ON THE VALIDITY OF THE RIEMANN HYPOTHESIS

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Abstract

In this paper, we have established a connection between The Dirichlet series with the Mobius function $M(s) = \sum_{n=1}^{\infty} \mu(n)/n^s$ and a functional representation of the zeta function $\zeta(s)$ in terms of its partial Euler product. For this purpose, the Dirichlet series $M(s)$ has been modified and represented in terms of the partial Euler product by progressively eliminating the numbers that first have a prime factor 2, then 3, then 5, ..up to the prime number p_r to obtain the series $M(s, p_r)$. It is shown that the series $M(s)$ and the new series $M(s, p_r)$ have the same region of convergence for every p_r. Unlike the partial sum of $M(s)$ that has irregular behavior, the partial sum of the new series exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series and to examine the validity of the Riemann Hypothesis.

Keywords: Riemann zeta function, Mobius function, Riemann hypothesis, conditional convergence, Euler product.

Classification: Number Theory, 11M26

1 Introduction

The Riemann zeta function $\zeta(s)$ satisfies the following functional equation over the complex plain [2]

$$\zeta(1-s) = 2(2\pi)^2 \cos(0.5s\pi)\Gamma(s)\zeta(s), \tag{1}$$

where, $s = \sigma + it$ is a complex variable and $s \neq 1$.

For $\sigma > 1$ (or $\Re(s) > 1$), $\zeta(s)$ can be expressed by the following series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2}$$

or by the following product over the primes p_i's

$$\frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \tag{3}$$

where, $p_1 = 2$, $\prod_{i=1}^{\infty}(1 - 1/{p_i}^s)$ is the Euler product and $\prod_{i=1}^{r}(1 - 1/{p_i}^s)$ is the partial Euler product. The above series and product representations of $\zeta(s)$ are absolutely convergent for $\sigma > 1$.

The region of the convergence for the sum in Equation (2) can be extended to $\Re(s) > 0$ by using the alternating series $\eta(s)$ where

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \tag{4}$$

and

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s). \tag{5}$$

One may notice that the term $1 - 2^{1-s}$ is zero at $s = 1$. This zero cancels the simple pole that $\zeta(s)$ has at $s = 1$ enabling the extension (or analog continuation) of the zeta function series representation over the critical strip where $0 < \Re(s) < 1$.

It is well known that all of the non-trivial zeros of $\zeta(s)$ are located in the critical strip. Riemann stated that all non-trivial zeros were very probably located on the critical line $\Re(s) = 0.5$ [11]. There are many equivalent statements for the Riemann Hypothesis (RH) and one of them involves the Dirichlet series with the Mobius function.

The Mobius function $\mu(n)$ is defined as follows
$\mu(n) = 1$, if $n = 1$.
$\mu(n) = (-1)^k$, if $n = \prod_{i=1}^{k} p_i$, p_i's are distinct primes.
$\mu(n) = 0$, if $p^2 | n$ for some prime number p.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}. \tag{6}$$

This series is absolutely convergent to $1/\zeta(s)$ for $\Re(s) > 1$ and conditionally convergent to $1/\zeta(s)$ for $\Re(s) = 1$. The Riemann hypothesis is equivalent to the statement that $M(s)$ is conditionally convergent to $1/\zeta(s)$ for $\Re(s) > 0.5$. It should be pointed out that our definition of $M(s)$ is different from Mertens function (defined in the literature as $M(x) = \sum_{1 \le n \le x} \mu(n)$). If we denote $M(s; 1, N)$ as partial sum of the series $M(s)$

$$M(s; 1, N) = \sum_{n=1}^{N} \frac{\mu(s)}{n^s}, \tag{7}$$

then the Mertens function is given by $M(0; 1, N)$. On RH, we then have [14]

$$M(0; 1, N) = O(N^{1/2+\epsilon}),$$

where ϵ is an arbitrary small number. By partial summation, on RH, we also have

$$M(1; 1, N) = O(N^{-1/2+\epsilon}).$$

The irregular behavior of the Mobius function $\mu(n)$ has so far hindered the attempts to estimate the asymptotic behavior of any of the above two sums as N approaches infinity.

The Riemann hypothesis is also equivalent to another statement that involves the prime number function $\pi(x)$ (defined by the the number of primes less than x). The prime counting function can be computed using Riemann Explicit Formula

$$\pi(x) + \sum_{n=2}^{\lfloor \log x/\log 2 \rfloor} \frac{\pi(x^{1/n})}{n} = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log(2) + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

and on RH,

$$\pi(x) = \mathrm{Li}(x) - \frac{\mathrm{Li}(x^{1/2})}{2} - \sum_{\rho} \mathrm{Li}(x^{\rho}) + \text{Lesser terms}$$

where $\mathrm{Li}(x)$ is the Logarithmic Integral of x and the sum $\sum_\rho \mathrm{Li}(x^\rho)$ is performed over the nontrivial zeros $\rho_i = \alpha_i + i\gamma_i$. This sum is conditionally convergent and it should be performed over the nontrivial zeros with $|\gamma_i| \leq T$ as T approaches infinity. The distribution of the prime number can be also analyzed by defining the function $\psi(x)$ as

$$\psi(x) = \sum_{{p_i}^m \leq x} \log p_i$$

and using Von Mangoldt formula given by

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - x^{-2})$$

It is well known that as x approaches infinity, the prime counting function is asymptotic to the function $\mathrm{Li}(x)$. Therefore, if we consider that $\pi(x)$ is comprised of two components, the regulator component given by $\mathrm{Li}(x)$ and the irregular component $J(x)$ given by

$$J(x) = \pi(x) - \mathrm{Li}(x) \tag{8}$$

then on RH, we have

$$J(x) < \frac{1}{8\pi}\sqrt{x}\log x \qquad \text{for } x > 2657$$

The irregular component $J(x)$ is also given by [12] (refer to lemmas 5 and 6)

$$J(x) = \frac{\psi(x) - x}{\log x} + O\left(\frac{\sqrt{x}}{\log x}\right)$$

or

$$J(x) = -\frac{1}{\log x}\sum_{\rho} \frac{x^{\rho}}{\rho} + O\left(\frac{\sqrt{x}}{\log x}\right) \tag{9}$$

Our method to examine the validity of the Riemann Hypothesis is based on representing the Dirichlet series $M(s)$ (defined by Equation (6)) in terms of the integral $\int dJ(x)/x$. In order to do that, we need to smooth the irregular behavior of the function $M(s)$ by introducing a method to represent the series $M(s)$ in terms of the partial Euler product. This task is achieved in section 2 by first eliminating the numbers that have the prime factor 2 to generate the series $M(s, 3)$ (i.e, the series $M(s, 3)$ is void of any number with a prime factor less than 3). For the series $M(s, 3)$, we then eliminate the numbers with the prime factor 3 to generate the series $M(s, 5)$, and so on, up to the prime number p_r. In other words, we have applied the sieving technique to modify the series $M(s)$ to include only the numbers with prime factors greater than or equal to p_r. In the literature [10], numbers with prime factors less than y are called y-smooth while numbers with prime factors greater than y are called y-rough. In essence, our approach is to compute the Dirichlet series over p_{r-1}-rough numbers. In section 3, we have shown that the series $M(s)$ and the new series $M(s, p_r)$ have the same region of convergence (Theorem 1).

We will then present two methods to represent the series $M(s, p_r)$ in terms of the integral $\int_{p_r}^{\infty} dJ(x)/x$. The first method is based on complex analysis (section 4). With this method, we have provided a functional equation for $\zeta(s)$ using its partial Euler product. The second method is described in section 5 and it is based on integration methods to represent the series $M(s, p_r)$ in terms of the integral $\int_{p_r}^{\infty} dJ(x)/x$.

Gonek, Hughes and Keating [5] have done an extensive research into establishing a relationship between $\zeta(s)$ and its partial Euler product for $\Re(s) < 1$. Gonek stated "Analytic number theorists believe that an eventual proof of the Riemann Hypothesis must use both the Euler product and functional equation of the zeta-function. For there are functions with similar functional equations but no Euler product, and functions with an Euler product but no functional equation." In section 4, we will present a functional equation for $\zeta(s)$ using its partial Euler product. The method is based on writing the Euler product formula as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i^s}\right)\prod_{r}^{\infty}\left(1 - \frac{1}{p_i^s}\right).$$

The above equation is valid for $\Re(s) > 1$. To be able to represent $\zeta(s)$ in term of its partial Euler product for $\Re(s) \leq 1$, we need to replace the term $\prod_r^{\infty}(1 - 1/p_i^s)$ with an equivalent one that allows the analytic continuation for the representation of $\zeta(s)$ for $\Re(s) \leq 1$. Thus, the new term (that we need to introduce to replace $\prod_r^{\infty}(1 - 1/p_i^s)$) must have a zero that cancels the pole that $\zeta(s)$ has at $s = 1$. In the section 4, we will use the complex analysis to compute this new term and then represent $\zeta(s)$ in terms of its partial Euler product. This functional representation is given by Theorem 2. We have then used this theorem to represent the series $M(s, p_r)$ in terms of the integral $\int_{p_r}^{\infty} dJ(x)/x$ (Theorem 3).

As mentioned before, the efforts to use the series $M(\sigma)$ to examine the validity of the Riemann Hypothesis have so far failed due to the irregular behavior of the partial sum of the series $M(\sigma)$ (due to the irregular behavior of the Mobius function $\mu(n)$). In section 5, we have shown that the partial sum of the new series $M(\sigma, p_r)$ exhibits regular behavior as p_r approaches infinity. This has allowed the use of integration methods to compute the partial sum of the new series. We have then shown that the partial sum of the series $M(1, p_r)$ can be decomposed into two terms (Theorem 4). The first term, that we have called the regular component, is generated by the regular component of the prime counting function $\mathrm{Li}(x)$. The second term is the remainder and we denote it as the irregular component.

In section 6, we have used Theorem 3 and the Fourier analysis to derive a second representation for the partial sum of the irregular component of the series $M(1, p_r)$. The two representations of the irregular component of the partial sum of the series $M(1, p_r)$ are then compared to examine the validity of the Riemann Hypothesis. This comparison analysis indicates that non-trivial zeros can be found arbitrary close to the line $\Re(s) = 1$.

2 Applying the Sieving Method to the Dirichlet Series $M(s)$.

The Dirichlet series $M(s)$ with the Mobius function is defined as

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s},$$

where $\mu(n)$ is the Mobius function. Thus,

$$M(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{1}{6^s}....$$

Next, we introduce the series $M(s,3)$ by eliminating all the numbers that have a prime factor 2 (or keeping only the numbers with prime factors greater than or equal to 3). Thus, $M(s,3)$ can be written as

$$M(s,3) = 1 - \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s} + \frac{0}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{15^s}.....$$

Our analysis to test the conditional convergence of these series ($M(s)$ and $M(s,3)$ for $\sigma \leq 1$) is based on comparing correspondent terms of these two series. Therefore, rearrangement and permutation of the terms may have a significant impact on analyzing the region of convergence of both series. Thus, it essential to have the same index for both series $M(s)$ and $M(s,3)$ refer to the same term. Hence, we will represent $M(s,3)$ as follows

$$M(s,3) = 1 + \frac{0}{2^s} - \frac{1}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

or

$$M(s,3) = \sum_{n=1}^{\infty} \frac{\mu(n,3)}{n^s}, \tag{10}$$

where
$\mu(n,3) = \mu(n)$, if n is an odd number,
$\mu(n,3) = 0$, if n is an even number.

The above series $M(s,3)$ can be further modified by eliminating all the numbers that have a prime factor 3 (or keeping only the numbers with prime factors greater than or equal to 5) to get the series $M(s,5)$ where

$$M(s,5) = 1 - \frac{1}{5^s} - \frac{1}{7^s} - \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} - \frac{1}{19^s} - \frac{1}{23^s} + \frac{0}{25^s}....,$$

or more conveniently

$$M(s,5) = 1 + \frac{0}{2^s} - \frac{0}{3^s} + \frac{0}{4^s} - \frac{1}{5^s} + \frac{0}{6^s} - \frac{1}{7^s} - \frac{0}{8^s}....,$$

and so on.

Let $I(p_r)$ represent, in ascending order, the integers with distinct prime factors that belong to the set $\{p_i : p_i \geq p_r\}$. Let $\{1, I(p_r)\}$ be the set of 1 and $I(p_r)$ (for example, $\{1, I(3)\}$ is the set of square-free odd numbers), then we define the series $M(s, p_r)$ as

$$M(s,p_r) = \sum_{n=1}^{\infty} \frac{\mu(n,p_r)}{n^s}, \tag{11}$$

where
$\mu(n,p_r) = \mu(n)$, if $n \in \{1, I(p_r)\}$,
otherwise, $\mu(n,p_r) = 0$.

It can be easily shown that, for every prime number p_r, the series $M(s,p_r)$ converges absolutely for $\Re(s) > 1$. Furthermore, it can be shown that, for $\Re(s) > 1$, $M(s,p_r)$ satisfies the following equation

$$M(s) = M(s,p_r) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right). \tag{12}$$

Since

$$M(s) = \frac{1}{\zeta(s)} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right),$$

therefore we conclude that, for $\Re(s) > 1$, $M(s,p_r)$ approaches 1 as p_r approaches infinity. It should be pointed out here that with this definition of $M(s,p_r)$, $M(2,s)$ is equal to $M(s)$.

3 The region of convergence for the series $M(s)$ and $M(s,p_r)$.

In this section, we will deal with the question of the relationship between the conditional convergence of the two series $M(s,p_r)$ and $M(s)$ over the strip $0.5 < \Re(s) \leq 1$. Theorem 1 establishes this relationship.

Theorem 1. *For $s = \sigma + it$, where $0.5 < \sigma \leq 1$ and for every prime number p_r, the series $M(s)$ converges conditionally if and only if the series $M(s,p_r)$ converges conditionally. Furthermore, within the region of convergence, $M(s)$ and $M(s,p_r)$ are related as follows*

$$M(s) = M(s,p_r) \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^s}\right). \tag{13}$$

The proof of this theorem can be achieved either by applying the Cauchy convergence criteria or more conveniently by applying the complex analysis where we take advantage of the fact that both functions $\zeta(s)$ and $\zeta(s)\prod_{i=1}^{r-1}(1 - 1/p_i^s)$ have the same zeros (and a simple pole at $s = 1$) to the right of the line $\Re(s) = 0$.

