta(1) + \frac{1}{2^{\sigma}} \delta(1) + \ldots + \frac{1}{2(L-3)^{\sigma}} \delta(1)\right|,
\]
or
\[
|\delta_2 + \delta_3 + \ldots + \delta_{L-1}| + \frac{1}{2^{\sigma}} |\delta_2 + \delta_3 + \ldots + \delta_{L-2}| + \ldots + \frac{1}{2(L-3)^{\sigma}} |\delta_2| \leq \frac{2^{\sigma}}{2^{\sigma} - 1} |\delta(1)|.
\]

Therefore
\[
\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_{L-1} = \left(1 + \frac{1}{2^{\sigma}} + \frac{1}{2^{2\sigma}} + \ldots + \frac{1}{2(L-2)^{\sigma}}\right) \epsilon_1 + \gamma_1,
\]

where \(\gamma_1\) is of the same order as that of \(\delta(1)\).

As \(L\) approaches infinity, we then obtain
\[
\sum_{i=1}^{\infty} \epsilon_i = \frac{2^{\sigma}}{2^{\sigma} - 1} \epsilon_1 + \gamma_1.
\]

Therefore, the sum \(M(\sigma, 2; N + 1, \infty)\) (which is equal to \(\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots\) ) is bounded by the sum \(M(\sigma, 2; N + 1, 2N)\) (which is equal to \(\epsilon_1\)).

The previous process can be repeated with \(2N\) is substituted for \(N\) and Equation (76) becomes
\[
A_2 + \delta_2 = B_2 + \epsilon_2 - \frac{1}{2^{\sigma}} B_2,
\]
where \(A_2 = M(\sigma; 1, 2N)\) and \(B_2 = M(\sigma; 2; 1, 2N)\). Thus,
\[
A_2 = B_2 - \frac{1}{2^{\sigma}} B_2 + \frac{1}{2^{\sigma}} \epsilon_1.
\]

Following the same process, we can show that that the sum \(M(\sigma, 2; 2N + 1, \infty)\) is given by
\[
\sum_{i=2}^{\infty} \epsilon_i = \frac{1}{2^{\sigma} - 1} \epsilon_1 + \gamma_2.
\]

where \(\gamma_2\) is of the same order as that of \(\delta(2)\).
If we repeat the process \( l \) times, we obtain

\[
A_l = B_l - \frac{1}{2\sigma}B_l + \frac{1}{2(l-1)\sigma}\epsilon_1, 
\]

where \( A_l = M(\sigma; 1, 2^lN) \) and \( B_l = M(\sigma, 2; 1, 2^lN) \) and the sum \( M(\sigma, 2; 2^lN + 1, \infty) \) is given by

\[
\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{2(l-2)\sigma} \frac{1}{2^\sigma - 1} \epsilon_1 + \gamma_l. 
\]

where \( \gamma_l \) is of the same order as that of \( \delta(l) \). Since by the virtue of the convergence of \( M(\sigma) \), \( \delta(l) \) tends to zero as \( l \) approaches infinity, therefore \( \gamma_l \) and the above sum approach zero as \( l \) approaches infinity.

Thus, we conclude that \( M(\sigma, 2; 2^lN + 1, \infty) \) (given by \( \sum_{i=l}^{\infty} \epsilon_i \)) approaches zero as \( l \) approaches infinity. Furthermore, as \( l \) approaches infinity, \( B = \lim_{l \to \infty} B_l \) approaches its limit given by

\[
\left(1 - \frac{1}{2^\sigma}\right) B = M(\sigma; 1, \infty). 
\]

Hence,

\[
\left(1 - \frac{1}{2^\sigma}\right) M(\sigma, 2) = M(\sigma). 
\]

Similarly, following the same steps, we can show that

\[
\left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3; 1, \infty) = M(\sigma, 2; 1, \infty). 
\]

or

\[
\left(1 - \frac{1}{2^\sigma}\right) \left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3; 1, \infty) = M(\sigma; 1, \infty). 
\]

