Introducing Spin

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Abstract

It is pointed out that imaginary numbers were originally introduced into our real number system without a foundation. With only a vague definition of i to work with ($\sqrt{-1}$), assumptions were made about imaginary numbers that seemed correct, but had conceptual distortions.

Just as real numbers appear on the real number line, imaginary numbers appear on the imaginary unit circle. This paper builds a foundational bridge linking these disparate number systems. Then spin is introduced based on this solid mathematical foundation.

Technically the lack of a mathematical foundation invalidates our current usage of complex numbers. It doesn't mean we are necessarily making mistakes, but if we want the consistency and certainty of our real number system to extend to include imaginary numbers, then we need to define precisely how imaginary numbers relate to real numbers. Fortunately, at the same time the validity of our current usage of imaginary numbers is called into question, a solid foundation for imaginary numbers is introduced herein with spin.

Spin equations were previously thought to be wave equations. Now we know complex spin equations generate real electromagnetic waves.

Introduction

There are two types of spin:
- imaginary spin based on the constant i
- real spin based on the functions: e^x, $\cos(x)$, and $\sin(x)$.

These two types of spin are integrated by the equation.

$$e^{ix} = \cos(x) + i\sin(x) \tag{1}$$

Remarkably, this equation was first written three centuries ago by Euler. Presumably Euler had no knowledge of spin, however he also was famous for his work with the power series, and it turns out that power series are closely related to spin. Ultimately this equation evolves into a complex spin equation generating a real wave.

$$e^{ix} + e^{-ix} = 2\cos(x) \tag{2}$$

There is a natural way to derive an energy and a momentum equation from this spin equation. There is also a way to convert the momentum equation to a kinetic energy equation, so we really have two equations to compute energy. These two equations are combined into one formula - the time dependent Schrodinger equation.

1 Background

1.1 Traditional View of Imaginary Numbers

The constant i (sometimes referred to as j) is currently defined solely as the square root of -1. Descartes was uncomfortable with this constant, because there was no way to conceptualize what it meant to be the square root of -1, and also because there were no practical applications that used this constant. In his book *La Géométrie* in 1637, he disparagingly coined the term "imaginary numbers", suggesting they were not real numbers.

Ironically, the diagrams used today to represent imaginary numbers are based on Descartes' Cartesian coordinate system. Argand diagrams use the x axis to represent real numbers and the y axis to represent imaginary numbers. Since the coordinate points on these diagrams have both a real component and an imaginary component, these numbers are said to be complex. Similarly, the intersection of the real and imaginary axes are sometimes referred to as the complex plane.

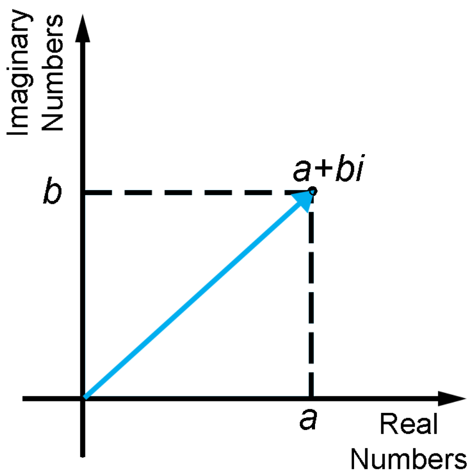


FIGURE 1: Argand Diagram

1.2 What Do We Really Know ?

Before imaginary numbers can enter the real number system, we need to know conceptually and mathematically how they relate. Yet the constant i is not decipherable and there are no mathematical operations or properties that allow us to relate imaginary numbers to real numbers.

We are told that i is equal to the square root of -1. but we still need to know:
– Conceptually how do we envision i ?
– What does it mean to multiply two imaginary numbers ?
– How does i relate to real numbers ?

The current definition of i is based on arbitrary answers to those questions. Without a foundation that relates imaginary and real numbers, complex terms such as $3i$ are mathematically meaningless - and not valid numbers.

1.3 Bridging Real and Imaginary Numbers

Conceptually i is a quarter of a turn on a circle. This competing concept is not new, but now it has evolved into the foundation for spin.

Spin equations are complex and they generate answers that contain a mixture of real and imaginary terms. When a spin equation is "paired" with its complex conjugate, the complex terms in the answer always appear with their conjugate and cancel out. This means "pairing" a complex spin equation with its conjugate yields an all real answer. This is the mathematical foundation linking imaginary and real numbers.

Note that we still have no way to relate i with a real number. So every point on the complex plane is still invalid. But we don't need to know how i directly relates to real numbers, because we know how to combine two imaginary numbers to get a real number. In the next section on imaginary spin, we will see this process represented mathematically.

2 Spin Based on Imaginary Numbers

2.1 Imaginary Unit Circle

Imaginary spin can be graphically represented on the imaginary unit circle using the expressions i^n and i^{-n}.