In the following, we will use the complex analysis to prove Theorem 1 by using a method similar to the one outlined by Littlewood theorem that shows that the Riemann Hypothesis is valid if and only if the sum $\sum_{n=1}^{\infty} \mu(n)/n^s$ is convergent to $1/\zeta(s)$ for every s with $\sigma > 0.5$. The prove of this theorem can be found in [14] (refer to Theorem 14.12) and it depends mainly on Lemma 3.12 of the same reference [14]. This Lemma states: Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$, where $\sigma > 1$, $a_n = O(\psi(n))$ being non-decreasing and $\sum_{n=1}^{\infty} |a_n|/n^\sigma = O(1/(\sigma-1)^\alpha)$ as $\sigma \to 1$. Then, if $c > 0$, $\sigma + c > 1$, x is not an integer and N is the integer nearest to x, we have

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w} dw + O\left(\frac{x^c}{T(\sigma+c-1)^\alpha}\right) + O\left(\frac{\psi(2x)x^{1-\sigma}\log x}{T}\right) + O\left(\frac{\psi(N)x^{1-\sigma}}{T|x-N|}\right)$$

To prove the first part of Theorem 1 (i.e. for $s = \sigma + it$ and $0.5 < \sigma \leq 1$, the series $M(s, p_r)$ converges conditionally if $M(s)$ converges conditionally), we note that for $\sigma > 1$,

$$M(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

and

$$M(s, p_r) = \sum_{n=1}^{\infty} \frac{\mu(n, p_r)}{n^s} = \frac{1}{\zeta(s) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right)}.$$

If we assume that $M(s)$ is convergent for $\sigma > h > 0.5$, then $\zeta(s)$ has no zeros in the complex plane to the right of the line $\Re(s) = h$ [14] (refer to Theorem 14.12). Consequently, the function $\zeta(s) \prod_{i=1}^{r-1}(1 - 1/p_i^s)$ has no zeros in the complex plane to the right of the line $\Re(s) = h$. Thus, we may apply Lemma 3.12 [14] with $a_n = \mu(n, p_r)$, $f(s) = 1/\Big(\zeta(s) \prod_{i=1}^{r-1}(1 - 1/p_i^s)\Big)$, $c = 2$ and x half an odd integer to obtain [14] (refer to Theorem 14.12)

$$\sum_{n<x} \frac{\mu(n, p_r)}{n^s} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^{s+w}}\right)} \frac{x^w}{w} dw + O\left(\frac{x^2}{T}\right)$$

However, by the calculus of residues we have

$$\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^{s+w}}\right)} \frac{x^w}{w} dw = \frac{1}{\zeta(s) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right)} +$$

$$\frac{1}{2\pi i} \left(\int_{2-iT}^{h-\sigma+\gamma-iT} + \int_{h-\sigma+\gamma-iT}^{h-\sigma+\gamma+iT} + \int_{h-\sigma+\gamma+iT}^{2+iT} \right) \frac{1}{\zeta(s+w) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^{s+w}}\right)} \frac{x^w}{w} dw$$

where, $0 < \gamma < \sigma - h$. Since, along the line of integration and for an arbitrary small ϵ, we have $1/\zeta(\sigma + iT) = O(T^\epsilon)$ [14], therefore the first and third integrals on right side of the above equation are given by $O(T^{-1+\epsilon} x^2)$ while the second integral is given by $O(x^{h-\sigma+\gamma} T^\epsilon)$. Hence

$$\sum_{n<x} \frac{\mu(n, p_r)}{n^s} = \frac{1}{\zeta(s) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right)} + O(T^{-1+\epsilon} x^2) + O(T^\epsilon x^{h-\sigma+\gamma})$$

Taking $T = x^3$, the $O-$terms tend to zero as x approaches infinity. Consequently, the partial sum $\sum_{n<x} \mu(n, p_r)/n^s$ is convergent as x approaches infinity and it is given by

$$M(s, p_r) = \sum_{n=1}^{\infty} \frac{\mu(n, p_r)}{n^s} = \frac{1}{\zeta(s) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right)}.$$

or

$$M(s) = M(s, p_r) \prod_{i=1}^{r-1} \left(1 - \frac{1}{{p_i}^s}\right).$$

Similarly, we can prove the second part of Theorem 1 (i.e. for $s = \sigma + it$ and $0.5 < \sigma \leq 1$, the series $M(s)$ converges conditionally if $M(s, p_r)$ converges conditionally). The second part of the theorem can be also proved by first defining $M(s, p_r; N_1, N_2)$ as the partial sum

$$M(s, p_r; N_1, N_2) = \sum_{n=N_1}^{N_2} \frac{\mu(n, p_r)}{n^s}, \tag{14}$$

where $N_2 \geq N_1 \geq p_r$. Then, we have

$$M(s, p_{r-1}; 1, Np_{r-1}) = M(s, p_r; 1, Np_{r-1}) - \frac{1}{p_{r-1}^s} M(s, p_r; 1, N). \tag{15}$$

Since the series $M(s, p_r)$ is conditionally convergent, then the partial sums $M(s, p_r; 1, Np_r)$ and $M(s, p_r; 1, N)$ are both convergent to $M(s, p_r)$ as N approaches infinity. Furthermore, the partial sum $M(s, p_r; Np_{r-1}, Np_{r-1} + k)$ (for any integer k in the range $1 \leq k \leq p_{r-1}$) approaches zero as N approaches infinity. Hence, as N approaches infinity, we obtain

$$M(s, p_{r-1}) = \lim_{x \to \infty} M(s, p_{r-1}; 1, x) = M(s, p_r) \left(1 - \frac{1}{p_{r-1}^s}\right).$$

By repeating this process $r - 1$ times, we then obtain

$$M(s) = M(s, p_r) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right).$$

4 Functional representation of $\zeta(s)$ using its partial Euler product.

In this section, we will use the prime counting function to derive a functional representation for $\zeta(s)$ using its partial Euler product. We will start this task by first writing $\zeta(s)$ for $\sigma > 1$ as follows

$$1/\zeta(s) = \prod_{i=1}^{\infty} \left(1 - \frac{1}{p_i^s}\right) = \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right) \prod_{r}^{\infty} \left(1 - \frac{1}{p_i^s}\right). \tag{16}$$

For $\sigma > 0.5$, we have

$$\log \prod_{i=r1}^{r2} \left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2} \log \left(1 - \frac{1}{{p_i}^s}\right) + 2\pi i N,$$

where N is zero, positive or negative integer to account for the ambiguity in the phase of the logarithm of complex numbers. Since $1/|p_i^s| < 1$, hence,

$$\log \prod_{i=r1}^{r2} \left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r1}^{r2} \left(-\frac{1}{{p_i}^s} - \frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - ...\right) + 2\pi i N.$$

Let $\delta(p_{r1}, p_{r2}, s)$ and $\delta(p_{r1}, s)$ be defined as the sums

$$\delta(p_{r1}, p_{r2}, s) = \sum_{i=r1}^{r2} \left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}} ...\right), \tag{17}$$

and

$$\delta(p_{r1}, s) = \sum_{i=r1}^{\infty} \left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}} ... \right). \tag{18}$$

Thus,

$$\log \prod_{i=r1}^{r2} \left(1 - \frac{1}{p_i^s}\right) = -\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} + \delta(p_{r1}, p_{r2}, s) + 2\pi i N. \tag{19}$$

Since $|\delta(p_{r1}, p_{r2}, s)| < \sum_{n=p_{r1}}^{\infty} \left(\frac{1}{2n^{2\sigma}} + \frac{1}{3n^{3s}} + \frac{1}{4n^{4s}} ... \right)$, thus $|\delta(p_{r1}, p_{r2}, s)| = O(p_{r1}^{1-2\sigma}/(2\sigma - 1))$. Furthermore, if $2\sigma - 1$ is a fixed positive number, then $|\delta(p_{r1}, p_{r2}, s)| = O(p_{r1}^{1-2\sigma})$.

Using the Prime Number Theorem (PNT) with a suitable constant $a > 0$, the number of primes less than x is given by [13] (refer to page 43)

$$\pi(x) = \text{Li}(x) + J(x), \tag{20}$$

where $\text{Li}(x)$ is the Logarithmic Integral of x and

$$J(x) = O\left(xe^{-a\sqrt{\log x}}\right), \tag{21}$$

or

$$J(x) = O\left(x/(\log x)^k\right), \tag{22}$$

where k is a number greater than zero.

Using Stieltjes integral [7], we may then write the sum $\sum_{i=r1}^{r2} 1/{p_i}^{\sigma}$ for $\sigma > 1$ as follows

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = \int_{x=p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^{\sigma}}. \tag{23}$$

Using Equation (22) for the representation of $\pi(x)$, we may then write the integral in Equation (23) as [7] (refer to Theorem 2, page 57)

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^{\sigma}} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r1})^k}\right), \tag{24}$$

where k is a number greater than zero. Therefore,

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = \int_{p_{r1}}^{\infty} \frac{1}{x^{\sigma}} \frac{1}{\log x} dx - \int_{p_{r2}}^{\infty} \frac{1}{x^{\sigma}} \frac{1}{\log x} dx + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{25}$$

Recalling that the Exponential Integral $E_1(r)$ is given by

$$E_1(r) = \int_r^{\infty} \frac{e^{-u}}{u} du,$$

and using the substitutions $u = (\sigma - 1)\log x$, $du = (\sigma - 1)dx/x$ and $x^{\sigma}/x = e^u$, then for $\sigma > 1$, we may write Equation (25) as

$$\sum_{i=r1}^{r2} \frac{1}{p_i^{\sigma}} = E_1\left((\sigma - 1)\log p_{r1}\right) - E_1\left((\sigma - 1)\log p_{r2}\right) + O\left(\frac{1}{(\log p_{r1})^k}\right). \tag{26}$$

Combining Equations (19) and ((26)) and noting that, for $\sigma > 1$, $E_1\left((\sigma-1)\log p_{r2}\right)$ approaches zero as p_{r2} approaches infinity, we may write Equation (16) for $s = \sigma$ and $\sigma > 1$ as

$$-\log\zeta(\sigma) = \sum_{i=1}^{r-1}\log\left(1-\frac{1}{{p_i}^{\sigma}}\right) - \sum_{i=r}^{\infty}\frac{1}{{p_i}^{\sigma}} + \delta(p_r,\sigma),$$

or

$$\log\zeta(\sigma) + \sum_{i=1}^{r-1}\log\left(1-\frac{1}{{p_i}^{\sigma}}\right) - E_1\left((\sigma-1)\log p_r\right) = \epsilon,$$

where $\epsilon = O(1/(\log p_{r1})^k)$ can be made arbitrarily small by setting p_r sufficiently large. Therefore, by taking the exponential of both sides of the above equation, we then have

$$\zeta(\sigma)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^{\sigma}}\right)\exp\left(-E_1((\sigma-1)\log p_r)\right) = 1+\epsilon+O(\epsilon^2). \tag{27}$$

As p_r approaches infinity, ϵ approaches zero. Hence, the right side of the above equation approaches 1 as p_r approaches infinity.

Similarly, for $\Re(s) > 1$, we can use the following expression for $E_1(s)$

$$E_1(s) = \int_1^{\infty}\frac{e^{-xs}}{x}dx,$$

to show that

$$\log\zeta(s) + \sum_{i=1}^{r-1}\log\left(1-\frac{1}{{p_i}^{s}}\right) - E_1\left((s-1)\log p_r\right) = \epsilon + 2\pi i N.$$

where $|\epsilon|$ can be made arbitrarily small by setting p_r sufficiently large. Taking the exponent of both sides and allowing r to approach infinity, we then have

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^{s}}\right)\exp\left(-E_1((s-1)\log p_r)\right)\right\} = 1. \tag{28}$$

Let the function $G(s,p_r)$ be defined as

$$G(s,p_r) = \zeta(s)\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^{s}}\right)\exp\left(-E_1((s-1)\log p_r)\right) \tag{29}$$

where, $G(s,p_r)$ is a regular function for $\Re(s) > 1$. Referring to Equation (28), the function $G(s,p_r)$ approaches 1 as p_r approaches infinity. It should be noted that, for every p_r, the function $\exp\left(-E_1((s-1)\log p_r)\right)$ is an entire function, the function $\zeta(s)$ is analytic everywhere except at $s = 1$ and the function $\prod_{i=1}^{r-1}(1-1/p_i^s)$ is analytic for $\Re(s) > 0$. Thus, for any $\sigma > 1$, the function $G(s,p_r)$ can be considered as a sequence of analytic functions. Furthermore, as p_r (or r) approaches infinity, this sequence is uniformly convergent over the half plane with $\sigma > 1+\epsilon$ (where, ϵ is an arbitrary small number). Therefore, by the virtue of the Weiestrass theorem, the limit is also analytic function [4] (Weiestrass theorem states that if the function sequence f_n is analytic over the region Ω and f_n is uniformly convergent to a function f, then

f is also analytic on Ω and ${f_n}'$ converges uniformly to f' on Ω). If we define this limit as $G(s)$, where

$$G(s) = \lim_{r\to\infty} G(s, p_r) \tag{30}$$

then, $G(s)$ is analytic over the half plane $\Re(s) > 1$ and it is equal to 1 by the virtue of Equation (28).

Our next task is to extend the previous results to the line $s = 1 + it$. We will then show that on RH and for the strip $s = \sigma + it$ (where $0.5 < \sigma < 1$), these results are also valid with the limit of $G(s, p_r)$ is 1 as p_r approaches infinity.

We will start this task by showing that although both $\zeta(s)$ and $E_1((s-1)\log p_r)$ have a singularity at $s = 1$, the product $G(s, p_r)$ has a removable singularity at $s = 1$ for every p_r. This can be shown by first expanding $\zeta(s)$ as a Laurent series about its singularity at $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2\frac{(s-1)^2}{2!} - \gamma_3\frac{(s-1)^3}{3!} + ..., \tag{31}$$

where γ is the Euler-Mascheroni constant and γ_i's are the Stieltjes constants. For $s = 1 + \epsilon$, where $\epsilon = \epsilon_1 + i\epsilon_2$, ϵ_1 and ϵ_2 are arbitrary small numbers, the above equation can be written as

$$\zeta(s) = \frac{1}{\epsilon} + \gamma - \gamma_1\epsilon + \gamma_2\frac{\epsilon^2}{2!} - \gamma_3\frac{\epsilon^3}{3!} + ... \tag{32}$$

Furthermore, using the definition of the Exponential Integral, we may write $E_1(s)$ as

$$E_1(s) = -\gamma - \log s + s - \frac{s^2}{2\,2!} + \frac{s^3}{3\,3!} - \frac{s^4}{4\,4!} + ... \tag{33}$$

Thus, for $s = 1 + \epsilon$, we have

$$\exp\left(-E_1((s-1)\log p_r)\right) = e^{\gamma}\epsilon\ \log p_r\ \exp\left(-\epsilon\log p_r + \frac{(\epsilon\log p_r)^2}{2\,2!} - \frac{(\epsilon\log p_r)^3}{3\,3!} + ...\right). \tag{34}$$

By taking the product $\zeta(s)\exp\left(-E_1((s-1)\log p_r)\right)$ and allowing $|\epsilon|$ to approach zero, we then have

$$\lim_{s\to 1}\left\{\zeta(s)\ \exp\left(-E_1((s-1)\log p_r)\right)\right\} = e^{\gamma}\ \log p_r. \tag{35}$$

However, it is well known that the partial Euler product at $s = 1$ can be written as [10]

$$\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i}\right) = \frac{e^{-\gamma}}{\log p_{r-1}} + O\left(\frac{1}{(\log p_{r-1})^2}\right). \tag{36}$$

Multiplying Equations (35) and (36), we then conclude that at $s = 1$, $G(s, p_r)$ approaches 1 as p_r approaches infinity. Furthermore, for $s = 1 + it$ and $t \neq 0$, the value of $\exp(-E_1(it\log p_r))$ approaches 1 as p_r approaches infinity and since

$$\lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1 - \frac{1}{{p_i}^s}\right)\right\} = 1,$$

therefore, for $s = 1 + it$, we have the following

$$\lim_{r\to\infty} G(s, p_r) = \lim_{r\to\infty}\left\{\zeta(s)\prod_{i=1}^{r-1}\left(1 - \frac{1}{p_i^s}\right)\exp\left(-E_1((s-1)\log p_r)\right)\right\} = 1.$$

So far, we have shown that the function $G(s, p_r)$ is uniformly convergent to 1 when $\Re(s) > 1 + \delta > 1$ (where δ is an arbitrary small number). We have also shown that $G(s, p_r)$ is convergent to 1 for $\Re(s) = 1$. In the following, we will show that, assuming the validity of the Riemann Hypothesis, the function $G(s, p_r)$ is uniformly convergent to 1 for every value of s with $\Re(s) > 0.5 + \epsilon$, where ϵ is an arbitrary small number. Toward this end, we will first show that on RH the function $G(s, p_r)$ is convergent for any value of s on the real axis with $\sigma > 0.5$. This can be achieved by first writing the expressions for $G(\sigma, p_{r1})$ and $G(\sigma, p_{r2})$ (where $r2$ is an arbitrary large number greater than $r1$)

$$G(\sigma, p_{r1}) = \zeta(\sigma) \exp\left(-E_1((\sigma - 1)\log p_{r1})\right) \prod_{i=1}^{r1-1} \left(1 - \frac{1}{p_i^{\sigma}}\right), \tag{37}$$

$$G(\sigma, p_{r2}) = \zeta(\sigma) \exp\left(-E_1((\sigma - 1)\log p_{r2})\right) \prod_{i=1}^{r2-1} \left(1 - \frac{1}{p_i^{\sigma}}\right). \tag{38}$$