This task can be achieved by first defining

\[
M(\sigma, 2; 1, N) = \sum_{n=1}^{N} \frac{\mu(n, 2)}{n^\sigma} = A_1, 
\]

\[
M(\sigma, 2; N + 1, 3N) = \sum_{n=N+1}^{3N} \frac{\mu(n, 2)}{n^\sigma} = \delta_1, 
\]

\[
M(\sigma, 2; 3N + 1, 3^2N) = \sum_{n=3N+1}^{3^2N} \frac{\mu(n, 2)}{n^\sigma} = \delta_2, 
\]

\[
M(\sigma, 2; 3L-1N + 1, 3^LN) = \sum_{n=3L-1N+1}^{3^LN} \frac{\mu(n, 2)}{n^\sigma} = \delta_{L-1}, 
\]

and
\[ M(\sigma, 3; 1, N) = \sum_{n=1}^{N} \frac{\mu(n, 3)}{n^\sigma} = B_1, \]

\[ M(\sigma, 3; N + 1, 3N) = \sum_{n=N+1}^{3N} \frac{\mu(n, 3)}{n^\sigma} = \epsilon_1, \]

\[ M(\sigma, 3; 3N + 1, 3^2N) = \sum_{n=3N+1}^{3^2N} \frac{\mu(n, 3)}{n^\sigma} = \epsilon_2, \]

\[ M(\sigma, 3; 3^{L-1}N + 1, 3^L N) = \sum_{n=3^{L-1}N+1}^{3^L N} \frac{\mu(n, 3)}{n^\sigma} = \epsilon_{L-1}. \]

Since
\[
\sum_{n=1}^{3N} \frac{\mu(n, 2)}{n^\sigma} = \sum_{n=1}^{3N} \frac{\mu(n, 3)}{n^\sigma} - \sum_{n=1}^{N} \frac{\mu(n, 3)}{(3n)^\sigma},
\]

thus
\[ M(\sigma, 2; 1, 3N) = M(\sigma, 3; 1, 3N) - \frac{1}{3^\sigma} M(\sigma, 3; 1, N) \]

Similarly,
\[ M(\sigma, 2; 3^l N + 1, 3^{l+1} N) = M(\sigma, 3; 3^l N + 1, 3^{l+1} N) - \frac{1}{3^\sigma} M(\sigma, 3; 3^{l-1} N + 1, 3^l N) \]

Following the same process, we can show that
\[ \sum_{i=1}^{\infty} \epsilon_i = \frac{3^\sigma}{3^\sigma - 1} \epsilon_1 + \gamma_1, \]

where \( \gamma_1 \) is of the same order as that of \( \delta(1) \) (\( \delta(l) \) is defined as the maximum of \( |\delta_t|, |\delta_{t+1}|, |\delta_{t+2}|, ..., |\delta_{L-1}|, |\delta_t + \delta_{t+1}|, |\delta_t + \delta_{t+1} + \delta_{t+2}|, ..., |\delta_t + \delta_{t+1} + ... + \delta_{L-1}| \)).

Similarly, if we define \( A_2 = M(\sigma, 2; 1, 3N) \) and \( B_2 = M(\sigma, 3; 1, 3N) \), then
\[ A_2 = B_2 - \frac{1}{3^\sigma} B_2 + \frac{1}{3^\sigma} \epsilon_1. \]

Therefore
\[ \sum_{i=2}^{\infty} \epsilon_i = \frac{1}{3^\sigma - 1} \epsilon_1 + \gamma_2. \]

where \( \gamma_2 \) is of the same order as that of \( \delta(2) \).
Repeating the steps \( l \) times, we then obtain
\[
\sum_{i=l}^{\infty} \epsilon_i = \frac{1}{3^{(l-2)\sigma}} \frac{1}{3^\sigma - 1} \epsilon_1 + \gamma_l.
\]
where \( \gamma_l \) is of the same order as that of \( \delta(l) \). Hence the above sum approaches zero as \( l \) approaches infinity.

Thus, we conclude that \( M(\sigma, 3; 3^{l}N + 1, \infty) \) (given by \( \sum_{i=l}^{\infty} \epsilon_i \)) approaches zero as \( l \) approaches infinity. Furthermore, as \( l \) approaches infinity, \( B = \lim_{l \to \infty} B_l \) approaches its limit given by
\[
\left(1 - \frac{1}{3^\sigma}\right) B = M(\sigma, 2; 1, \infty).
\]
Hence,
\[
\left(1 - \frac{1}{3^\sigma}\right) M(\sigma, 3) = M(\sigma, 2).
\]

Repeating the process \( r \) times, we then conclude
\[
M(\sigma) = M(\sigma, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^\sigma}\right).
\]

The second part of the theorem can be proved by recalling
\[
M(s, p_{r-1}; 1, Np_r) = M(s, p_r; 1, Np_r) - \frac{1}{p_r^s} M(s, p_r; 1, N).
\]
If both series \( M(s, p_{r-1}) \) and \( M(s, p_r) \) are convergent, then as \( N \) approaches infinity, we obtain
\[
M(s, p_{r-1}) = M(s, p_r) \left(1 - \frac{1}{p_r^s}\right).
\]
Repeating the process \( r \) times, we then conclude
\[
M(\sigma) = M(\sigma, p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^\sigma}\right).
\]

