Introducing Unitary Circles

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Abstract

Unitary circles model the property of spin, and are based on imaginary numbers.

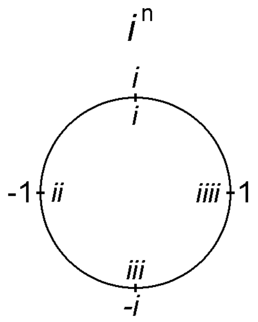


FIGURE 1 – A unitary circle

The imaginary constant i is not a real number, and it has not been apparent how it relates to real numbers. With unitary circles, the role of imaginary numbers can be clearly understood. And the definition of i changes.

1 Background

The constant i (sometimes referred to as j) is currently defined as the square root of -1. Descartes was uncomfortable with this constant, because there was no way to conceptualize what it meant to be the square root of -1, and also because there were no practical applications that used this constant. In his book *La Géométrie* in 1637, he disparagingly coined the term "imaginary numbers", suggesting they were not real numbers.

Ironically, the diagrams used today to represent imaginary numbers are based on Descartes' Cartesian coordinate system. Argand diagrams use the x axis to represent real numbers and the y axis to represent imaginary numbers. Since the coordinate points on these diagrams have both a real component and an imaginary component, these numbers are said to be complex. Similarly, the intersection of the real and imaginary axes are sometimes referred to as the complex plane.

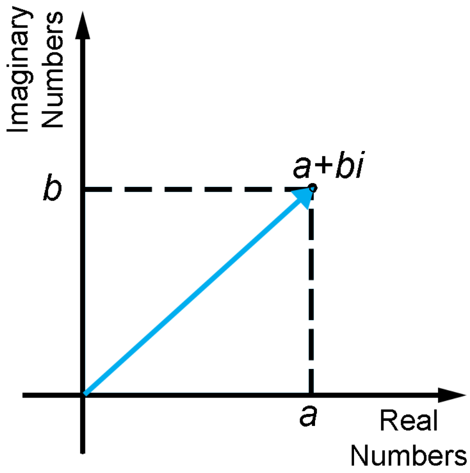


FIGURE 2 – Argand Diagram

While there is consistency in viewing imaginary numbers with Argand diagrams, there still is no practical value to this approach. There are no real world problems that can be solved by working with imaginary numbers in this way. There are applications that use imaginary numbers, but they involve working with circles and the property of spin.

2 Unitary Circles

Unitary circles model the property of spin. They spin around (a full revolution) in one, two, or four steps. What makes them all unitary circles is the characteristic that the step unit, raised to the number of steps, equals one.

Types of Unitary Circles

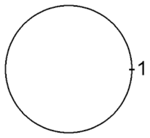
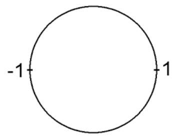
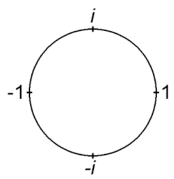
Number of Steps	Unitary Circle	Step Unit	Unitary Formula
1		1	$1^1 = 1$
2		-1	$-1^2 = 1$
4		*i*	$i^4 = 1$

FIGURE 3 – Unitary circles

The last circle, with a step unit of i, is called the imaginary unit circle.

3 Imaginary Unit Circle

The imaginary unit circle is a graphical representation of the expressions i^n and i^{-n}.

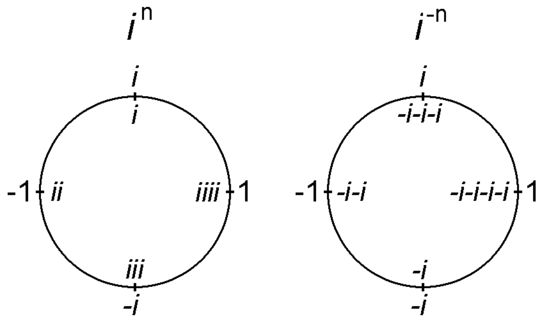


Figure 4 – Both halves of the imaginary unit circle

As n grows larger, i^n steps around the circle in a counter-clockwise direction. Similarly, the expression i^{-n} steps around the circle in a clockwise direction. The expressions i^n and i^{-n} generate both real and complex terms. It is, therefore, problematic to use these expressions in formulas, because real world problems require only real answers.

However i^n can be "paired" with i^{-n}, so that the complex terms appear with their conjugate and cancel out. Pairing binds together i^n and i^{-n} so that they can be treated as one mathematical entity. (Some form of brackets will be needed to denote their bound status.) The resultant value will always be a real number (either 2, 0, or -2).