Since the function $G(s, p_r)$ is analytic and not equal to 0 for $\sigma > 0.5$, hence we can divide Equation (38) by Equation (37) and then take the logarithm to obtain

$$\log\left(\frac{G(\sigma, p_{r2})}{G(\sigma, p_{r1})}\right) = E_1\left((\sigma - 1)\log p_{r1}\right) - E_1\left((\sigma - 1)\log p_{r2}\right) + \log\left(\prod_{i=r1}^{r2-1}\left(1 - \frac{1}{{p_i}^{\sigma}}\right)\right) + 2i\pi N_1. \tag{39}$$

where N_1 is zero, positive or negative integer. To compute the logarithm of the partial Euler product in Equation (39), we recall Equation (19) with $s = \sigma + i0$ and $\sigma > 0.5$ to obtain

$$\log \prod_{r1}^{r2-1} \left(1 - \frac{1}{p_i^{\sigma}}\right) = -\sum_{i=r1}^{r2-1} \frac{1}{{p_i}^{\sigma}} + \delta(p_{r1}, p_{r2-1}, \sigma),$$

where $\delta(p_{r1}, p_{r2-1}, \sigma) = O(p_{r1}^{1-2\sigma}/(2\sigma - 1))$. Furthermore, we have

$$\pi(x) = \mathrm{Li}(x) + J(x), \tag{40}$$

where, on RH, $J(x)$ is given by

$$J(x) = O\left(\sqrt{x}\, \log x\right). \tag{41}$$

Using the above equation for the representation of the prime counting function, we may then have (refer to Appendix 1)

$$\sum_{i=r1}^{r2-1} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma - 1)\log p_{r1}) - E_1((\sigma - 1)\log p_{r2-1}) + \varepsilon(p_{r1}, p_{r2-1}, \sigma),$$

where

$$\varepsilon(p_{r1}, p_{r2}, s) = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^s,$$

$$\varepsilon(p_{r1}, s) = \int_{p_{r1}}^{\infty} dJ(x)/x^s,$$

and on RH, $|\varepsilon(p_{r1}, p_{r2}, s)| = O\left({p_{r1}}^{0.5-\sigma} \log p_{r1}/(\sigma - 0.5)^2\right)$. Hence, on RH and for $\sigma > 0.5$, Equation (39) can be written as

$$\log\frac{G(\sigma,p_{r2})}{G(\sigma,p_{r1})}=-\varepsilon(p_{r1},p_{r2-1},\sigma)+\delta(p_{r1},p_{r2-1},\sigma)+E_1((\sigma-1)\log p_{r2-1})-E_1((\sigma-1)\log p_{r2})+2i\pi N_1.$$

Taking the exponential of both sides, we then have

$$\frac{G(\sigma,p_{r2})}{G(\sigma,p_{r1})}=\exp\left(-\varepsilon(p_{r1},p_{r2-1},\sigma)+\delta(p_{r1},p_{r2-1},\sigma)+E_1((\sigma-1)\log p_{r2-1})-E_1((\sigma-1)\log p_{r2})\right).$$

However, the difference $E_1((\sigma-1)\log p_{r2-1})-E_1((\sigma-1)\log p_{r2})$ approaches zero as p_{r2} approaches infinity. This follows from Cramer's theorem on the gap between primes. This theorem states that on RH, the gap between the prime number p_{r-1} and p_r is less than $k\sqrt{p}_r\log p_r$ for some constant k [3]. Therefore,

$$\lim_{p_{r2}\to\infty}\frac{G(\sigma,p_{r2})}{G(\sigma,p_{r1})}=e^{-\varepsilon(p_{r1},\sigma)+\delta(p_{r1},\sigma)}.$$

It should be emphasized here that for the above equation, p_{r1} was kept fixed while p_{r2} was allowed to approach infinity. Therefore, $G(\sigma,p_r)$ is bounded for any arbitrary large p_r. Furthermore, on RH and for $\sigma>0.5+\epsilon$, the term $-\varepsilon(p_{r1},\sigma)+\delta(p_{r1},\sigma)$ can be made arbitrary small by choosing p_{r1} arbitrary large, thus the limit of $G(\sigma,p_r)$ as p_r approaches infinity exists and it is given by

$$G(\sigma)=\lim_{r\to\infty}G(\sigma,p_r)\tag{42}$$

This proves that, on RH, $G(\sigma,p_r)$ is convergent as p_r approaches infinity and thus $G(\sigma)$ exists for $\sigma>0.5$.

Similarly, we can follow the same steps to show that $G(s,p_r)$ is convergent as p_r approaches infinity and thus $G(s)$ exists for $\Re(s)>0.5$. Therefore, on RH and for $\sigma>0.5+\epsilon$, we have (refer to Equation (39))

$$\log\left(\frac{G(s,p_{r2})}{G(s,p_{r1})}\right)=E_1\left((s-1)\log p_{r1}\right)-E_1\left((s-1)\log p_{r2}\right)-\sum_{i=r1}^{r2-1}\frac{1}{{p_i}^s}+\delta(p_{r1},p_{r2-1},s)+2\pi iN_1,\tag{43}$$

where N_1 is zero, positive or negative integer. In Appendix 2, we have shown that, on RH and for $\Re(s)>0.5$, we also have

$$\sum_{i=r1}^{r2}\frac{1}{{p_i}^s}=E_1((s-1)\log p_{r1})-E_1((s-1)\log p_{r2})+\varepsilon(p_{r1},p_{r2},s),\tag{44}$$

where $\varepsilon(p_{r1},p_{r2},s)=\int_{p_{r1}}^{p_{r2}}dJ(x)/x^s$, $|\varepsilon(p_{r1},p_{r2},s)|=O\left(\frac{|s|}{(\sigma-0.5)^2}\,{p_{r1}}^{0.5-\sigma}\log p_{r1}\right)$ (on RH and for $\sigma>0.5$) and $\varepsilon(p_{r1},s)=\int_{p_{r1}}^{\infty}dJ(x)/x^s$. Hence

$$\lim_{p_{r2}\to\infty}\frac{G(s,p_{r2})}{G(s,p_{r1})}=e^{-\varepsilon(p_{r1},s)+\delta(p_{r1},s)}.$$

Therefore, the limit of $G(s,p_r)$ as p_r approaches infinity exists and it is given by

$$G(s)=\lim_{r\to\infty}G(s,p_r)\tag{45}$$

It should be noted that, while the function sequence $G(s, p_r)$ is not uniformly convergent when the region of convergence is extended all the way to the line $\sigma = 0.5$, it is however uniformly convergent for any rectangle extending from $-iT$ to iT (for any arbitrary large T) and with $\sigma > 0.5 + \epsilon$ (for any arbitrary small ϵ). This follows from the fact that, on RH, $\varepsilon(s, p_r)$ is bounded for any rectangle extending from $-iT$ to iT (for any arbitrary large T) and with $\sigma > 0.5 + \epsilon$ (for any arbitrary small ϵ). . Since $G(s, p_r)$ is analytic for $\Re(s) > 0$ and it is uniformly convergent for $\Re(s) > 0.5 + \epsilon$, thus $G(s)$ is analytic for the half right complex plain with $\Re(s) > 0.5 + \epsilon$ (Weiestrass theorem [4]). Since we have shown that $G(s) = 1$ for $\Re(s) \geq 1$, thus on RH, $G(s) = 1$ for $\Re(s) > 0.5 + \epsilon$. Consequently, we have the following theorem

Theorem 2. *On RH and for $\sigma > 0.5$, we have*

$$\lim_{r \to \infty} \left\{ \zeta(s) \prod_{i=1}^{r-1} \left(1 - \frac{1}{p_i^s}\right) \exp\left(-E_1((s-1)\log p_r)\right) \right\} = 1. \tag{46}$$

$$\lim_{r \to \infty} \left\{ M(s, p_r) \exp\left(E_1((s-1)\log p_r)\right) \right\} = 1. \tag{47}$$

It should be also pointed out that Theorem 2 can be generalized for the case where there are no non-trivial zeros for values of s with $\Re(s) > h$ (where $h > 0.5$). For this case, Equation (46) is valid for every s with $\Re(s) > h$ and $|\varepsilon(p_{r1}, s)|$ in Appendix 2 is given by $O\left(\frac{|s|}{(\sigma-h)^2} {p_{r1}}^{h-\sigma} \log p_{r1}\right)$.

Equation (46) of Theorem 2 can be written as follows

$$\log \zeta(s) + \log \prod_{i=1}^{r2-1} \left(1 - \frac{1}{p_i^s}\right) - E_1\left((s-1)\log p_{r2}\right) + 2\pi i N_2 = 0 \quad \text{as } r_2 \to \infty$$

where N_2 is zero, positive or negative number. Notice that the equality of both sides of the above equation is attained as r_2 (or p_{r2}) approaches infinity (or more appropriately, the right side can be made arbitrary close to zero by choosing p_{r2} sufficiently large). For $r < r2$, the above equation can be then written as

$$\log \zeta(s) = E_1\left((s-1)\log p_{r2}\right) - \sum_{i=1}^{r-1} \log\left(1 - \frac{1}{p_i^s}\right) - \sum_{i=r}^{r2-1} \log\left(1 - \frac{1}{p_i^s}\right) + 2\pi i N_3 \quad \text{as } r_2 \to \infty$$

where N_3 is zero, positive or negative number and

$$-\sum_{i=r}^{r2-1} \log\left(1 - \frac{1}{p_i^s}\right) = \sum_{i=r}^{r2-1} \frac{1}{{p_i}^s} - \delta(p_r, p_{r2-1}, s) + 2\pi i N_4$$

where N_4 is zero, positive or negative number. For the region of convergence of the series $M(s, p_r)$, we have (refer to Appendix 2)

$$\sum_{i=r}^{r2-1} \frac{1}{{p_i}^s} = E_1\left((s-1)\log p_r\right) - E_1\left((s-1)\log p_{r2-1}\right) + \varepsilon(p_r, p_{r2-1}, s)$$

Therefore, $\zeta(s)$ can be written as

$$\zeta(s)=\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i^s}\right)^{-1}\lim_{p_{r2}\to\infty}e^{E_1((s-1)\log p_r)+E_1((s-1)\log p_{r2})-E_1((s-1)\log p_{r2-1})+\varepsilon(p_r,p_{r2},s)-\delta(p_r,p_{r2},s)} \tag{48}$$

where for sufficiently large p_r, $|\delta(p_r,s)|$ is negligible compared to $|\varepsilon(p_r,s)|$ (in fact, $|\delta(p_r,s)|$ is of the same order of magnitude as $|\varepsilon(p_r,s)|^2$). Consequently, $M(s,p_r)$ can be represented by the following theorem

Theorem 3. *For the region of convergence of the series $M(s,p_r)=\sum_1^\infty \mu(n,p_r)/n^s$, we have*

$$M(s,p_r)=e^{-E_1((s-1)\log p_r)-\varepsilon(p_r,s)+\delta(p_r,s)}, \tag{49}$$

where $\varepsilon(p_r,s)=\int_{p_r}^\infty dJ(x)/x^s$, $J(x)=\pi(x)-\mathrm{Li}(x)$ and $\delta(p_r,s)=\sum_{i=r}^\infty\left(-\frac{1}{2{p_i}^{2s}}-\frac{1}{3{p_i}^{3s}}-\frac{1}{4{p_i}^{4s}}\cdots\right)$. Furthermore, on RH and for sufficiently large p_r, we have for $\sigma>0.5$

$$M(\sigma,p_r)=e^{-E_1((\sigma-1)\log p_r)}\left(1-\varepsilon(p_r,\sigma)+O\left(\varepsilon(p_r,\sigma)^2\right)+\delta(p_r,s)\right). \tag{50}$$

While in this section we have used the complex analysis to compute $M(s,p_r)$, in the next section, we will employ integration methods to compute the partial sum $M(s,p_r;1,p_r^a)$. The results obtained in this section and the following section will be then combined (using the Fourier analysis methods) in section (6) to examine the validity of the Riemann Hypothesis.

5 The series $M(\sigma,p_r)$ at $\sigma=1$.

In this section, we will compute the partial sum $M(1,p_r;1,{p_r}^a)$ using integration methods and noting that $M(1,p_r)$ equals zero for every p_r (in other words, for every p_r, $M(1,p_r;1,{p_r}^a)$ approaches zero as a approaches infinity).

Before we present the details of our method, it is important to mention that the partial sum $M(1,p_r;1,{p_r}^a)$ can be also generated using y-smooth numbers. The y-smooth numbers are the numbers that have only prime factors less than or equal to y. These numbers have been extensively analyzed in the literature [6] [8]. In [6], a method was presented to generate the partial sum $M(1,p_r;1,{p_r}^a)$. With this method and using the inclusion-exclusion principle [6] (refer to page 248), one can then provide an estimate for the partial sum $M(1,p_r;1,{p_r}^a)$. In this section, we will provide a more general approach to compute $M(1,p_r;1,{p_r}^a)$. The main advantage of our approach is the ability to extend it to compute the partial sum for values of s other than 1. We will present our method in the following two steps.

- In the first step of our approach, we will show that, for every a and as p_r approaches infinity, the partial sum $M(1,p_r;1,{p_r}^a)$ approaches a function that is dependent on only a (independent of p_r).

Toward this end, we define the function $f(a,p_r)$ as

$$f(a,p_r)=M(1,p_r;1,{p_r}^a)=\sum_{n=1}^{{p_r}^a}\frac{\mu(n,p_r)}{n}.$$

We will then show that, for every a and as p_r approaches infinity, the function $f(a, p_r)$ approaches a deterministic function $\rho(a)$. In other words; if we plot $M(1, p_r; 1, N)$ (where $N = {p_r}^a$) as a function of $a = \log N/\log p_r$, then for each value of a and as p_r approaches infinity, $f(a, p_r)$ approaches a unique value $\rho(a)$. This is equivalent to the statement

$$\rho(a) = \lim_{p_r \to \infty} f(a, p_r) = \lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a).$$

This result can be achieved by first noting that the partial sum $M(1, p_r; 1, {p_r}^a)$ for $1 < a < 2$ is given by

$$M(1, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \le p_i < {p_r}^a} \frac{1}{p_i}.$$

If we define $M_1(1, p_r; 1, {p_r}^a)$ as

$$M_1(1, p_r; 1, {p_r}^a) = \sum_{p_r \le p_i < {p_r}^a} \frac{1}{p_i},$$

then, using Stieltjes integral, we obtain

$$M(1, p_r; 1, {p_r}^a) = 1 - M_1(1, p_r; 1, {p_r}^a) = 1 - \int_{x=p_r}^{{p_r}^a} \frac{d\pi(x)}{x} = 1 - \int_{y=1}^{a} \frac{d\pi({p_r}^y)}{{p_r}^y}.$$

Since

$$d\pi({p_r}^y) = d\mathrm{Li}({p_r}^y) + dJ(p_r^y),$$

therefore

$$d\pi({p_r}^y) = \frac{1}{\log({p_r}^y)} d{p_r}^y + dJ({p_r}^y) = \frac{p_r^y}{y} dy + dJ({p_r}^y),$$

where on RH, $J(p_r^y) = O(\sqrt{{p_r}^y} \log({p_r}^y))$. Hence, for $1 < a < 2$, we have

$$M(1, p_r; 1, {p_r}^a) = 1 - \int_1^a \frac{dy}{y} - \int_1^a \frac{dJ({p_r}^y)}{{p_r}^y} = 1 - \log(a) + g_1(p_r, a), \tag{51}$$

where

$$g_1(p_r, a) = - \int_1^a \frac{dJ({p_r}^y)}{{p_r}^y}. \tag{52}$$

As p_r approaches infinity, $g_1(p_r, a)$ approaches zero. Consequently,

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - \log a.$$

The terms of the partial sum $M(1, p_r; 1, p_r{}^a)$ for a in the range $1 < a < 3$ are either a reciprocal of a prime or a reciprocal of the product of two primes. Therefore, for $1 < a < 3$, we have

$$M(1, p_r; 1, {p_r}^a) = 1 - \sum_{p_r \le p_i < {p_r}^a} \frac{1}{p_i} + \sum_{p_r \le p_{i1} < p_{i2} < p_{i1} p_{i2} < {p_r}^a} \frac{1}{p_{i1} p_{i2}},$$

where p_{i1} and p_{i2} are two distinct primes that are greater than or equal to p_r. Let $M_2(1, p_r; 1, {p_r}^a)$ be defined as

$$M_2(1, p_r; 1, {p_r}^a) = \sum_{p_r \le p_{i1} < p_{i2} < p_{i1}p_{i2} < {p_r}^a} \frac{1}{p_{i1}p_{i2}} = \frac{1}{2} \sum_{p_r \le p_i < {p_r}^{a-1}} \frac{1}{p_i} M_1(1, p_r; 1, p_r^a/p_i) + r_2.$$

Note that, for the second sum (i.e. $\sum_{p_r \le p_i < {p_r}^{a-1}} \frac{1}{p_i} M_1(1, p_r; 1, p_r^a/p_i)$), the factor of half was added since each term of the form $1/(p_{i1}p_{i2})$ is repeated twice. Furthermore, this sum includes non square-free terms (notice that, there is no repetition in any of the non square-free terms). The term r_2 was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or r_2) approaches zero as p_r approaches infinity. Using Stieltjes integral, we then have

$$M_2(1, p_r; 1, {p_r}^a) = \frac{1}{2} \int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y} \left(\log(a-y) + g_1(p_r, a-y)\right) + r_2.$$

Hence

$$M(1, p_r; 1, {p_r}^a) = 1 - \log(a) + g_1(p_r, a) + \frac{1}{2} \int_1^{a-1} \frac{\log(a-y)}{y} dy + g_2(p_r, a),$$

where

$$g_2(p_r, a) = \frac{1}{2} \int_1^{a-1} \frac{g_1(p_r, a-y)}{y} dy + \frac{1}{2} \int_1^{a-1} \log(a-y) \frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{2} \int_1^{a-1} g_1(p_r, a-y) \frac{dJ({p_r}^y)}{p_r^y} + r_2.$$

It can be easily shown that, for any fixed value of a, the three integrals on the right side of the above equation approach zero as p_r approaches infinity. We will also show later that r_2 approaches zero as p_r approaches infinity. Thus, for $1 \le a < 3$, we have

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - \log a + \frac{1}{2} \int_1^{a-1} \frac{\log(a-y)}{y} dy$$

Therefore, as p_r approaches infinity, $M(1, p_r; 1, {p_r}^a)$ approaches a function that is dependent on only a.