**Appendix 2**

Assuming RH is valid and for \( \sigma > 0.5 \), to show that
\[
\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) + \epsilon
\]
where, \( \epsilon = O \left( \frac{t}{(\sigma-0.5)\pi} p_{r_1}^{1/2-\sigma} \log p_{r_1} \right) \), we first recall that
\[
\sum_{i=r_1}^{r_2} \frac{1}{p_i^\sigma} = \int_{p_{r_1}}^{p_{r_2}} \frac{d\pi(x)}{x^\sigma} = \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx + \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma} dO \left( \sqrt{x \log x} \right).
\]
We will first compute the integral with the $O$ notation. This can be done by integration by parts to obtain

$$\int_{p_{r_1}}^{p_2} \frac{1}{x^\sigma} dO (\sqrt{x} \log x) = O \left( \frac{\sqrt{p_{r_2} \log p_{r_2}}}{p_{r_2}^\sigma} \right) - O \left( \frac{\sqrt{p_{r_1} \log p_{r_1}}}{p_{r_1}^\sigma} \right) - \int_{p_{r_1}}^{p_2} O (\sqrt{x} \log x) \left( \frac{1}{x^\sigma} \right)$$

Since $x > 0$, thus

$$\int_{p_{r_1}}^{p_2} \frac{1}{x^\sigma} dO (\sqrt{x} \log x) = O \left( \frac{\sqrt{p_{r_2} \log p_{r_2}}}{p_{r_2}^\sigma} \right) - O \left( \frac{\sqrt{p_{r_1} \log p_{r_1}}}{p_{r_1}^\sigma} \right) - O \left( \int_{p_{r_1}}^{p_2} \sqrt{x} \log x \left( \frac{1}{x^\sigma} \right) \right)$$

With the substitution of variables $y = \log x$, we then obtain

$$\int_{p_{r_1}}^{p_2} \sqrt{x} \log x \left( \frac{1}{x^\sigma} \right) = - \int_{p_{r_1}}^{p_2} \sigma ye^{(\frac{1}{\sigma})y} dy.$$ 

Since

$$\int xe^{ax} dx = \left( \frac{x}{a} - \frac{1}{a^2} \right) e^{ax},$$

therefore

$$\int_{p_{r_1}}^{p_2} \sqrt{x} \log x \left( \frac{1}{x^\sigma} \right) = - \sigma \left( \frac{\log p_{r_2}}{0.5 - \sigma} - \frac{1}{(0.5 - \sigma)^2} \right) p_{r_2}^{0.5 - \sigma} + \sigma \left( \frac{\log p_{r_1}}{0.5 - \sigma} - \frac{1}{(0.5 - \sigma)^2} \right) p_{r_1}^{0.5 - \sigma}.$$ 

Hence, for $\sigma > 0.5$, we have

$$\int_{p_{r_1}}^{p_2} \frac{1}{x^\sigma} dO (\sqrt{x} \log x) = O \left( \frac{p_{r_1}^{0.5 - \sigma} \log p_{r_1}}{(\sigma - 0.5)^2} \right)$$ (77) 

For $\sigma \geq 1$, the integral $\int_{p_{r_1}}^{p_2} \frac{1}{x^\sigma \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(r) = \int_{r}^{\infty} \frac{e^{-u}}{u} du$ (where $r \geq 0$) to obtain

$$\int_{p_{r_1}}^{p_2} \frac{1}{x^\sigma \log x} dx = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2})$$

To compute the integral $\int_{p_{r_1}}^{p_2} \frac{1}{x^\sigma \log x} dx$ for $\sigma < 0$, we first use the substantiation $y = \log x$ to obtain

$$\int_{p_{r_1}}^{p_2} \frac{1}{x^\sigma \log x} dx = \int_{\log p_{r_1}}^{\log p_{r_2}} \frac{e^{(1-\sigma)y}}{y} dy = \int_{\log p_{r_1}}^{\log p_{r_2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{\epsilon}^{\log p_{r_1}} \frac{e^{(1-\sigma)y}}{y} dy$$

where $\epsilon$ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r_1}$ and $z_2 = y/\log p_{r_2}$, we then obtain