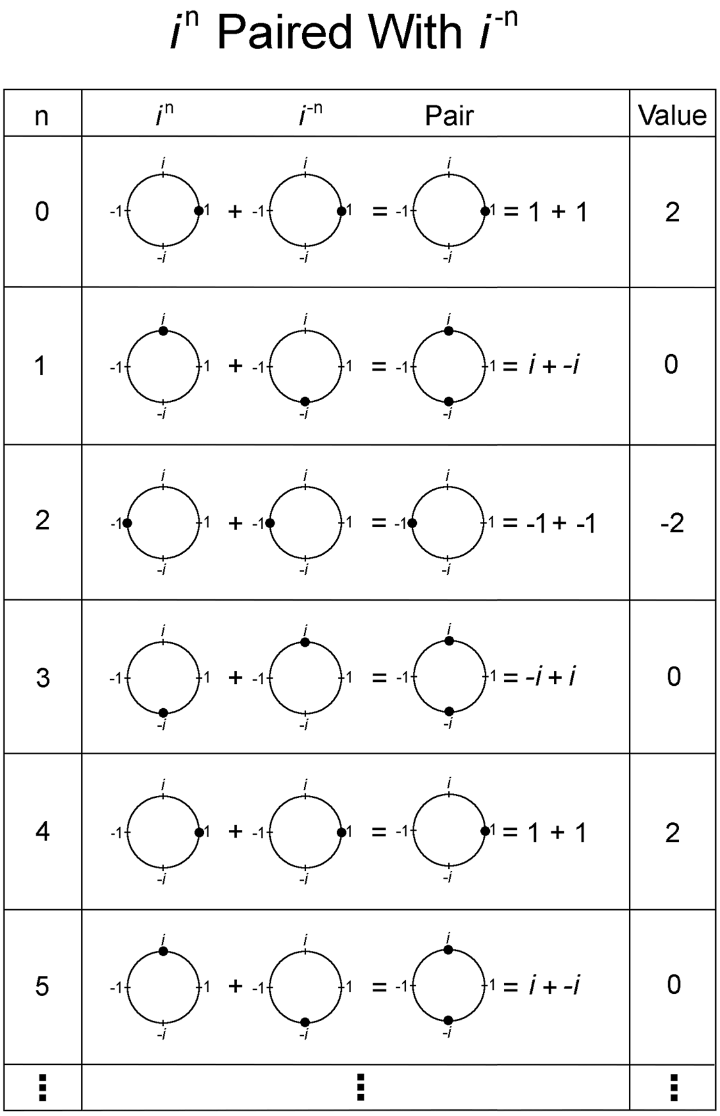


FIGURE 5 – How imaginary terms are bound together

Imaginary terms that are not paired, have no way of relating to real numbers. i is equivalent to the square root of -1, but we know now that -1 refers to a half of a spin on a circle (roughly speaking). i is a quarter of a spin. So, for example,

$$(3i)^2 \neq -9 \tag{1}$$

The units don't match - it's mixing apples and oranges.

4 Unitary Circles and Pi

Two of the unitary circles appear to have a natural connection to the constant pi. One of them represents the formula -1^n, and the other represents the formula $i^n + i^{-n}$.

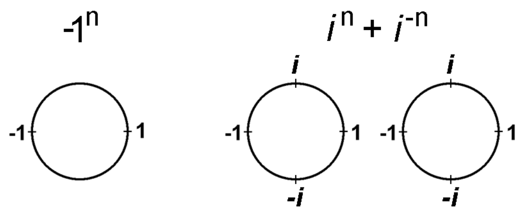


FIGURE 6 – Two unitary circles

Below are two pi formulas based on these unitary circles.

$$\sum_{n=0}^{\infty} \frac{-1^n}{2n-1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots = \frac{\Pi}{4} \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{i^n + i^{-n}}{n+1} = \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} \cdots = \frac{\Pi}{2} \tag{3}$$

Formula 2 is the famous Leibniz pi formula. Formula 3, introduced with this paper, is one of the most elegant pi formulas extant.

5 A Symmetric Pi Formula

The imaginary pi formula (Formula 3 above) can be rewritten in an equivalent form, so that n starts at 1 instead of 0. Since n starts higher by one, the formula must be adjusted so that each occurrence of n is lowered by 1 (however i^{-n} spins backwards, so it must be raised by 1). The new formula is displayed below in Formula 4.

$$\sum_{n=0}^{\infty} \frac{i^n + i^{-n}}{n+1} = \sum_{n=1}^{\infty} \frac{i^{(n-1)} + i^{(-n+1)}}{n} \tag{4}$$

Formula 5 illustrates that the new formula (with n starting at 1) is symmetric. That is, running the formula backwards from -1 to -infinity also yields pi divided by 2.

$$\sum_{n=1}^{\infty} \frac{i^{(n-1)} + i^{(-n+1)}}{n} = \sum_{n=-1}^{-\infty} \frac{i^{(n-1)} + i^{(-n+1)}}{n} = \frac{\Pi}{2} \tag{5}$$

or expressed in another way

$$\sum_{n=1}^{\infty} \frac{i^{(n-1)} + i^{(-n+1)}}{n} + \sum_{n=-1}^{-\infty} \frac{i^{(n-1)} + i^{(-n+1)}}{n} = \Pi \tag{6}$$

6 Summary

Unitary circles model the property of spin, and have the characteristic that the step unit, raised to the number of steps, equals 1.

$$\text{Step unit}^{\text{Number of steps}} = 1 \tag{7}$$

The imaginary unit circle is a unitary circle, with i as the step unit, and the number of steps is four.

$$i^4 = 1 \tag{8}$$

The new definition of i is:

The step unit in the imaginary unit circle; equal in value to the fourth root of 1.

$$i = \sqrt[4]{1} \tag{9}$$

Pairing imaginary terms brings unitary circles into the real number system. The elegant pi formulas suggest that these circles will play an important role.