Repeating the previous process $\lfloor a \rfloor$ times (where $\lfloor x \rfloor$ is the integer value of x) and by using the induction method, we can show that, as p_r approaches infinity, the partial sum $M(1, p_r; 1, {p_r}^a)$ approaches a function that is dependent on only a. Specifically, we first write the partial sum $M(1, p_r; 1, {p_r}^a)$ as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - M_1(1, p_r; 1, {p_r}^a) + M_2(1, p_r; 1, {p_r}^a) - ... + (-1)^j M_j(1, p_r; 1, {p_r}^a) + ... + (-1)^{\lfloor a \rfloor - 1} M_{\lfloor a \rfloor - 1}(1, p_r; 1, {p_r}^a) + (-1)^{\lfloor a \rfloor} M_{\lfloor a \rfloor}(1, p_r; 1, {p_r}^a),$$

where

$$M_j(1, p_r; 1, {p_r}^a) = \sum_{p_r \le p_{i1} < p_{i2} < .. < p_{ij} < p_{i1}p_{i2}..p_{ij} < {p_r}^a} \frac{1}{p_{i1}p_{i2}...p_{ij}}.$$

and $p_{i1}, p_{i2}, ..., p_{ij}$ are j distinct prime numbers greater than or equal to p_r. If we assume that $M_{j-1}(1, p_r; 1, {p_r}^a)$ is given by

$$M_{j-1}(1, p_r; 1, {p_r}^a) = h_{j-1}(a) + g_{j-1}(p_r, a)$$

where $h_{j-1}(a)$ is a function of a and $g_{j-1}(p_r, a)$ approaches zero as p_r approaches infinity, then

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \sum_{p_r \leq p_i < {p_r}^{a-1}} \frac{1}{p_i} M_{j-1}(1, p_r; p_r, p_r^a/p_i) + r_j,$$

where the factor of $1/j$ was added since each term of the form $1/(p_{i1}p_{i2}...p_{ij})$ is repeated j times. It should be also noted that the sum of the above equation includes non square-free terms. The term r_j was added to offset the contribution by these non square-free terms. We will show later that the contribution by these terms (or r_j) approaches zero as p_r approaches infinity. Using Stieltjes integral, we then have

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \int_1^{a-1} \frac{d\pi({p_r}^y)}{{p_r}^y} \left(h_{j-1}(a-y) + g_{j-1}(p_r, a-y)\right) + r_j.$$

Hence

$$M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy + g_j(p_r, a),$$

where the first term is a definite integral with only one variable y integrated over the range $1 \leq y \leq a-1$. Thus, the definite integral is a function of only a. We define this function as $h_j(a)$. The second term is given by

$$g_j(p_r, a) = \frac{1}{j} \int_1^{a-1} \frac{g_{j-1}(p_r, a-y)}{y} dy + \frac{1}{j} \int_1^{a-1} h_{j-1}(a-y) \frac{dJ({p_r}^y)}{p_r^y} + \frac{1}{j} \int_1^{a-1} g_{j-1}(p_r, a-y) \frac{dJ({p_r}^y)}{p_r^y} + r_j.$$

It can be easily shown that, for a fixed value of a, the three integrals on the right side of the above equation approach zero as p_r approaches infinity. We will also show later that r_j approaches zero as p_r approaches infinity. Hence, as p_r approaches infinity, we have

$$\lim_{p_r \to \infty} M_j(1, p_r; 1, {p_r}^a) = \frac{1}{j} \int_1^{a-1} \frac{h_{j-1}(a-y)}{y} dy = h_j(a)$$

where $h_1(a) = \log(a)$. Hence, for every a and as p_r approaches infinity, we have

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = 1 - h_1(a) + h_2(a) - h_3(a) + ... + (-1)^{\lfloor a \rfloor} h_{\lfloor a \rfloor}(a) = \rho(a). \tag{53}$$

It should be pointed out that the above equation implies that the partial sums $M(1, p_r; 1, {p_r}^a)$ and $M(1, p_r^y; 1, {p_r}^{ay})$ (where, p_r^y is a prime number) have the same limit as p_r approaches infinity. Hence,

$$\lim_{p_r \to \infty} M(1, p_r; 1, {p_r}^a) = \lim_{p_r \to \infty} M(1, p_r^y; 1, {p_r}^{ay}) = \rho(a). \tag{54}$$

Equation (54) will be used in the next step to estimate the asymptotic behavior of the function $\rho(a)$ as a approaches infinity.

As mentioned earlier, the partial sum $M(1,p_r;1,{p_r}^a)$ constructed by this process included non square-free terms (i.e r_i's). In the following, we will show that, for every a and as p_r approaches infinity, the total contribution by these non square-free terms approaches zero as well. Toward this end, let S_0 be the sum of the terms with the factor $1/p_r^2$. Therefore, S_0 can be expressed as K_0/p_r^2. Let S_1 be the sum of the remaining terms with the factor $1/(p_{r+1})^2$. Therefore, S_1 can be expressed as $K_1/(p_{r+1})^2$. Let S_2 be the sum of the remaining terms with the factor $1/(p_{r+2})^2$ where S_2 can be expressed as $K_2/(p_{r+2})^2$, and so on. Let S be sum of all the terms associated with non square-free terms. Thus, S is given by

$$S = \frac{1}{{p_r}^2}K_0 + \frac{1}{{p_{r+1}}^2}K_1 + ... + \frac{1}{p_{r+L}^2}K_L,$$

where p_{r+L} is the largest prime that satisfies the condition $p_{r+L}^2 \leq {p_r}^a$. Furthermore, since there is no repetition in any of the non square-free terms, therefore

$$|K_0|, |K_1|, ..., |K_L| < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{{p_r}^a},$$

and

$$|K_0|, |K_1|, ..., |K_L| = O(a \log p_r).$$

Thus,

$$S = \left(\frac{1}{{p_r}^2} + \frac{1}{{p_{r+1}}^2} + ... + \frac{1}{p_{r+L}^2}\right) O(a \log p_r).$$

Hence, the contribution by the non square-free terms S is given by,

$$S = O(a \log p_r / p_r).$$

Consequently, for every a and as p_r approaches infinity, S (or the contribution by the non square-free terms) approaches zero.

- In the second step, we write the partial sum $M(1,p_r;1,{p_r}^a)$ as the sum of two components. The first one is the deterministic or regular component and it is given by $\rho(a)$ (= $\lim_{p_r \to \infty} M(1,p_r;1,{p_r}^a)$). The second one is the irregular component $R(1,p_r;1,{p_r}^a)$ given by $M(1,p_r;1,{p_r}^a) - \rho(a)$. We will then show that the function $\rho(a)$ is the Dickman function that has been extensively used to analyze the properties of y-smooth numbers.

Toward this end, we write the partial sum $M(1,p_r;1,{p_r}^a)$ as the following sum

$$M(1,p_r;1,{p_r}^a) = 1 - \sum_{p_r \leq p_i < {p_r}^{a/2}} \frac{1}{p_i} M(1,p_{i+1};1,{p_r}^a/p_i) - \sum_{{p_r}^{a/2} \leq p_i < {p_r}^a} \frac{1}{p_i}. \quad (55)$$

The second sum was added since the first sum is void of the terms $1/p_i$'s for ${p_i}^{a/2} \leq p_i \leq p_r^a$. It can be easily shown that every term on the right side of Equation (55) is a term on the left side of the equation and vice versa. Furthermore, there is no repetition of any term on the right side of Equation (55). The first sum on the right side of the above equation can be written as follows (refer to Equation (15))

$$\sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i} M(1, p_{i+1}; 1, {p_r}^a/p_i) = \sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i} \left(M(1, p_i; 1, {p_r}^a/p_i) + \frac{1}{p_i} M(1, p_{i+1}; 1, {p_r}^a/p_i^2) \right)$$

Let $Q(p_r, a)$ be defined as

$$Q(p_r, a) = \sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i^2} M(1, p_{i+1}; 1, {p_r}^a/p_i^2).$$

Hence

$$\sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i} M(1, p_{i+1}; 1, {p_r}^a/p_i) = \sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i} M(1, p_i; 1, {p_r}^a/p_i) + Q(p_r, a).$$

For sufficiently large p_r, the contribution by the term $Q(p_r, a)$ becomes negligible compared to the sum on the right side of the above equation. In fact, it can be easily shown that the term $Q(p_r, a)$ is given by $O(p_r^{-1})$. In Appendix 3, we have shown that

$$|M(1, p_{i+1}; 1, {p_r}^a/p_i^2)| \le 2.$$

Thus,

$$|Q(p_r, a)| = \left| \sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i^2} M(1, p_{i+1}; 1, {p_r}^a/p_i^2) \right| \le 2 \sum_{p_r \le p_i < {p_r}^{a/2}} \frac{1}{p_i^2} = O(p_r^{-1}).$$

Using Stieltjes integral, we can write Equation (55) as follows

$$M(1, p_r; 1, {p_r}^a) = 1 - \int_1^{a/2} \frac{d\pi({p_r}^y)}{p_r^y} M(1, {p_r}^y; 1, p_r^a/p_r^y) - \int_{a/2}^{a} \frac{d\pi({p_r}^y)}{p_r^y} + Q(p_r, a), \tag{56}$$

where $d\pi({p_r}^y) = d\mathrm{Li}({p_r}^y) + dJ({p_r}^y)$. It should pointed out that while Equations (55) and (56) provide the value of the partial sum $M(s, p_r; 1, p_r^a)$ at $s = 1$, they can be easily modified to compute the partial sum for any value of s to the right of the line $\Re(s) = 1$ (and on RH, to the right of the line $\Re(s) = 0.5$).

For any fixed a, as p_r approaches infinity, $M(1, {p_r}^y; 1, p_r^{a-y})$ approaches $\rho(a/y - 1)$ (refer to Equation (54)). Therefore, as p_r approaches infinity, we have

$$\rho(a) = 1 - \int_1^{a/2} \frac{\rho\left(\frac{a}{y} - 1\right)}{y} dy - \int_{a/2}^{a} \frac{dy}{y}. \tag{57}$$

In the following, we will show that $\rho(a)$ is the Dickman function that has been extensively used in the analysis of the y-smooth numbers. This task will be achieved by using Equation (57) to compute the difference $\rho(a + \Delta a) - \rho(a)$ (where, Δa is an arbitrary small number) to obtain

$$\rho(a+\Delta a)-\rho(a) = -\int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y}-1\right)}{y} dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y}-1\right)}{y} dy - \int_{(a+\Delta a)/2}^{(a+\Delta a)} \frac{dy}{y} + \int_{a/2}^{a} \frac{dy}{y}.$$

Since the third integral of the above equation is equal to the fourth integral, therefore

$$\rho(a+\Delta a)-\rho(a) = -\int_1^{(a+\Delta a)/2} \frac{\rho\left(\frac{a+\Delta a}{y}-1\right)}{y} dy + \int_1^{a/2} \frac{\rho\left(\frac{a}{y}-1\right)}{y} dy.$$

If we define $z = y/(1+\Delta a/a)$, then we have

$$\rho(a+\Delta a)-\rho(a) = -\int_{1/(1+\Delta a/a)}^{((a+\Delta a)/2)/(1+\Delta a/a)} \frac{\rho\left(\frac{a}{z}-1\right)}{z} dz + \int_1^{a/2} \frac{\rho\left(\frac{a}{y}-1\right)}{y} dy.$$

Thus,

$$\rho(a+\Delta a)-\rho(a) = -\int_{1/(1+\Delta a/a)}^{1} \frac{\rho\left(\frac{a}{z}-1\right)}{z} dz.$$

Dividing both sides of the above equation by Δa and letting Δa approach zero, we then obtain

$$\frac{d\rho(a)}{da} = -\frac{\rho(a-1)}{a}, \tag{58}$$

where $\rho(a) = 1-\log(a)$ for $1 \leq a \leq 2$. Equation (58) is a first order delay differential equation that has been extensively analyzed in the literature [6] [8]. The function $\rho(a)$ is known as the Dickman function. As a approaches infinity, $\rho(a)$ can be given by the following estimate [6]

$$\rho(a) = \left(\frac{e+o(1)}{a\log a}\right)^a. \tag{59}$$

For sufficiently large values of a, we have $\rho(a) < a^{-a}$.

To compute the irregular component of $M(1,p_r;1,p_r^a)$, we notice that $R(1,p_r;1,{p_r}^a)$ is given by

$$R(1,p_r;1,{p_r}^a) = M(1,p_r;1,p_r^a) - \rho(a).$$

Thus, $R(1,p_r;1,{p_r}^a)$ can be computed by subtracting Equation (57) from Equation (56) to obtain the following theorem

Theorem 4. *The partial sum $M(1,p_r;1,p_r^a) = \sum_{n=1}^{\lfloor p_r^a \rfloor} u(n,p_r)/n$ can be expressed as*

$$M(1,p_r;1,p_r^a) = \rho(a) + R(1,p_r;1,{p_r}^a) \tag{60}$$

where $\rho(a)$ is Dickman function. The regular component of $M(1,p_r;1,p_r^a)$ is given by

$$\rho(a) = \lim_{p_r\to\infty} M(1,p_r;1,p_r^a). \tag{61}$$

$R(1,p_r;1,{p_r}^a)$ is defined as the irregular component of $M(1,p_r;1,p_r^a)$ and it is given by

$$R(1,p_r;1,{p_r}^a) = -\int_1^{\frac{a}{2}} \rho\left(a/y-1\right) \frac{dJ({p_r}^y)}{p_r^y} - \int_{\frac{a}{2}}^{a} \frac{dJ({p_r}^y)}{p_r^y} - \int_1^{\frac{a}{2}} R(1,{p_r}^y;1,p_r^{a-y}) \frac{d\pi(p_r^y)}{p_r^y} + Q(p_r,a) \tag{62}$$

where $Q(p_r,a)$ is given by $O(p_r^{-1})$

Since the partial sum $M(s, p_r; 1, N)$ is given by the sum $\sum_{n=1}^{N} u(n, p_r)/n^s$, therefore we can write it as follows

$$M(s, p_r; 1, N) = 1 + \int_{x=p_r}^{N} \frac{x}{x^s} dM(1, p_r; 1, x),$$

or

$$M(s, p_r; 1, p_r^a) = 1 + \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} dM(1, p_r; 1, p_r^y).$$

Consequently

$$M(s, p_r; 1, p_r^a) = 1 + \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} d\rho(y) + \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} dR(1, p_r; 1, {p_r}^y). \tag{63}$$

Therefore, for any s, the partial sum $M(s, p_r; 1, p_r^a)$ has two components. The first one is the deterministic or regular component given by $1 + \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} d\rho(y)$. The second one is the irregular component given by $\int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} dR(1, p_r; 1, {p_r}^y)$. Therefore if we define α as

$$\alpha = (s-1) \log p_r$$

and the regular component of $M(s, p_r; 1, p_r^a)$ as $F(\alpha, a)$, then

$$F(\alpha, a) = 1 + \int_{1}^{a} \frac{{p_r}^x}{{p_r}^{sx}} d\rho(x) = 1 + \int_{1}^{a} {p_r}^{(1-s)x} \rho'(x) dx,$$

or,

$$F(\alpha, a) = 1 + \int_{1}^{a} e^{-\alpha x} \rho'(x) dx, \tag{64}$$

while the irregular component is given by

$$R(s, p_r; 1, p_r^a) = M(s, p_r; 1, p_r^a) - F(\alpha, a).$$

Notice that for $s = 1$, we have $\alpha = 0$ and $F(0, a) = \rho(a)$. We also notice that the regular component exists for any value of s with $\Re(s) > 0$. This is expected since the regular components of both the prime counting function and $M(s, p_r; 1, p_r^a)$ are not determined by the location of the non-trivial zeros within the critical strip.