$$\int_{p_{r_1}}^{p_2} \frac{1}{x^\sigma \log x} dx = \int_{\epsilon/\log p_{r_2}}^{1} \frac{e^{(1-\sigma)(\log p_{r_2})z_2}}{z_2} dz_2 - \int_{\epsilon/\log p_{r_1}}^{1} \frac{e^{(1-\sigma)(\log p_{r_1})z_1}}{z_1} dz_1.$$
With the variable substantiations $w_1 = (1 - \sigma)(\log p_{r_1})z_1$ and $w_2 = (1 - \sigma)(\log p_{r_2})z_1$ and by adding and subtracting the terms $\int_{(1-\sigma)\epsilon}^{1} \frac{d}{dw_2} w_2 - \int_{(1-\sigma)\epsilon}^{1} \frac{d}{dw_1} w_1$, we then have

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx = \int_{(1-\sigma)\epsilon}^{1} \frac{d}{dw_2} w_2 - \int_{(1-\sigma)\epsilon}^{1} \frac{d}{dw_1} w_1.$$ 

Using the following identity [9, page 230]

$$\int_0^a \frac{e^t - 1}{t} dt = -E_1(-a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1,$

$$\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2})$$

Hence, for $\sigma > 0.5$, we have

$$\sum_{i=1}^{r^2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) + \varepsilon$$

It should be pointed out that in general, if there are no non-trivial zeros for values of $s$ with $\Re(s) > a$, then by following the same steps, we may also show that for $\sigma > a$, we have

$$\sum_{i=1}^{r^2} \frac{1}{p_i^\sigma} = E_1((\sigma - 1) \log p_{r_1}) - E_1((\sigma - 1) \log p_{r_2}) + \varepsilon$$

where, $\varepsilon = O \left( \frac{t}{\sigma-a} p_{r_1}^{a-\sigma} \log p_{r_1} \right)$.

**Appendix 3**

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=1}^{r^2} \frac{1}{p_i^s} = E_1((s - 1) \log p_{r_1}) - E_1((s - 1) \log p_{r_2}) + \varepsilon$$

where, $\varepsilon = O \left( \frac{t+1}{(\sigma-0.5)} p_{r_1}^{1/2-\sigma} \log p_{r_1} \right)$, we first recall that

$$\sum_{i=1}^{r^2} \frac{1}{p_i^s} = \int_{p_{r_1}}^{p_{r_2}} \frac{d\pi(x)}{x^s} = \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx + \int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s} dO (x \log x).$$
We will first compute the integral with the $O$ notation. This can be done by integration by parts to obtain

$$
\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s} dO (\sqrt{x} \log x) = O \left( \sqrt[p_{r_2}]{\log p_{r_2}} \right) - O \left( \sqrt[p_{r_1}]{\log p_{r_1}} \right) - \int_{p_{r_1}}^{p_{r_2}} O (\sqrt{x} \log x) \, d \left( \frac{1}{x^s} \right)
$$

The integral on the right side of the above equation can be then written as

$$
\int_{p_{r_1}}^{p_{r_2}} O (\sqrt{x} \log x) \, d \left( \frac{1}{x^s} \right) = -s \int_{p_{r_1}}^{p_{r_2}} O (\sqrt{x} \log x) \, x^{-s-1} dx.
$$

Hence,

$$
\left| \int_{p_{r_1}}^{p_{r_2}} O (\sqrt{x} \log x) \, d \left( \frac{1}{x^s} \right) \right| \leq |s| \int_{p_{r_1}}^{p_{r_2}} O (\sqrt{x} \log x) \, |x^{-s-1}| dx.
$$

For sufficiently large $t$, we can write $|s|=t$ and consequently

$$
\left| \int_{p_{r_1}}^{p_{r_2}} O (\sqrt{x} \log x) \, d \left( \frac{1}{x^s} \right) \right| = O \left( t \frac{p_{r_1}^{0.5-\sigma} \log p_{r_1}}{(\sigma - 0.5)^2} \right).
$$

Hence,

$$
\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s} dO (\sqrt{x} \log x) = O \left( (t + 1) \frac{p_{r_1}^{0.5-\sigma} \log p_{r_1}}{(\sigma - 0.5)^2} \right).
$$

For $\Re(s) \geq 1$, the integral $\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(z) = \int_1^{\infty} e^{-xz} \, dt$ (where $\Re(z) \geq 0$) to obtain

$$
\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx = E_1((s - 1) \log p_{r_1}) - E_1((s - 1) \log p_{r_2})
$$