We now define $F(\alpha)$ as

$$F(\alpha) = \lim_{a \to \infty} F(\alpha, a) = 1 + \int_{1}^{\infty} e^{-\alpha x} \rho'(x) dx. \tag{65}$$

Thus, for $\Re(s) \geq 1$, α is a complex variable in the complex plane to the right of the line $\Re(s) = 1$. Hence, the integral $\int_1^{\infty} e^{-\alpha x} \rho'(x) dx$ is the Laplace transform of the function $\rho^{'}(x)$ and is given by $F(\alpha) - 1$ (where $F(\alpha)$ is the regular component of the series $M(s, p_r)$, i.e. $M(s, p_r; 1, \infty)$). Since the Laplace transform of $\rho(x)$ multiplied by s is given by $e^{-E_1(s)}$ [9] (refer to page 569) [8] and the Laplace transform of $\rho^{'}(x)$ is given by $s\mathcal{L}(\rho(x)) - \rho(0)$, therefore

$$F(\alpha) = e^{-E_1(\alpha)}.$$

Remarkably, these results agree with what we have obtained in Theorem 2. In Theorem 2, we have shown that

$$\lim_{r\to\infty} \{M(s,p_r)\exp\left(E_1((s-1)\log p_{r+1})\right)\} = 1,$$

or referring to Theorem (3), we have

$$M(s,p_r) = e^{-E_1(\alpha)-\varepsilon(p_r,s)+\delta(p_r,s)}, \tag{66}$$

where $\varepsilon(p_r,s) = \int_{p_r}^{\infty} dJ(x)/x^s$ and $J(x) = \pi(x) - \mathrm{Li}(x)$. Consequently, we have the following theorem

Theorem 5. *For the region of convergence of the series $M(s,p_r)$, $M(s,p_r)$ can be expressed as*

$$M(s,p_r) = F(\alpha) + R(s,p_r) \tag{67}$$

where $\alpha = (s-1)\log p_r$ and $F(\alpha)$ is regular component of $M(s,p_r)$ given by

$$F(\alpha) = e^{-E_1(\alpha)}. \tag{68}$$

and $R(s,p_r)$ is the irregular component of $M(s,p_r)$ and it is given by

$$R(s,p_r) = e^{-E_1(\alpha)}(e^{-\varepsilon(p_r,s)+\delta(p_r,s)} - 1). \tag{69}$$

Furthermore, on RH, $M(s,p_r)$ can be written as

$$M(s,p_r) = F(\alpha)\left(1 - \varepsilon(p_r,s) + O(p_r^{-1}\log^2 p_r)\right). \tag{70}$$

It should be emphasized here that the regular component $F(\alpha)$ is the value of $M(s,p_r)$ due to the $\mathrm{Li}(x)$ component of the prime counting function $\pi(x)$. It is also important to note that the irregular component is not the same as the difference between the partial sum $M(s,p_r;1,{p_r}^a)$ and the series $M(s,p_r)$.Therefore, except for $s=1$ (where the irregular component $F(0)(e^{-\varepsilon(p_r,1)+\delta(p_r,1)}-1)$ is zero for every p_r), $F(\alpha)(e^{-\varepsilon(p_r,s)+\delta(p_r,s)}-1)$ may have values different from zero although it approaches zero as p_r approaches infinity

In the following section, we will use Theorem 5 and the Fourier analysis to obtain an alternative representation for $R(1,p_r;1,p_r^a)$. This representation will then be compared with Equation (62) of Theorem 4 to examine the validity of the Riemann Hypothesis.

6 The irregular component of $M(1,p_r;1,p_r^a)$ and the Riemann Hypothesis.

The irregular component of $M(1,p_r;1,p_r^a)$ for values of $a>1$ is given by Equation (62) of Theorem 4

$$R(1,p_r;1,{p_r}^a) = -\int_1^{\frac{a}{2}} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_{\frac{a}{2}}^{a}\frac{dJ({p_r}^y)}{p_r^y} - \int_1^{\frac{a}{2}} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} + Q(p_r,a)$$

In the following, we will find an alternative representation for $R(1,p_r;1,{p_r}^a)$ using Equation (69) of Theorem 5

$$R(s,p_r;1,\infty) = e^{-E_1(\alpha)}(e^{-\varepsilon(p_r,s)+\delta(p_r,s)} - 1),$$

or

$$R(s, p_r; 1, \infty) = -e^{-E_1(\alpha)}\varepsilon(p_r, s) + e^{-E_1(\alpha)}r(p_r, s),$$

where $\varepsilon(p_r, s) = \int_{y=1}^{\infty} dJ(p_r^y)/p_r^{sy} = \int_{y=1}^{\infty} e^{-\alpha y} dJ(p_r^y)/p_r^y$. Also, on RH, $|r(p_r, s)|$ is given by $O(p_r^{-1}\log^2 p_r)$. However, using Stieltjes integral, we can write $R(s, p_r; 1, {p_r}^a)$ in term of $R(1, p_r; 1, {p_r}^a)$ as follows

$$R(s, p_r; 1, {p_r}^a) = \int_{y=1}^{a} \frac{p_r^y}{p_r^{ys}} dR(1, p_r; 1, {p_r}^y) = \int_{y=1}^{a} e^{-\alpha y} dR(1, p_r; 1, {p_r}^y).$$

Hence

$$\int_{y=1}^{\infty} e^{-\alpha y} dR(1, p_r; 1, {p_r}^y) = -e^{-E_1(\alpha)}\varepsilon(p_r, s) + e^{-E_1(\alpha)}r(p_r, s),$$

or

$$\int_{y=1}^{\infty} e^{-\alpha y} dR(1, p_r; 1, {p_r}^y) = -e^{-E_1(\alpha)} \int_{y=1}^{\infty} e^{-\alpha y} \frac{dJ(p_r^y)}{p_r^y} + e^{-E_1(\alpha)}r(p_r, s). \tag{71}$$

To compute $R(1, p_r; 1, {p_r}^a)$ using the above equation, we first ignore the term $r(p_r, s)$ (the contribution by the term $r(p_r, s)$ is analyzed in Appendix 4). For $y \geq 1$, let $f_1(y)$ and $f_2(y)$ be defined as

$$f_1(y) = \frac{dR(1, p_r; 1, {p_r}^y)}{dy},$$

and

$$f_2(y) = \frac{dJ(p_r^y)/p_r^y}{dy}.$$

while $f_1(y) = f_2(y) = 0$ for $y < 1$. Thus, after ignoring the term $r(p_r, s)$, we can write Equation (71) as follows

$$\mathcal{L}f_1(y) = -e^{-E_1(\alpha)}\mathcal{L}f_2(y).$$

Since $\mathcal{L}^{-1}e^{-E_1(\alpha)} = \rho'(y) + \delta(y)$, therefore

$$f_1(y) = -\left((\rho' + \delta) * f_2\right)(y)$$

Since $f_1(y)$, $f_2(y)$ and $\rho'(y)$ are zero for $y < 1$, hence

$$f_1(y) = -\int_{x=1}^{y-1} \rho'(y - x) f_2(x) dx - f_2(y)$$

Consequently,

$$\int_{y=1}^{a} f_1(y) dy = -\int_{y=2}^{a} dy \int_{x=1}^{y-1} \rho'(y - x) f_2(x) dx - \int_{y=1}^{a} f_2(y) dy$$

Thus,

$$\int_{y=1}^{a} dR(1, p_r; 1, {p_r}^y) = -\int_{y=2}^{a} dy \int_{x=1}^{y-1} \rho'(y - x) f_2(x) dx - \int_{x=1}^{a} f_2(x) dx. \tag{72}$$

The right side of the above equation can be written as the following sums,

$$\int_{y=2}^{a} dy \int_{x=1}^{y-1} \rho'(y-x) f_2(x) dx + \int_{x=1}^{a} f_2(x) dx =$$

$$\lim_{N\to\infty} \left(\sum_{j=2N}^{\lfloor aN \rfloor} \Delta y \sum_{i=N}^{j-N} \rho^{'}(y_j - x_i) \frac{J(p_r^{x_{i+1}}) - J(p_r^{x_i})}{p_r^{x_i}} + \sum_{i=N}^{\lfloor aN \rfloor} \frac{J(p_r^{x_{i+1}}) - J(p_r^{x_i})}{p_r^{x_i}} \right)$$

where $\Delta y = 1/N$, $y_j = j/N$, $\Delta x = 1/N$ and $x_i = i/N$. From the above sum, we notice that, for every x_i, the term $\left(J(p_r^{x_{i+1}}) - J(p_r^{x_i})\right)/p_r^{x_i}$ is multiplied by $\rho^{'}(y_j - x_i)$'s for values of y_j's in the range $x_i \le y_j \le a$. Thus, by noting that $\rho'(x)$ and $f_2(x)$ are zero for $x < 1$, the order of integration of equation (72) can be changed as follows

$$\int_{y=1}^{a} dR(1, p_r; 1, {p_r}^y) = -\int_{x=1}^{a} f_2(x) \left(1 + \int_{y=x+1}^{a} \rho'(y-x) dy\right) dx,$$

or

$$\int_{y=1}^{a} dR(1, p_r; 1, {p_r}^y) = -\int_{x=1}^{a} \frac{dJ(p_r^x)}{p_r^x} \left(1 + \int_{y=x+1}^{a} \rho'(y-x) dy\right). \tag{73}$$

Since $\rho(z) = 1 + \int_1^z \rho'(x)dx$, thus $\rho(a-x) = 1 + \int_{x+1}^a \rho'(y-x)dy$. Hence, ignoring the term $r(p_r, s)$, we then have

$$R(1, p_r; 1, {p_r}^a) = -\int_{x=1}^{a} \rho(a-x) \frac{dJ(p_r^x)}{p_r^x}.$$

In Appendix 4, we have shown that the contribution by the term $r(p_r, s)$ is given by $O(\varepsilon^2(p_r, 1)) + \delta(p_r, 1))$. Consequently, we have the following theorem

Theorem 6. *For sufficiently large N and for every $p_r > N$, the relationship between the irregular component $R(1, p_r; 1, p_r^a)$ of the partial sum $M(1, p_r; 1, p_r^a)$ and $J(x)$ is given by*

$$R(1, p_r; 1, {p_r}^a) = -\int_{x=1}^{a} \rho(a-x) \frac{dJ(p_r^x)}{p_r^x} + O(\varepsilon^2(p_r, 1) + \delta(p_r, 1)). \tag{74}$$

where $R(1, p_r; 1, p_r^a) = M(1, p_r; 1, p_r^a) - \rho(a)$, $J(x) = \pi(x) - \mathrm{Li}(x)$ and on RH, $O(\varepsilon^2(p_r, 1) + \delta(p_r, 1))$ is given by $p_r^{-1} \log^2 p_r$.

In the following, we will examine the validity of the Riemann Hypothesis by analyzing Equations (74) (of Theorem 6) and (62) (of Theorem 4) for sufficiently large values of p_r. For sufficiently large values of p_r, the integral $\int_{y=1}^{\infty} dJ(p_r^y)/p_r^y$ is determined by the values of y in the vicinity of one. More specifically, referring to Appendix 1, on RH, we have

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = O\left({p_r}^{-1/2} \log p_r\right).$$

Furthermore, by the virtue of Equation (9), we also have on RH

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = \Omega\left({p_r}^{-1/2+\epsilon}\right).$$

where ϵ can be made arbitrary small by choosing p_r sufficiently large. Therefore, for sufficiently large N and for some constant k, there are an infinite number of p_r's (that are greater than N) such that

$$\left|\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y}\right| > k{p_r}^{-1/2+\delta} > 0.$$

Furthermore, for any positive number h, we also have

$$\int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = O\left((1+h){p_r}^{-h}{p_r}^{-1/2}\log p_r\right) = O\left({p_r}^{-h}{p_r}^{-1/2}\log p_r\right).$$

Thus,

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = \int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y} + O\left({p_r}^{-h}{p_r}^{-1/2}\log p_r\right).$$

Therefore, on RH and for sufficiently small h, we can always find infinitely many p_r's so that the integral $\int_{y=1}^{\infty} dJ(p_r^y)/p_r^y$ is determined by values of y in the vicinity of one. In other words; we have

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = \int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y} + \int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y}.$$

where,

$$\left|\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y}\right| > k{p_r}^{-1/2+\epsilon} > 0,$$

and

$$\left|\int_{y=1+h}^{\infty} \frac{dJ(p_r^y)}{p_r^y}\right| < k_1{p_r}^{-h}{p_r}^{-1/2}\log p_r,$$

for some constant k_1. Therefore, for any h and for sufficiently large p_r, we have

$$\int_{y=1}^{\infty} \frac{dJ(p_r^y)}{p_r^y} = (1+\delta_1)\int_{y=1}^{1+h} \frac{dJ(p_r^y)}{p_r^y}, \tag{75}$$

where δ_1 is given by $O({p_r}^{-h})$ and it can be made arbitrary close to zero by choosing p_r sufficiently large.

After analyzing the integral $\int_{y=1}^{\infty} dJ(p_r^y)/p_r^y$, we now turn our attention to the analysis of the term $R(1, p_r; 1, {p_r}^a)$. Using Equation (62) of Theorem 4, we have unconditionally

$$R(1,p_r;1,{p_r}^a) = -\int_1^{\frac{a}{2}} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_{\frac{a}{2}}^{a} \frac{dJ({p_r}^y)}{p_r^y} - \int_1^{\frac{a}{2}} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} + Q(p_r,a)$$

where $Q(p_r, a)$ is given by $O(p_r^{-1})$. The term $R(1, p_r; 1, {p_r}^a)$ can be also computed using Equation (74) of Theorem 6

$$R(1,p_r;1,{p_r}^a) = -\int_{x=1}^{a} \rho(a-x)\frac{dJ(p_r^x)}{p_r^x} + O(\varepsilon^2(p_r,1)+\delta(p_r,1)),$$

where, on RH, $O(\varepsilon^2(p_r, 1) + \delta(p_r, 1))$ is given by $O({p_r}^{-1}\log^2 p_r)$. Consequently, we have the following theorem

Theorem 7. *On RH, the difference between the representation of $R(1,p_r;1,{p_r}^a)$ by Equation (62) without the term $Q(p_r,a)$ and the representation of $R(1,p_r;1,{p_r}^a)$ by Equation (74) without the O term is given by $O({p_r}^{-1}\log^2 p_r)$. In other words;*

$$\left(-\int_1^{\frac{a}{2}} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_{\frac{a}{2}}^a \frac{dJ({p_r}^y)}{p_r^y} - \int_1^{\frac{a}{2}} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y}\right) - \left(-\int_{x=1}^a \rho(a-x)\frac{dJ(p_r^x)}{p_r^x}\right) = O({p_r}^{-1}\log^2 p_r). \quad (76)$$

In the following, we will compute this difference for values of a in three intervals. The first interval is $1\le a\le 2$ where we will show that this difference is zero. The second interval is $2\le a\le 3$ where we will show that, on RH, this difference is given by $O({p_r}^{-1}\log^2 p_r)$. The third interval is $3\le a\le 4$. Our method to examine the validity of the Riemann Hypothesis is based on analyzing this difference in the third interval.