To compute the integral $\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx$ for $\Re(z) < 1$, we first write the integral as follows

$$
\int_{p_{r_1}}^{p_{r_2}} \frac{1}{x^s \log x} dx = \int_{p_{r_1}}^{p_{r_2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} \, dx - i \int_{p_{r_1}}^{p_{r_2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} \, dx.
$$

The first integral on the right side $\int_{p_{r_1}}^{p_{r_2}} e^{-\sigma \log x} \cos(t \log x) \, dx$ can be computed by using the substitution $y = \log x$ to obtain

$$
\int_{p_{r_1}}^{p_{r_2}} e^{-\sigma \log x} \cos(t \log x) \, dx = \int_{p_{r_1}}^{p_{r_2}} e^{(1-\sigma)y} \cos(ty) \, dy,
$$

or

$$
\int_{p_{r_1}}^{p_{r_2}} e^{-\sigma \log x} \cos(t \log x) \, dx = \int_{p_{r_1}}^{p_{r_2}} e^{(1-\sigma)y} \cos(ty) \, dy + \int_{p_{r_1}}^{p_{r_2}} e^{(1-\sigma)y} \, dy - \int_{p_{r_1}}^{p_{r_2}} e^{(1-\sigma)y} \, dy.
$$

Hence,

$$
\int_{p_{r_1}}^{p_{r_2}} e^{-\sigma \log x} \cos(t \log x) \, dx = \int_{p_{r_1}}^{p_{r_2}} e^{(1-\sigma)y} \left( 1 - \cos(ty) \right) \, dy - \int_{p_{r_1}}^{p_{r_2}} e^{(1-\sigma)y} \left( 1 - \cos(ty) \right) \, dy.
$$
\[
\int_{\epsilon}^{p_1} e^{(1-\sigma)y} \frac{dy}{y} + \int_{\epsilon}^{p_2} e^{(1-\sigma)y} \frac{dy}{y}
\]

where, \(\epsilon\) is an arbitrary small positive number. With the variable substantiations \(z_1 = y / \log p_{r1}\) and \(z_2 = y / \log p_{r2}\), we then obtain

\[
\int_{\epsilon / \log p_{r1}}^{1} e^{(1-\sigma)(\log p_{r1})z_1} (1 - \cos(t(\log p_{r1})z_1)) dz_1 -
\int_{\epsilon / \log p_{r2}}^{1} e^{(1-\sigma)(\log p_{r2})z_2} (1 - \cos(t(\log p_{r2})z_2)) dz_2 -
\int_{\epsilon / \log p_{r1}}^{1} e^{(1-\sigma)(\log p_{r1})z_1} dz_1 + \int_{\epsilon / \log p_{r2}}^{1} e^{(1-\sigma)(\log p_{r2})z_2} dz_2
\]

By the virtue of the following identity ([9], page 230)

\[
\int_{0}^{1} e^{t(1 - \cos(bt))} \frac{dt}{t} = \frac{1}{2} \log(1 + b^2/a^2) + \text{Li}(a) + \Re[E_1(-a + ib)],
\]

where \(a > 0\), we then obtain the following

\[
\int_{p_{r1}}^{p_2} e^{-\sigma \log x} \cos(t \log x) \frac{dx}{\log x} = \Re[E_1((s - 1) \log p_{r1})] + \text{Li}((1 - \sigma) \log p_{r1}) - \Re[E_1((s - 1) \log p_{r2})] - \text{Li}((1 - \sigma) \log p_{r2}) -
\int_{\epsilon / \log p_{r1}}^{1} e^{(1-\sigma)(\log p_{r1})z_1} dz_1 + \int_{\epsilon / \log p_{r2}}^{1} e^{(1-\sigma)(\log p_{r2})z_2} dz_2
\]

With the variable substantiations \(w_1 = (1 - \sigma)(\log p_{r1})z_1\) and \(w_1 = (1 - \sigma)(\log p_{r1})z_1\) and by adding and subtracting the terms \(-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\epsilon} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\epsilon} \frac{dw_1}{w_1}\), we then have

\[
\int_{p_{r1}}^{p_2} e^{-\sigma \log x} \cos(t \log x) \frac{dx}{\log x} = \Re[E_1((s - 1) \log p_{r1})] + \text{Li}((1 - \sigma) \log p_{r1}) - \Re[E_1((s - 1) \log p_{r2})] - \text{Li}((1 - \sigma) \log p_{r2}) +
\int_{(1-\sigma)\epsilon}^{(1-\sigma)\epsilon} \frac{e^{w_2} - 1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\epsilon} \frac{e^{w_1} - 1}{w_1} dw_1 +
\int_{(1-\sigma)\epsilon}^{(1-\sigma)\epsilon} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\epsilon} \frac{dw_1}{w_1}.
\]