For the interval $1\le a\le 2$, it can be easily shown that Equations (62) and (74)) provide the same value for $R(1,p_r;1,{p_r}^a)$ (this follows from the fact that for $0\le u\le 1$, $\rho(u)=1$ and $\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})d\pi(p_r^y)/p_r^y = 0$ for $1\le a\le 2$).

For the interval $2\le a\le 3$, the difference between the two representations of the term $R(1,p_r;1,{p_r}^a)$ can be computed by first noting that

$$\int_1^{a/2} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_1^a \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = \int_1^{a/2}\left(\rho\left(\frac{1}{y}(a-y)\right)-\rho(a-y)\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_{a/2}^a \rho(a-y)\frac{dJ(p_r^y)}{p_r^y}.$$

We also have $\rho(u)=1-\log(u)$ for $1\le u\le 2$. Thus

$$\int_1^{a/2} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_1^a \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = \int_1^{a/2}\log y\frac{dJ(p_r^y)}{p_r^y} - \int_{a/2}^{a-1}(1-\log(a-y))\frac{dJ(p_r^y)}{p_r^y} - \int_{a-1}^a \frac{dJ(p_r^y)}{p_r^y},$$

or

$$\int_1^{a/2} \rho\left(a/y-1\right)\frac{dJ({p_r}^y)}{p_r^y} - \int_1^a \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = \int_1^{a/2}\log y\frac{dJ(p_r^y)}{p_r^y} + \int_{a/2}^{a-1}\log(a-y)\frac{dJ(p_r^y)}{p_r^y} - \int_{a/2}^a \frac{dJ(p_r^y)}{p_r^y}.$$

The third integral on the right side of Equation (62) is given by

$$\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} = \int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})\frac{dy}{y} + \int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})\frac{dJ(p_r^y)}{p_r^y}.$$

Using the method of integration by parts, we then have on RH

$$\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} = -\int_1^{a/2}\log y\ dR(1,{p_r}^y;1,p_r^{a-y}) + O({p_r}^{-1}\log^2 p_r).$$

where, on RH and for $a \le 3$, $\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})dJ(p^y)/p^y = O({p_r}^{-1}\log^2 p_r)$. To compute $dR(1,p_r^y;1,{p_r}^{a-y})$, we note that the change in $R(1,p_r^y;1,{p_r}^{a-y})$ due to the change in y by Δy is given by

$$\Delta R(1,p_r^y;1,{p_r}^{a-y}) = R(1,{p_r}^{y+\Delta y};1,{p_r}^{a-y-\Delta y}) - R(1,p_r^y;1,{p_r}^{a-y})$$

However, referring to Equation (52), for $1 \le h < 2$, we have

$$R(1,p_r;1,{p_r}^{1+h}) = \int_{y=1}^{1+h} dR(1,p_r;1,{p_r}^y) = -\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y}. \tag{77}$$

Thus, for $1 \le y \le 2$, we have

$$dR(1,p_r;1,{p_r}^y) = -\frac{dJ(p_r^y)}{p_r^y}. \tag{78}$$

Consequently, for $1 \le \frac{a-y}{y} \le 2$, we obtain

$$\Delta R(1,p_r^y;1,{p_r}^{a-y}) = -\int_{z=y+\Delta y}^{a-y-\Delta y}\frac{dJ(p_r^z)}{p_r^z} + \int_{z=y}^{a-y}\frac{dJ(p_r^z)}{p_r^z},$$

or

$$\Delta R(1,p_r^y;1,{p_r}^{a-y}) = \int_{z=y}^{y+\Delta y}\frac{dJ(p_r^z)}{p_r^z} + \int_{z=a-y-\Delta y}^{a-y}\frac{dJ(p_r^z)}{p_r^z}.$$

Hence for $2 \le a \le 3$,

$$dR(1,p_r^y;1,{p_r}^{a-y}) = \frac{dJ(p_r^y)}{p_r^y} + \frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}}.$$

Therefore, on RH and for $2 \le a \le 3$, we conclude that

$$\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} = -\int_1^{a/2}\log y\left(\frac{dJ(p_r^y)}{p_r^y} + \frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}}\right) + O({p_r}^{-1}\log^2 p_r), \tag{79}$$

or

$$\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} = -\int_1^{a/2}\log y\frac{dJ(p_r^y)}{p_r^y} + \int_{a-1}^{a/2}\log(a-z)\frac{dJ({p_r}^z)}{{p_r}^z} + O({p_r}^{-1}\log^2 p_r).$$

Thus, on RH, the difference between the two representations of the term $R(1,p_r;1,{p_r}^a)$ is given by

$$R(1,p_r;1,p_r^a) + \int_1^a \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = O({p_r}^{-1}\log^2 p_r) +$$

$$-\int_{a-1}^{a/2}\log(a-y)\frac{dJ(p_r^y)}{p_r^y} - \int_{a/2}^{a-1}\log(a-y)\frac{dJ(p_r^y)}{p_r^y} - \int_{a/2}^{a}\frac{dJ(p_r^y)}{p_r^y} + \int_{a/2}^{a}\frac{dJ(p_r^y)}{p_r^y}.$$

Hence, on RH and for $2 \leq a \leq 3$, we have

$$R(1, p_r; 1, p_r^a) + \int_1^a \rho(a-y) \frac{dJ(p_r^y)}{p_r^y} = O({p_r}^{-1} \log^2 p_r). \tag{80}$$

For the interval $3 < a \leq 4$, the representation of the functions $\rho(a-y)$ and $\rho((a-y)/y)$ is dependent on the value of y. For values of y in the range $1 \leq y \leq a/3$, we have [7]

$$\rho(a-y) = 1 - \log(a-y) + \int_2^{a-y} \log(v-1) \frac{dv}{v},$$

and

$$\rho\left(\frac{1}{y}(a-y)\right) = 1 - \log(a-y) + \log y + \int_2^{(a-y)/y} \log(v-1) \frac{dv}{v},$$

Thus, for values of y in the range $1 \leq y \leq a/3$, we have

$$-\int_1^{a/3} \rho\left(a/y - 1\right) \frac{dJ({p_r}^y)}{p_r^y} + \int_1^{a/3} \rho(a-y) \frac{dJ(p_r^y)}{p_r^y} =$$
$$-\int_1^{a/3} \log y \frac{dJ(p_r^y)}{p_r^y} + \int_{y=1}^{a/3} \frac{dJ(p_r^y)}{p_r^y} \int_{v=(a-y)/y}^{a-y} \log(v-1) \frac{dv}{v}. \tag{81}$$

For values of y in the range $a/3 \leq y \leq a-2$, we have

$$\rho(a-y) = 1 - \log(a-y) + \int_2^{a-y} \log(v-1) \frac{dv}{v},$$

and

$$\rho\left(\frac{1}{y}(a-y)\right) = 1 + \log y - \log(a-y).$$

Thus, for values of y in the range $a/3 \leq y \leq a-2$, we have

$$-\int_{a/3}^{a-2} \rho\left(a/y - 1\right) \frac{dJ({p_r}^y)}{p_r^y} + \int_{a/3}^{a-2} \rho(a-y) \frac{dJ(p_r^y)}{p_r^y} =$$
$$-\int_{a/3}^{a-2} \log y \frac{dJ(p_r^y)}{p_r^y} + \int_{a/3}^{a-2} \frac{dJ(p_r^y)}{p_r^y} \int_2^{a-y} \log(v-1) \frac{dv}{v}. \tag{82}$$

Similarly, for values of y in the range $a-2 \leq y \leq a/2$, we have

$$-\int_{a-2}^{a/2} \rho\left(a/y - 1\right) \frac{dJ({p_r}^y)}{p_r^y} + \int_{a-2}^{a/2} \rho(a-y) \frac{dJ(p_r^y)}{p_r^y} = -\int_{a-2}^{a/2} \log y \frac{dJ(p_r^y)}{p_r^y}. \tag{83}$$

For values of y in the range $a/2 \leq y \leq a-1$, we have

$$\int_{a/2}^{a-1} \rho(a-y) \frac{dJ(p_r^y)}{p_r^y} = \int_{a/2}^{a-1} (1 - \log(a-y)) \frac{dJ(p_r^y)}{p_r^y}.$$

while for values of y in the range $a-1 \le y \le a$, we have

$$\int_{a-1}^{a} \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = \int_{a-1}^{a} \frac{dJ(p_r^y)}{p_r^y}.$$

Thus, for values of y in the range $a/2 \le y \le a$, we have

$$\int_{a/2}^{a} \rho(a-y)\frac{dJ(p_r^y)}{p_r^y} = -\int_{a/2}^{a-1} \log(a-y)\frac{dJ(p_r^y)}{p_r^y} + \int_{a/2}^{a} \frac{dJ(p_r^y)}{p_r^y}. \tag{84}$$

To compute the integral $\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})d\pi(p^y)/p^y$, we refer to Equation (74) of Theorem 6. On RH and for $a \le 3$, we then have

$$R(1,p_r;1,p_r^a) = -\int_1^a \rho(a-x)\frac{dJ(p_r^x)}{p_r^x} + O({p_r}^{-1}\log^2 p_r).$$

Therefore, for $a \le 4$ and $1 \le y \le a/2$, we have

$$R(1,{p_r}^y;1,p_r^{a-y}) = -\int_{x=1}^{\frac{a-y}{y}} \rho\left(\frac{a-y}{y} - x\right)\frac{dJ((p_r^y)^x)}{(p_r^y)^x} + O({p_r}^{-1}\log^2 p_r).$$

Defining $z = yx$, we then have

$$R(1,{p_r}^y;1,p_r^{a-y}) = -\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z} + O({p_r}^{-1}\log^2 p_r),$$

and

$$\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} = -\int_1^{a/2}\left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z} + O({p_r}^{-1}\log^2 p_r)\right)\frac{d\pi(p_r^y)}{p_r^y}.$$

By noting that $d\pi(p_r^y)/p_r^y = d\log y + dJ(p_r^z)/p_r^z$, $0 \le d\pi(p_r^y) \le dp_r^y$ and by using the method of integration by parts, we then have

$$\int_1^{a/2} R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y} = \int_1^{a/2} \log y \ \ d\left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right) + O({p_r}^{-1}\log^3 p_r).$$

The change in the integral $\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}$ due to the change in y by Δy is given by

$$\Delta\left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right) =$$

$$\int_{z=y+\Delta y}^{a-y-\Delta y} \rho\left(\frac{a-z}{y+\Delta y} - 1\right)\frac{dJ({p_r}^z)}{{p_r}^z} - \int_{z=y}^{a-y} \rho\left(\frac{a-z}{y} - 1\right)\frac{dJ(p_r^z)}{p_r^z},$$

or

$$\Delta\left(\int_{z=y}^{a-y} \rho\left(\frac{a-y}{y} - \frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right) = -\int_{z=y}^{y+\Delta y} \rho\left(\frac{a-z}{y} - 1\right)\frac{dJ(p_r^z)}{p_r^z} -$$

$$\int_{z=a-y-\Delta y}^{a-y} \rho\left(\frac{a-z}{y} - 1\right)\frac{dJ({p_r}^{a-z})}{{p_r}^{a-z}} + \int_{z=y}^{a-y}\left(\rho\left(\frac{a-z}{y+\Delta y} - 1\right) - \rho\left(\frac{a-z}{y} - 1\right)\right)\frac{dJ(p_r^z)}{p_r^z},$$

where

$$\rho\left(\frac{a-z}{y+\Delta y}-1\right)-\rho\left(\frac{a-z}{y}-1\right)=\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\Delta y.$$

Consequently

$$\Delta\left(\int_{z=y}^{a-y}\rho\left(\frac{a-y}{y}-\frac{z}{y}\right)\frac{dJ(p_r^z)}{p_r^z}\right)=-\rho(a-2)\frac{dJ(p_r^y)}{p_r^y}-\rho(0)\frac{dJ({p_r}^{a-y})}{{p_r}^{a-y}}+$$

$$\Delta y\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z},$$

and

$$\int_1^{a/2}R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y}=-\rho(a-2)\int_1^{a/2}\log y\frac{dJ(p_r^z)}{p_r^z}-\int_1^{a/2}\log y\frac{dJ({p_r}^{a-z})}{{p_r}^{a-z}}+$$

$$\int_1^{a/2}\log y\left(\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z}\right)dy,$$

or

$$\int_1^{a/2}R(1,{p_r}^y;1,p_r^{a-y})\frac{d\pi(p_r^y)}{p_r^y}=-\rho(a-2)\int_1^{a/2}\log y\frac{dJ(p_r^z)}{p_r^z}-\int_{a/2}^{a-1}\log(a-y)\frac{dJ(p_r^y)}{p_r^y}+$$

$$\int_1^{a/2}\log y\left(\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z}\right)dy+O({p_r}^{-1}\log^3 p_r). \tag{85}$$

Thus, on RH and for $3<a\leq 4$, the difference of Theorem 7 can be obtained by combining Equations (81), (82), (83), (84) and (85) to get

$$R(1,p_r;1,p_r^a)-Q(p_r,a)+\int_1^a\rho(a-y)\frac{dJ(p_r^y)}{p_r^y}=$$

$$-\int_1^{a/2}\log y\frac{dJ(p_r^y)}{p_r^y}+\int_1^{a/3}\frac{dJ(p_r^y)}{p_r^y}\int_{(a-y)/y}^{a-y}\log(v-1)\frac{dv}{v}+\int_{a/3}^{a-2}\frac{dJ(p_r^y)}{p_r^y}\int_2^{a-y}\log(v-1)\frac{dv}{v}+$$

$$\rho(a-2)\int_1^{a/2}\log y\frac{dJ(p_r^y)}{p_r^y}-\int_1^{a/2}\log y\int_{z=y}^{a-y}\rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^z)}{p_r^z}+O({p_r}^{-1}\log^3 p_r). \tag{86}$$

Therefore, by the virtue of Theorem 7, for the Riemann Hypothesis to be valid, the left side of the above equation should be given by $O({p_r}^{-1}\log^2 p_r)$. Referring to Equation (75), on RH and for any arbitrary small h and for any $b>h$, we can find infinitely many p_r satisfying the following equation

$$\int_{y=1}^{b}\frac{dJ(p_r^y)}{p_r^y}=(1+\delta)\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y},$$

To examine Equation (86) for the validity of RH, we split the terms on the right side of the equation into two terms. We denote the first one as A and it comprises the terms with values of y in the vicinity of 1. We denote the second term as B and it is the sum of the remaining terms. Thus

$$A = (-1+\rho(a-2)) \int_1^{a/2} \log y \frac{dJ(p_r^y)}{p_r^y} + \int_1^{a/3} \frac{dJ(p_r^y)}{p_r^y} \int_{(a-y)/y}^{a-y} \log(v-1)\frac{dv}{v} -$$

$$\int_{y=1}^{a/2} \log y \int_{z=y}^{a-y} \rho'\left(\frac{a-z}{y}-1\right) \frac{a-z}{y^2} \frac{dJ(p_r^z)}{p_r^z}, \tag{87}$$

and

$$B = \int_{a/3}^{a-2} \frac{dJ(p_r^y)}{p_r^y} \int_2^{a-y} \log(v-1)\frac{dv}{v} + O({p_r}^{-1} \log^2 p_r),$$

where on RH and for $a > 3$, it can be easily shown that

$$B = O({p_r}^{-a/6} \log p_r).$$

To analyze the terms of A, we note that in the vicinity of $y = 1$, the first integral on the right side of Equation (87) can be written as

$$\int_1^{a/2} \log y \frac{dJ(p_r^y)}{p_r^y} = \int_1^{a/2} \frac{dJ(p_r^y)}{p_r^y} \left((y-1) - \frac{1}{2}(y-1)^2 + O((y-1)^3) \right),$$

For the second integral, we first define the integral $\int_{z=y}^{a-y} \rho'\left(\frac{a-z}{y}-1\right)$ as $g(y)$. Expanding the function $g(y)$ as a Taylor series in the vicinity of $y = 1$ yields

$$\int_{(a-y)/y}^{a-y} \log(v-1)\frac{dv}{v} = \log(a-2)(y-1) - \frac{1}{2}\left(\log(a-2) + \frac{a-1}{a-2}\right)(y-1)^2 + O((y-1)^3),$$

and

$$\int_1^{a/3} \frac{dJ(p_r^y)}{p_r^y} \int_{(a-y)/y}^{a-y} \log(v-1)\frac{dv}{v} =$$

$$\int_1^{a/3} \frac{dJ(p_r^y)}{p_r^y} \left(\log(a-2)(y-1) - \frac{1}{2}\left(\log(a-2) + \frac{a-1}{a-2}\right)(y-1)^2 + O((y-1)^3) \right).$$