Using the following identity [9, page 230]

\[
\int_{0}^{a} e^{\frac{t}{a} - 1} dt = \text{Ei}(a) - \log(a) - \gamma
\]

where \(a > 0\), we then obtain for \(\sigma < 1\),

\[
\int_{p_{r1}}^{p_2} e^{-\sigma \log x} \cos(t \log x) \frac{dx}{\log x} = \Re[E_1((s - 1) \log p_{r1})] - \Re[E_1((s - 1) \log p_{r2})]
\]
Similarly, using the identity [9, page 230]

\[
\int_{\rho_0}^{1} \frac{e^{at} \sin(bt)}{t} dt = \pi - \arctan(b/a) + \Im[E_1(-a + ib)],
\]
where \(a > 0\), we can show that for \(\sigma < 1\), we have

\[
- \int_{p_{r1}}^{p_{r2}} e^{-\sigma \log x} \frac{\sin(t \log x)}{\log x} dx = \Im[E_1((s - 1) \log p_{r1})] - \Im[E_1((s - 1) \log p_{r2})].
\]

Therefore, for \(\Re(s) > 0.5\), we have

\[
\sum_{i=r1}^{2} \frac{1}{p_{ri}^s} = E_1((s - 1) \log p_{r1}) - E_1((s - 1) \log p_{r2}) + \varepsilon,
\]
where, \(\varepsilon = O\left(\frac{t+1}{(\sigma-0.5) \log p_{r1}^{1/2-\sigma}}\right).

**Appendix 4**

In Appendix 4, we will show that the sum \(\sum_{\rho} E_1((s - \rho) \log p_{r})\) is convergent if \(|s - \rho| > 0\) for every \(\rho\). Furthermore, we will show that the sum approaches zeros as \(p_{r}\) approaches infinity. This task will be achieved by noting that, for sufficiently large \(p_{r}\), \(E_1((s - \rho) \log p_{r})\) can be written as

\[
E_1((s - \rho) \log p_{r}) = \frac{e^{-(s-\rho) \log p_{r}}}{(s - \rho) \log p_{r}} \left(1 + O\left(\frac{1}{|s - \rho| \log p_{r}}\right)\right)
\]
(78)

Therefore, if the sum \(\sum_{\rho} E_1((s - \rho) \log p_{r})\) is convergent, then it will be given by

\[
\sum_{\rho} E_1((s - \rho) \log p_{r}) = \sum_{\rho} \frac{e^{-(s-\rho) \log p_{r}}}{(s - \rho) \log p_{r}} + \varepsilon,
\]
(79)

where \(\varepsilon\) is the contribution by the sum of the \(O\) terms in Equation (78). It can be easily shown that if \(|s - \rho| \geq \varepsilon > 0\) for every \(\rho\), then \(\varepsilon\) in Equation (79) tends to zero as \(p_{r}\) approaches infinity. This result can be deduced by noting that \(O(\varepsilon) = (p_{r}^{\min \Re(s-\rho)}/(\log p_{r})^2) \sum_{\rho} 1/|s - \rho|^2\). Since the sum \(\sum_{\rho} 1/|s - \rho|^2\) is bounded, therefore Equation (79) can be further simplified to

\[
\sum_{\rho} E_1((s - \rho) \log p_{r}) = \frac{p_{r}^{-s}}{\log p_{r}} \sum_{\rho} \frac{p_{r}^\rho}{s - \rho} + O(p_{r}^{\min \Re(s-\rho)}/(\log p_{r})^2).
\]
(80)

To show the sum \(\sum_{\rho} E_1((s - \rho) \log p_{r})\) is convergent, let \(s = \sigma + iT\) and \(\rho_i = \beta_i + i\gamma_i\). We split \(\rho_i\)'s into two groups. The first group comprises of the non-trivial zeros with \(\gamma_i\)'s less than or equal to \(mT\), where \(m > 1\). The rest of the non-trivial zeros belong to the second group.
Since the first group has a finite number of $\rho_i$’s, thus the sum $\sum_{|\gamma_i|\leq mT} E_1 ((s - \rho) \log p_r)$ is bounded. Since $|p_{r^{-s}} p_r| < 1$ for every $\rho$, therefore

$$\left| \sum_{|\gamma_i|\leq mT} E_1 ((s - \rho) \log p_r) \right| = \frac{1}{1 - \log p_r} \sum_{|\gamma_i|\leq mT} \frac{1}{s - \rho}.$$  