For the third integral on the right side of Equation (87), we first rearrange the double integral as follows

$$\int_{y=1}^{a/2} \log y \int_{z=y}^{a-y} \rho'\left(\frac{a-z}{y}-1\right) \frac{a-z}{y^2} \frac{dJ(p_r^z)}{p_r^z} =$$

$$\int_{z=1}^{\frac{a}{2}} \frac{dJ(p_r^z)}{p_r^z} \int_{y=1}^{z} \log y \; \rho'\left(\frac{a-z}{y}-1\right) \frac{a-z}{y^2} dy + \int_{z=\frac{a}{2}}^{a} \frac{dJ(p_r^z)}{p_r^z} \int_{y=1}^{a-z} \log y \; \rho'\left(\frac{a-z}{y}-1\right) \frac{a-z}{y^2} dy$$

where on RH, the second integral on the right side of the above equation is given by $O({p_r}^{-a/4} \log p_r)$. For sufficiently large p_r, the second integral becomes negligible compared to the first integral. Thus

$$\int_1^{a/2} \log y \int_{z=y}^{a-y} \rho'\left(\frac{a-z}{y}-1\right) \frac{a-z}{y^2} \frac{dJ(p_r^z)}{p_r^z} =$$

$$\int_{y=1}^{a/2} \frac{dJ(p_r^y)}{p_r^y} \int_{x=1}^{y} \rho'\left(\frac{a-y}{x}-1\right) \frac{a-y}{x^2} \left((x-1) + O((x-1)^2)\right) dx + O({p_r}^{-a/4} \log p_r),$$

or

$$\int_1^{a/2} \log y \int_{z=y}^{a-y} \rho'\left(\frac{a-z}{y}-1\right)\frac{a-z}{y^2}\frac{dJ(p_r^y)}{p_r^y} =$$

$$\int_{y=1}^{a/2} \frac{dJ(p_r^y)}{p_r^y}(a-1)\rho'(a-2)\left(\frac{(y-1)^2}{2}+O((y-1)^3)\right)+O(p_r^{-a/4}\log p_r).$$

Consequently, on RH and for sufficiently large p_r, we have

$$A = (1+\delta)\int_1^{\infty}\left(D\,(y-1)^2+O((y-1)^3)\right)\frac{dJ(p_r^y)}{p_r^y}+O(p_r^{-a/4}\log p_r)$$

where δ can be made arbitrary small by choosing p_r sufficiently large and

$$D = -\frac{3}{2}\log(a-2)-\frac{a+1}{2(a-1)}-\frac{(a-1)\rho'(a-2)}{2}.$$

Therefore, on RH and for values of a in the range $3 \leq a \leq 4$, we can find infinitely many p_r satisfying the following equation

$$O(p_r^{-1}\log^2 p_r) = (1+\delta)\int_1^{\infty}\left(D\,(y-1)^2+O((y-1)^3)\right)\frac{dJ(p_r^y)}{p_r^y}+O(p_r^{-a/6}\log p_r)$$

or

$$O(p_r^{-a/6}\log p_r) = (1+\delta)\int_1^{\infty}\left(D\,(y-1)^2+O((y-1)^3)\right)\frac{dJ(p_r^y)}{p_r^y} \tag{88}$$

where δ can be made arbitrary small by choosing p_r sufficiently large and $|D|> 0$. However, in Appendix 5 we have shown that $\int_1^{\infty}(y-1)^2 dJ(p_r^y)/p_r^y = \Omega({p_r}^{-1/2}\log^{-3} p_r)$. Therefore, on RH and for sufficiently large p_r, we have

$$O(p_r^{-a/6}\log p_r) = \Omega\left(\frac{{p_r}^{-1/2}}{\log^3 p_r}\right) \tag{89}$$

However, the above equation will eventually lead to a contradiction for sufficiently large p_r. This contradiction points to the invalidity of the Riemann Hypothesis. Similar results are also attained if we assume that there are no zeros to the right of the line $\Re(s) = c$ for any $c < 1$. This follows from the fact if there are no zeros to right of the line $\Re(s) = c$ for any $c < 1$, then $J(x)$ is given by $O(x^{1-c}\log x)$ and this will lead to similar contradiction. This indicates that non-trivial zeros can be found arbitrary close to the line $\Re(s) = 1$.

Furthermore, Equation (88) can be used to estimate where the distribution of the prime number deviates or starts to deviate from what has been predicted by the Riemann hypotheses. As mentioned earlier, we don't expect to have inconsistent results with RH for values of a less than 3. Hence, we need to set a greater than 3. In the following, we will set a equal to 4. For $a = 4$, the left side of Equation (89) is less than $k_1 p_r^{-2/3}\log p_r$ for some constant k_1 while the right side of the equation is greater than $k_2 {p_r}^{-1/2}/\log^3 p_r$ for some constant k_2. Therefore, if p_{r1} satisfies the following equation

$$k_1 \log^4 p_{r1} = k_2 p_{r1}^{1/6}, \tag{90}$$

then for prime numbers $p_{r2} > p_{r1}$, the sum $R(1, p_{r2}; 1, p_{r2}^4) + \int_1^4 \rho(4-y) dJ(p_{r2}^y)/p_{r2}^y$ is greater than $O({p_{r2}}^{-1} \log^2 p_{r2})$. Consequently, we expect that the prime numbers greater than p_{r1}^4 do not follow the distribution predicted by the Riemann hypothesis. Notice that the estimation of p_{r1} depends on accurate estimation of the constants k_1 and k_2

Appendix 1

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^\sigma} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon(p_{r1}, p_{r2}, \sigma)$$

where, $\varepsilon(p_{r1}, p_{r2}, \sigma) = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^\sigma = O\left(\frac{1}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$ and $J(x) = \pi(x) - \mathrm{Li}(x)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^\sigma} = \int_{p_{r1}}^{p_{r2}} \frac{d\mathrm{Li}(x)}{x^\sigma} + \int_{p_{r1}}^{p_{r2}} \frac{dJ(x)}{x^\sigma},$$

or

$$\sum_{i=r1}^{r2} \frac{1}{p_i^\sigma} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^\sigma} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right).$$

We will first compute the integral with the O notation. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^\sigma} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^\sigma} - \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) d\left(\frac{1}{x^\sigma}\right)$$

Since $x > 0$, thus

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = \frac{O\left(\sqrt{p_{r2}}\log p_{r2}\right)}{{p_{r2}}^\sigma} - \frac{O\left(\sqrt{p_{r1}}\log p_{r1}\right)}{{p_{r1}}^\sigma} - O\left(\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^\sigma}\right)\right)$$

With the substitution of variables $y = \log x$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^\sigma}\right) = -\int_{\log p_{r1}}^{\log p_{r2}} \sigma y e^{(\frac{1}{2}-\sigma)y} dy.$$

Since

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax},$$

therefore

$$\int_{p_{r1}}^{p_{r2}} \sqrt{x}\log x \, d\left(\frac{1}{x^\sigma}\right) = -\sigma\left(\frac{\log p_{r2}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) {p_{r2}}^{0.5-\sigma} + \sigma\left(\frac{\log p_{r1}}{0.5-\sigma} - \frac{1}{(0.5-\sigma)^2}\right) {p_{r1}}^{0.5-\sigma}.$$

Hence, for $\sigma > 0.5$, we have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma} dO\left(\sqrt{x}\log x\right) = O\left(\frac{{p_{r1}}^{0.5-\sigma}\log p_{r1}}{(\sigma-0.5)^2}\right) \tag{91}$$

For $\sigma \geq 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(r) = \int_r^\infty \frac{e^{-u}}{u} du$ (where $r \geq 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

It should be pointed out that although the functions $E_1((\sigma-1)\log p_{r1})$ and $E_1((\sigma-1)\log p_{r2})$ have a singularity at $\sigma = 1$, the difference has a removable singularity at $\sigma = 1$. This follows from the fact that as σ approaches 1, the difference can be written as

$$E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) = -\log((1-\sigma)\log p_{r1}) - \gamma + \log((1-\sigma)\log p_{r2}) + \gamma$$

or,

$$\lim_{\sigma\to 1} \int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \lim_{\sigma\to 1} \{E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})\} = -\log\log p_{r1} + \log\log p_{r2}$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx$ for $\sigma < 0$, we first use the substantiation $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy = \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2})z_2}}{z_2} dz_2 - \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1})z_1}}{z_1} dz_1.$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_2 = (1-\sigma)(\log p_{r2})z_1$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2} dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1} dw_1 +$$
$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [1] (refer to page 230)

$$\int_0^a \frac{e^t - 1}{t} dt = -E_1(-a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^\sigma \log x} dx = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2})$$

Hence, for $\sigma > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon(p_{r1}, p_{r2}, \sigma)$$

In general, if there are no non-trivial zeros for values of s with $\Re(s) > a$, then by following the same steps, we can also show that for $\sigma > a$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{\sigma}} = E_1((\sigma-1)\log p_{r1}) - E_1((\sigma-1)\log p_{r2}) + \varepsilon(p_{r1}, p_{r2}, \sigma)$$

where, $\varepsilon(p_{r1}, p_{r2}, \sigma) = \int_{p_{r1}}^{p_{r2}} dJ(x)/x^{\sigma} = O\left({p_{r1}}^{a-\sigma} \log p_{r1}/(\sigma-a)^2\right)$.

Appendix 2

Assuming RH is valid and for $\sigma > 0.5$, to show that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{s}} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon(p_{r1}, p_{r2}, s)$$

where, $|\varepsilon(p_{r1}, p_{r2}, s)| = O\left(\frac{|s|}{(\sigma-0.5)^2}\, {p_{r1}}^{1/2-\sigma} \log p_{r1}\right)$, we first recall that

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^{s}} = \int_{p_{r1}}^{p_{r2}} \frac{d\pi(x)}{x^s} = \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx + \int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dJ(x).$$

We will first compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dJ(x)$. This can be done by integration by parts to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dJ(x) = \frac{J(p_{r2})}{p_{r2}^s} - \frac{J(p_{r1})}{p_{r1}^s} - \int_{p_{r1}}^{p_{r2}} J(x) d\left(\frac{1}{x^s}\right)$$

The integral on the right side of the above equation can be then written as

$$\int_{p_{r1}}^{p_{r2}} J(x) d\left(\frac{1}{x^s}\right) = -s \int_{p_{r1}}^{p_{r2}} J(x) x^{-s-1} dx.$$

Hence,

$$\left|\int_{p_{r1}}^{p_{r2}} J(x) d\left(\frac{1}{x^s}\right)\right| \le |s| \int_{p_{r1}}^{p_{r2}} O\left(\sqrt{x}\log x\right) |x^{-s-1}| dx.$$

Consequently,

$$\left|\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s} dJ(x)\right| = O\left(|s|\ \frac{{p_{r1}}^{0.5-\sigma} \log p_{r1}}{(\sigma-0.5)^2}\right).$$

For $\Re(s) \ge 1$, the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ can be computed directly from the definition of the Exponential Integral $E_1(z) = \int_1^{\infty} \frac{e^{-tz}}{t} dt$ (where $\Re(z) \ge 0$) to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2})$$

To compute the integral $\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx$ for $\Re(z) < 1$, we first write the integral as follows

$$\int_{p_{r1}}^{p_{r2}} \frac{1}{x^s \log x} dx = \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx - i \int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \sin(t \log x)}{\log x} dx.$$

The first integral on the right side $\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx$ can be computed by using the substitution $y = \log x$ to obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy,$$

or

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y} \cos(ty)}{y} dy + \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy - \int_{\log p_{r1}}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy.$$

Hence,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}(1 - \cos(ty))}{y} dy - \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}(1 - \cos(ty))}{y} dy -$$

$$\int_{\epsilon}^{\log p_{r1}} \frac{e^{(1-\sigma)y}}{y} dy + \int_{\epsilon}^{\log p_{r2}} \frac{e^{(1-\sigma)y}}{y} dy$$

where, ϵ is an arbitrary small positive number. With the variable substantiations $z_1 = y/\log p_{r1}$ and $z_2 = y/\log p_{r2}$, we then obtain

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1}) z_1}(1 - \cos(t(\log p_{r1}) z_1))}{z_1} dz_1 -$$

$$\int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2}) z_2}(1 - \cos(t(\log p_{r2}) z_2))}{z_2} dz_2 -$$

$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1}) z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2}) z_2}}{z_2} dz_2$$

By the virtue of the following identity [1] (refer to page 230)

$$\int_0^1 \frac{e^{at}(1 - \cos(bt))}{t} dt = \frac{1}{2} \log(1 + b^2/a^2) + \mathrm{Li}(a) + \Re[E_1(-a + ib)],$$

where $a > 0$, we then obtain the following

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma \log x} \cos(t \log x)}{\log x} dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) -$$

$$\Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) -$$

$$\int_{\epsilon/\log p_{r1}}^{1} \frac{e^{(1-\sigma)(\log p_{r1}) z_1}}{z_1} dz_1 + \int_{\epsilon/\log p_{r2}}^{1} \frac{e^{(1-\sigma)(\log p_{r2}) z_2}}{z_2} dz_2$$

With the variable substantiations $w_1 = (1-\sigma)(\log p_{r1})z_1$ and $w_2 = (1-\sigma)(\log p_{r2})z_2$ and by adding and subtracting the terms $-\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} + \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}$, we then have

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx = \Re[E_1((s-1)\log p_{r1})] + \mathrm{Li}((1-\sigma)\log p_{r1}) -$$

$$\Re[E_1((s-1)\log p_{r2})] - \mathrm{Li}((1-\sigma)\log p_{r2}) +$$

$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{e^{w_2}-1}{w_2}dw_2 - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{e^{w_1}-1}{w_1}dw_1 +$$

$$\int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r2}} \frac{dw_2}{w_2} - \int_{(1-\sigma)\epsilon}^{(1-\sigma)\log p_{r1}} \frac{dw_1}{w_1}.$$

Using the following identity [1] (refer to page 230)

$$\int_0^a \frac{e^t-1}{t}dt = \mathrm{Ei}(a) - \log(a) - \gamma$$

where $a > 0$, we then obtain for $\sigma < 1$,

$$\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\cos(t\log x)}{\log x}dx = \Re[E_1((s-1)\log p_{r1})] - \Re[E_1((s-1)\log p_{r2})]$$

Similarly, using the identity [1] (refer to page 230)

$$\int_0^1 \frac{e^{at}\sin(bt)}{t}dt = \pi - \arctan(b/a) + \Im[E_1(-a+ib)],$$

where $a > 0$, we can show that for $\sigma < 1$, we have

$$-\int_{p_{r1}}^{p_{r2}} \frac{e^{-\sigma\log x}\sin(t\log x)}{\log x}dx = \Im[E_1((s-1)\log p_{r1})] - \Im[E_1((s-1)\log p_{r2})].$$

Therefore, for $\Re(s) > 0.5$, we have

$$\sum_{i=r1}^{r2} \frac{1}{{p_i}^s} = E_1((s-1)\log p_{r1}) - E_1((s-1)\log p_{r2}) + \varepsilon(p_{r1}, p_{r2}, s)$$

where, $\varepsilon(p_{r1}, p_{r2}, s) = \int_{p_{r1}}^{p_{r2}} \frac{dJ(x)}{x^s}$ and on RH, $|\varepsilon(p_{r1}, p_{r2}, s)| = O\left(\frac{|s|}{(\sigma-0.5)^2} {p_{r1}}^{1/2-\sigma}\log p_{r1}\right)$.

It is worth mentioning here that the term $\varepsilon(p_r, s)$ can be represented in terms of the non-trivial zero if von Mangoldt function is used in this analysis instead of using the prime counting function.

Appendix 3

To show that

$$\left|\sum_{n=1}^{N} \frac{\mu(n,p_r)}{n}\right| \le 2$$

we first note that

$\sum_{d/n} \mu(d,p_r) = 1$, if $n = 1$,
$\sum_{d/n} \mu(d,p_r) = 1$, if all the prime factors of n are less than p_r,
$\sum_{d/n} \mu(d,p_r) = 0$, if any of the prime factors of n is greater than p_r.

Adding all the terms $\sum_{d/n} \mu(d,p_r)$ for $1 \le n \le N$, we then obtain

$$0 < \sum_{n=1}^{N} \mu(n,p_r) \left\lfloor \frac{N}{n} \right\rfloor \le N,$$

where $\lfloor x \rfloor$ refers to the integer value of x. Define r_n as

$$r_n = \frac{N}{n} - \left\lfloor \frac{N}{n} \right\rfloor,$$

where $0 \le r_n < 1$. Hence, we have

$$\sum_{n=1}^{N} \mu(n,p_r) r_n < \sum_{n=1}^{N} \mu(n,p_r) \left\lfloor \frac{N}{n} \right\rfloor + \sum_{n=1}^{N} \mu(n,p_r) r_n \le N + \sum_{n=1}^{N} \mu(n,p_r) r_n.$$

Since $0 \le r_n < 1$, therefore

$$-N \le \sum_{n=1}^{N} \mu(n,p_r) \left(r_n + \left\lfloor \frac{N}{n} \right\rfloor \right) \le 2N.$$

Thus, for every p_r we have

$$-N < \sum_{n=1}^{N} \mu(n,p_r) \frac{N}{n} \le 2N,$$

or

$$-1 < \sum_{n=1}^{N} \frac{\mu(n,p_r)}{n} \le 2.$$

Appendix 4

Referring to Equation (69), we have

$$R(s,p_r;1,\infty) = e^{-E_1(\alpha)}(e^{-\varepsilon(p_r,s)+\delta(p_r,s)} - 1)$$

where

$$\alpha = (s-1)\log p_r,$$

$$R(\alpha,p_r) = R(s,p_r;1,\infty) = \int_{y=1}^{\infty} e^{-\alpha y} dR(1,p_r;1,{p_r}^y),$$

$$\varepsilon_{p_r}(\alpha) = \varepsilon(p_r, s) = \int_1^\infty e^{-\alpha y} \frac{dJ(p_r^y)}{p_r^y},$$

and

$$\delta_{p_r}(\alpha) = \delta(p_r, s) = \sum_{i=r}^{\infty} \left(-\frac{1}{2{p_i}^{2s}} - \frac{1}{3{p_i}^{3s}} - \frac{1}{4{p_i}^{4s}} ... \right) = -\sum_{n=2}^{\infty} \int_1^\infty e^{-n\alpha y} \frac{d\pi(p_r^y)}{np_r^{ny}}.$$

Notice that for $s = 1 + it$, $\alpha = it \log p_r$ and if we define $\omega = t \log p_r$, then

$$R(\omega, p_r) = R(1 + it, p_r; 1, \infty) = \int_{y=1}^\infty e^{-i\omega y} dR(1, p_r; 1, {p_r}^y)$$

or, $R(\omega, p_r)$ is the Fourier transform of the function $dR(1, p_r; 1, {p_r}^y)/dy$. Similarly,

$$\varepsilon_{p_r}(\omega) = \int_1^\infty e^{-i\omega y} \frac{dJ(p_r^y)}{p_r^y},$$

and

$$\delta_{p_r}(\omega) = -\sum_{n=2}^{\infty} \int_1^\infty e^{-in\omega y} \frac{d\pi(p_r^y)}{np_r^{ny}}.$$

Therefore,

$$\frac{dR(1, p_r; 1, {p_r}^y)}{dy} = \mathcal{L}^{-1} R(\alpha, p_r),$$

or

$$\frac{dR(1, p_r; 1, {p_r}^y)}{dy} = \mathcal{F}^{-1} R(\omega, p_r).$$

Thus,

$$R(1, p_r; 1, {p_r}^a) = \int_1^a \mathcal{L}^{-1} R(\alpha, p_r)\, dy,$$

or

$$R(1, p_r; 1, {p_r}^a) = \int_1^a \mathcal{F}^{-1} R(\omega, p_r)\, dy.$$

Hence,

$$R(1, p_r; 1, {p_r}^a) = \int_1^a \left((\mathcal{L}^{-1} e^{-E_1(\alpha)} * \mathcal{L}^{-1} e^{-\varepsilon_{p_r}(\alpha)} * \mathcal{L}^{-1} e^{\delta_{p_r}(\alpha)})(y) - \mathcal{L}^{-1} e^{-E_1(\alpha)} \right) dy,$$

where, $\mathcal{L}^{-1} e^{-E_1(\alpha)} = \rho'(y) + \delta(y)$ and

$$R(1, p_r; 1, {p_r}^a) = \int_1^a \left((\mathcal{F}^{-1} e^{-E_1(\omega)} * \mathcal{F}^{-1} e^{-\varepsilon_{p_r}(\omega)} * \mathcal{F}^{-1} e^{\delta_{p_r}(\omega)})(y) - \mathcal{F}^{-1} e^{-E_1(\omega)} \right) dy$$

where, $\mathcal{F}^{-1} e^{-E_1(\omega)} = \rho'(y) + \delta(y)$.

To compute the inverse Fourier transform of $e^{-\varepsilon_{p_r}(\omega)}$ (= $e^{-\int_1^\infty e^{-i\omega y} \frac{dJ(p_r^y)}{p_r^y}}$) and $e^{-\delta_{p_r}(\omega)}$, we first note

$$e^{-\varepsilon_{p_r}(\omega)} = 1 - \varepsilon_{p_r}(\omega) + \frac{\varepsilon_{p_r}^2(\omega)}{2!} - ... ,$$

and

$$e^{-\delta_{p_r}(\omega)} = 1 - \delta_{p_r}(\omega) + \frac{\delta_{p_r}^2(\omega)}{2!} - ... ,$$

The dominant term in the computation of $R(1, p_r; 1, {p_r}^y)$ is given by

$$-(\mathcal{F}^{-1}e^{-E_1(\omega)} * \mathcal{F}^{-1}\epsilon_{p_r}(\omega))(y) = -\left((\rho' + \delta) * f_2\right)(y).$$

Referring to section 6, we then have

$$-\int_{y=1}^{a}(\mathcal{F}^{-1}e^{-E_1(\omega)} * \mathcal{F}^{-1}\epsilon_{p_r}(\omega))(y)dy = -\int_{x=1}^{a}\rho(a-x)\frac{dJ(p_r^x)}{p_r^x}.$$

As it mentioned in section (6), for any h and for sufficiently large p_r, we have

$$\int_{y=1}^{\infty}\frac{dJ(p_r^y)}{p_r^y} = (1+\delta)\int_{y=1}^{1+h}\frac{dJ(p_r^y)}{p_r^y},$$

where δ can be made arbitrary close to zero by choosing p_r sufficiently large. Therefore, for any h and for sufficiently large p_r, we have

$$-\int_{y=1}^{a}(\mathcal{F}^{-1}e^{-E_1(\omega)} * \mathcal{F}^{-1}\epsilon_{p_r}(\omega))(y)dy = -(1+\delta)\rho(a-1)\int_{x=1}^{\infty}\frac{dJ(p_r^x)}{p_r^x}.$$

The second dominant term in the computation of $R(1, p_r; 1, {p_r}^y)$ is given by

$$\frac{1}{2}(\mathcal{F}^{-1}e^{-E_1(\omega)} * \mathcal{F}^{-1}\epsilon_{p_r}(\omega) * \mathcal{F}^{-1}\epsilon_{p_r}(\omega))(y) = \frac{1}{2}\left((\rho' + \delta) * f_2 * f_2\right)(y).$$

If we denote the convolution $(\mathcal{F}^{-1}e^{-E_1(\omega)} * \mathcal{F}^{-1}\epsilon_{p_r}(\omega))(y)$ as $F_1(y)$, the convolution $(\mathcal{F}^{-1}e^{-E_1(\omega)} * \mathcal{F}^{-1}\epsilon_{p_r}(\omega) * \mathcal{F}^{-1}\epsilon_{p_r}(\omega))(y)$ as $F_2(y)$ and so on, then

$$\frac{1}{2}F_2(y) = \frac{1}{2}\int_{x=2}^{y}\rho^{'}(y-x)(f_2 * f_2)(x)dx + \frac{1}{2}(f_2 * f_2)(y)$$

and the contribution of the term $F_2(y)$ to $R(1, p_r; 1, {p_r}^y)$ is given by

$$\frac{1}{2}\int_{y=1}^{a}F_2(y)dy = \frac{1}{2}\int_{y=1}^{a}\left(\int_{x=2}^{y}\rho^{'}(y-x)(f_2 * f_2)(x)dx + (f_2 * f_2)(y)\right)dy,$$

or

$$\frac{1}{2}\int_{y=1}^{a}F_2(y)dy = \frac{1}{2}\int_{x=1}^{a}(f_2 * f_2)(x)\left(1 + \int_{y=x+2}^{a}\rho^{'}(y-x)dy\right)dx.$$

Thus

$$\frac{1}{2}\int_{y=1}^{a}F_2(y)dy = \frac{1}{2}\int_{x=1}^{a}\rho(a-x-1)(f_2 * f_2)(x)dx.$$

For sufficiently large p_r, we then have

$$\frac{1}{2}\int_{y=1}^{a}F_2(y)dy = \frac{1+\delta}{2}\rho(a-2)\int_{x=1}^{\infty}(f_2 * f_2)(x)dx$$

By noting that if $F(s)$ is the Laplace transform of $f(t)$ then $F(s)/s$ is the Laplace transform of $\int_0^t f(t)dt$ and using the final value theorem (which states that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$), we then obtain

$$\frac{1}{2}\int_{y=1}^{a}F_2(y)dy = \frac{1+\delta}{2}\rho(a-2)\left(\int_{z=1}^{\infty}\frac{dJ(p_r^z)}{p_r^z}\right)^2$$

where δ can be made arbitrary small by choosing p_r sufficiently large.

Similarly, for sufficiently large p_r we can show that the integral

$$\frac{1}{k!}\int_{y=1}^{a} F_n(y)dy = \frac{1}{k!}\int_{x=1}^{a} \rho(a-x-k+1)(f_2 * f_2 * .... * f_2)(x)dx,$$

is given by

$$\frac{1}{k!}\int_{y=1}^{a} F_n(y)dy = \frac{1+\delta}{k!}\rho(a-k)\left(\int_{z=1}^{\infty} \frac{dJ(p_r^z)}{p_r^z}\right)^k$$

where δ can be made arbitrary small by choosing p_r sufficiently large.

Following the same steps, we can also compute the contribution by the term $e^{\delta(p_r,s)}$ to $R(1,p_r;1,{p_r}^a)$. This contribution is dominated by the integral $\int_{y=1}^{a}(\mathcal{F}^{-1}e^{-E_1(\omega)} * \mathcal{F}^{-1}\delta_{p_r}(\omega))(y)dy$. For sufficiently large p_r, we then have

$$\int_{y=1}^{a}(\mathcal{F}^{-1}e^{-E_1(\omega)} * \mathcal{F}^{-1}\delta_{p_r}(\omega))(y)dy = (1+\epsilon)\rho(a-1)\int_{y=1}^{a} \frac{d\pi(p_r^y)}{p_r^{2y}} = O(p_r^{-1}).$$

where ϵ can be made arbitrary small by choosing p_r sufficiently large. As expected, in examining the validity of the Riemann hypothesis, the contibution by the term $\delta(p_r,s)$ can be ignored. This can be also conluded using Theorem 3. In Theorem 3, it is clear that the term $\delta(p_r,s)$ has no impact on the region of convergence of the series $M(s,p_r)$

Appendix 5

On RH, to show that

$$\int_{1}^{\infty}(y-1)^2\ \frac{dJ(p_r^y)}{p_r^y} = \Omega\left(\frac{\sqrt{p}_r}{\log p_r^3}\right),$$

we first note that by method of integration by parts, we have

$$\int_{1}^{\infty}(y-1)^2\ \frac{dJ(p_r^y)}{p_r^y} = -2\int_{1}^{\infty}(y-1)\frac{J(p_r^y)}{p_r^y}dy + \log p_r \int_{1}^{\infty}(y-1)^2\ \frac{J(p_r^y)}{p_r^y}dy.$$

Furthermore, $J(p_r^y)$ can be written as

$$J(p_r^y) = -\frac{1}{y\log p_r}\sum_{\rho}\frac{p_r^{y\rho_i}}{\rho_i} + O\left(\frac{\sqrt{p_r^y}}{y\log p_r}\right)$$

where the sum $\sum_{\rho} p_r^{y\rho_i}/\rho_i$ is performed over the nontrivial zeros $\rho_i = \alpha_i + i\gamma_i$. This sum is conditionally convergent and it should be performed over the nontrivial zeros with $|\gamma_i| \leq T$ as T approaches infinity. Referring to [12] (refer to lemmas 5 and 6), we can easily show that the O term in the above equation is dominated by sum $\sum_{m=2}^{\lfloor y\log p_r/\log 2\rfloor} \pi(p_r^{y/m})/m$ and the term $J(p_r^y)$ can be written as

$$J(p_r^y) = -\frac{1}{y\log p_r}\sum_{\rho}\frac{p_r^{y\rho_i}}{\rho_i} - \frac{1}{2}\mathrm{Li}(p_r^{y/2}) + \text{Lesser terms}$$

For sufficiently large p_r, we have

$$\int_1^\infty (y-1)\frac{p_r^{\rho_i y}}{yp_r^y}dy = -\frac{2p_r^{\rho_i}}{(\rho_i-1)^2 p_r \log^2 p_r}$$

and

$$\int_1^\infty (y-1)^2\ \frac{p_r^{\rho_i y}}{yp_r^y}dy = \frac{p_r^{\rho_i}}{(\rho_i-1)^3 p_r \log^3 p_r}$$

Therefore, on RH, we have

$$\int_1^\infty (y-1)^2 \frac{dJ(p_r^y)}{p_r^y} = -\frac{8}{\sqrt{p_r}\log^3 p_r} - \frac{1}{\sqrt{p_r}\log^3 p_r}\sum_\rho \frac{4-1/(1-\rho_i)}{\rho_i(1-\rho_i)^2} + \text{Lesser terms}$$

References

[1] Abramowitz and Stegun , *Handbook of Mathematical Functions.*, Tenth Edition, United States Printing Office, Washington DC, 1972.

[2] Edwards H. M., *Riemann Zeta Function.*, Academic Press, 1974.

[3] Cramer H., , *On the order of magnitude of the difference between consecutive prime numbers*, Acta Arithmetica, no.2, 23-46, 1936.

[4] Freitag E. and Busam R., *Complex Analysis*, Second Edition, Springer, 2005.

[5] Gonek S. M., Hughes C.P., and Keating J.P, *A Hybrid Euler-Hadamard Product for the Riemann Zeta Function*, Duke Mathematical Journal, Vol. 136, No. 3,c 2007.

[6] Granville A., *Smooth numbers: Computational number theory and beyond.*, Algorithmic Number Theory, Vol. 44, 267-323, 2008.

[7] Green D. H. and Knuth D. E., *Mathematics for the analysis of Algorithms.*, Third Edition, Birkhauser. 1990.

[8] Hilderrand A. and Tenenbaum G., *Integers without large prime factors*, Journal de Theorie des Numbers de Bordeaux, vol. 5, no. 2, 411-484, 1993.

[9] Lagarias J. C., *Euler constant: Euler work and modern developments.*, Bulletin (New Series) American Mathematical Society, Vol. 50, No. 4, 527-628, Oct. 2013.

[10] LeVeque W. J. *Fundamentals of Number Theory.*, LeVeque, W. L., Dover Duplications, 1996.

[11] Riemann R., *On the Number of Prime Numbers less than a Given Quantity.*, Nov. 1856. Translated by Wilkins, D. R., Dec. 1998.

[12] Skewes S., *On the difference* $\pi(x0 - \mathrm{li(x)(II)}$, Proc. London Math. Soc. (3) 5, 1955.

[13] Tenenbaum G. and Michel Mendes France, Translated by Philip G Spain, *The prime numbers and their distribution.*, Student Medical Library, Volume 6, AMS, USA, 2000.

[14] Titchmarsh E. C., *The Theory of the Riemann Zeta-function.*, Second Revised Edition (Heath-Brown), Oxford Science Publications, 1986, Reprinted 2007.