Hence

$$\sum_{|\gamma_i|\leq mT} E_1 ((s - \rho) \log p_r) = O\left(\frac{1}{\log p_r}\right).$$

The sum over the second group can be expanded as follows

$$\sum_{|\gamma_i|> mT} E_1 ((s - \rho) \log p_r) = -\frac{p_r^{-s}}{\log p_r} \left( \sum_{|\gamma_i|> mT} p_r^\rho_i + s \sum_{|\gamma_i|> mT} p_r^\rho_i + s^2 \sum_{|\gamma_i|> mT} p_r^\rho_i + \ldots \right) + \epsilon.$$

The first sum $\sum_{|\gamma_i|> mT} p_r^\rho_i / \rho_i$ is convergent by the virtue of Equation (47). The upper bound for the second term $(p_r^{-s} / \log p_r) s \sum_{|\gamma_i|> mT} p_r^\rho_i / \rho_i^2$ can be determined as follows

$$\left| \frac{p_r^{-s}}{\log p_r} \sum_{|\gamma_i|> mT} \frac{p_r^\rho_i}{\rho_i^2} \right| \leq \left| \frac{p_r^{-s} s}{\log p_r} \sum_{|\gamma_i|> mT} \frac{|p_r^\rho_i|}{\rho_i^2} \right|.$$

Since for sufficiently large $T$, $|s|$ is given by $T$ and the density of the non-trivial zeros is given by $O(\log t)$ (note that if there are roots off the critical line then their density is given by Bohr Landau theorem [1] and it is less than $O(\log t)$), thus

$$\left| \frac{p_r^{-s} s}{\log p_r} \sum_{|\gamma_i|> mT} \frac{p_r^\rho_i}{\rho_i^2} \right| \leq \frac{p_r^{-s + \max \beta_t} T}{\log p_r} \int_{mT}^{\infty} \frac{O(\log t)}{t^2} dt.$$

Hence

$$\left| \frac{p_r^{-s} s}{\log p_r} \sum_{|\gamma_i|> mT} \frac{p_r^\rho_i}{\rho_i^2} \right| \leq \frac{p_r^{-s + \max \beta_t} O(\log T)}{\log p_r} \cdot \frac{1}{m}.$$

Similarly, we can show that

$$\left| \frac{p_r^{-s} s^2}{\log p_r} \sum_{|\gamma_i|> mT} \frac{p_r^\rho_i}{\rho_i^3} \right| \leq \frac{p_r^{-s + \max \beta_t} O(\log T)}{\log p_r} \cdot \frac{1}{m^2},$$

and,

$$\left| \frac{p_r^{-s} s^i}{\log p_r} \sum_{|\gamma_i|> mT} \frac{p_r^\rho_i}{\rho_i^{i+1}} \right| \leq \frac{p_r^{-s + \max \beta_t} O(\log T)}{\log p_r} \cdot \frac{1}{m^i}.$$

Therefore,

$$\left| \frac{p_r^{-s}}{\log p_r} \left( s \sum_{|\gamma_i|> mT} \frac{p_r^\rho_i}{\rho_i^2} + s^2 \sum_{|\gamma_i|> mT} \frac{p_r^\rho_i}{\rho_i^3} + \ldots \right) \right| \leq \frac{p_r^{-s + \max \beta_t} O(\log T)}{\log p_r} \sum_{i=1}^{\infty} \frac{1}{m^i}.$$
Since $\sum_{i=1}^{\infty} 1/m_i$ is convergent, hence $(p_r^{-\sigma+\max\beta_i} O(\log T)/\log p_r) \sum_{i=1}^{\infty} 1/m_i$ is convergent and it is given by

$$\left| \frac{p_r^{-s}}{\log p_r} \left( s \sum_{|\gamma_i| > mT} \frac{p_r^{\rho_i}}{\rho_i^2} + s^2 \sum_{|\gamma_i| > mT} \frac{p_r^\rho}{\rho_i^3} + \ldots \right) \right| = O(p_r^{-\sigma+\max\beta_i} \log(T)/\log p_r).$$

Hence

$$\sum_{|\gamma_i| > mT} E_1 ((s - \rho) \log p_r) = -\frac{p_r^{-s}}{\log p_r} \left( \sum_{|\gamma_i| > mT} \frac{p_r^{\rho_i}}{\rho_i} \right) + O(p_r^{-\sigma+\max\beta_i} \log(T)/\log p_r).$$

Thus

$$\sum_{|\gamma_i| > mT} E_1 ((s - \rho) \log p_r) = -\frac{p_r^{-s}}{\log p_r} \left( \sum_{|\gamma_i| > mT} \frac{p_r^{\rho_i}}{\rho_i} \right) + O(p_r^{-\sigma+\max\beta_i} \log(T)/\log p_r).$$

Consequently, $\sum_{\rho} E_1 ((s - \rho) \log p_r)$ is convergent and it is given by

$$\sum_{\rho} E_1 ((s - \rho) \log p_r) = \frac{p_r^{-s}}{\log p_r} \sum_{\rho} \frac{p_r^\rho}{s - \rho} + O(1/\log p_r).$$

In the remaining of this Appendix, we will derive a formula to show the dependence of the sum $\sum_{\rho} E_1 ((s - \rho) \log p_r)$ on $T$ (where, $s = \sigma + iT$). On RH, we have

$$\sum_{|\gamma_i| > mT} E_1 ((s - \rho) \log p_r) = -\frac{p_r^{-s}}{\log p_r} \left( \sum_{|\gamma_i| > mT} \frac{p_r^{\rho_i}}{\rho_i} \right) + O \left( p_r^{0.5-\sigma} \log(T)/\log p_r \right).$$

Thus

$$\left\| \sum_{|\gamma_i| > mT} E_1 ((s - \rho) \log p_r) \right\| = O \left( p_r^{0.5-\sigma} \log p_r \right) + O \left( p_r^{0.5-\sigma} \log(T)/\log p_r \right).$$

Since the density of the roots on the critical line is given by $\log T$, thus the sum over the roots with $|\gamma_i| \leq mT$ can be given by the following integral

$$\left\| \sum_{|\gamma_i| \leq mT} E_1 ((s - \rho) \log p_r) \right\| = \frac{p_r^{0.5-\sigma}}{\log p_r} \int_{-mT}^{mT} \frac{O(\log t)}{\sqrt{(t - T)^2 + (\sigma - 0.5)^2}} dt.$$ 

Thus, for fixed $\sigma > 0.5 + \epsilon$, we have

$$\left\| \sum_{|\gamma_i| \leq mT} E_1 ((s - \rho) \log p_r) \right\| = p_r^{0.5-\sigma} O((m \log T)^2)/\log p_r.$$ 

Therefore, on RH, we have

$$\left\| \sum_{\rho} E_1 ((s - \rho) \log p_r) \right\| = O \left( p_r^{0.5-\sigma} \log p_r (\log T)^2 \right).$$ (81)
Appendix 5

To show that

\[ \left| \sum_{n=1}^{N} \frac{\mu(n, p_r)}{n} \right| \leq 2 \]

we first note that

- \( \sum_{d|n} \mu(d, p_r) = 1 \), if \( n = 1 \),
- \( \sum_{d|n} \mu(d, p_r) = 1 \), if all the prime factor of \( n \) are less than \( p_r \),
- \( \sum_{d|n} \mu(d, p_r) = 0 \), if any of the prime factor of \( n \) is greater than \( p_r \).

Adding all the terms \( \sum_{d|n} \mu(d, p_r) \) for \( 1 \leq n \leq N \), we then obtain

\[ 0 < \sum_{n=1}^{N} \mu(n, p_r) \left\lfloor \frac{N}{n} \right\rfloor \leq N, \]

where \( \lfloor x \rfloor \) refers to the integer value of \( x \). Define \( r_n \) as

\[ r_n = \frac{N}{n} - \left\lfloor \frac{N}{n} \right\rfloor, \]

where \( 0 \leq r_n < 1 \). Hence, we have

\[ \sum_{n=1}^{N} \mu(n, p_r) r_n < \sum_{n=1}^{N} \mu(n, p_r) \left\lfloor \frac{N}{n} \right\rfloor + \sum_{n=1}^{N} \mu(n, p_r) r_n \leq \sum_{n=1}^{N} \mu(n, p_r) r_n \]

Since

\[ -N \leq \sum_{n=1}^{N} \mu(n, p_r) r_n \leq N, \]

thus, for every \( p_r \) we have

\[ -N < \sum_{n=1}^{N} \mu(n, p_r) \frac{N}{n} \leq 2N, \]

or

\[ -1 < \sum_{n=1}^{N} \frac{\mu(n, p_r)}{n} \leq 2. \]

References