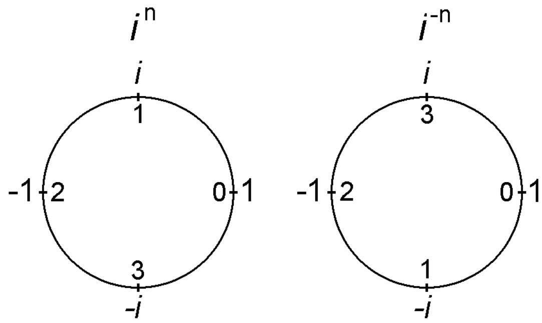


FIGURE 2: The imaginary unit circle

The expression i^n starts at the origin (with a step number of 0), and steps around the circle in a counterclockwise direction. This endless looping process is spin. Similarly, the expression i^{-n} starts at the origin, but steps around the circle in a clockwise direction.

The table below illustrates that spin loops endlessly (the values repeat every four numbers).

n	i^n	i^{-n}
0	1	1
1	i	$-i$
2	-1	-1
3	$-i$	i
4	1	1
5	i	$-i$
6	-1	-1
7	$-i$	i
8	1	1

TABLE 1: Generated spin terms

The equation i^n spins around taking multiplicative steps. For example two steps of i is equal to $i \times i$ (i^2 is equal to -1). Four steps of i is equal to $i \times i \times i \times i$. Those four steps bring you around and back to the origin with a value of 1 ($i^4 = 1$).

2.2 Complex Conjugates

The expressions i^n and i^{-n} generate both real and complex terms. It is therefore problematic to use either one of these expressions in formulas, because real world problems can not be solved with imaginary solutions (there is no way to interpret the result).

However i^n and i^{-n} are complex conjugates (the same complex equation spinning in the opposite direction). If they are "paired" together, then the real terms add together and the complex terms cancel out. We can see in the table below that no matter what value n has, $i^n + i^{-n}$ always yields a real value.

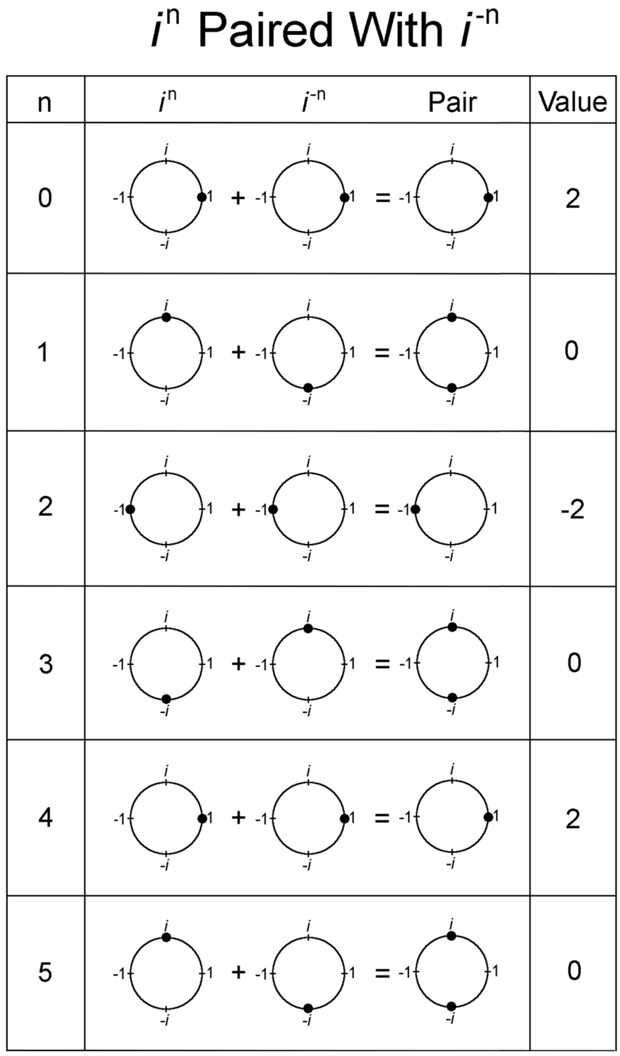
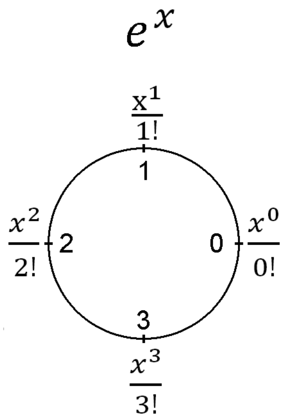


FIGURE 3: Complex conjugates paired together

3 Spin Based on Real Numbers

3.1 e^x Used in Spin Equations

The most fundamental real spin equation is e^x. e^x (or $\frac{x^n}{n!}$) can be graphically represented by starting at the origin ($n = 0$) and stepping counterclockwise around the circle (spinning). With each step, a new term in the answer is created (see formula 3).



n	$x^n/n!$
0	$x^0/0!$
1	$x^1/1!$
2	$x^2/2!$
3	$x^3/3!$
4	$x^4/4!$
5	$x^5/5!$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \tag{3}$$

The series generated by the spin equation e^x, is identical to the e^x power series.

3.2 Derivatives

At first glance it appears that e^x deals with exponents - x does grow exponentially. However there is also the n factorial term in the denominator that needs to be explained.

$$e^x = \frac{x^n}{n!} \tag{4}$$

The explanation involves derivatives. The derivative of any term in the e^x series, is equal to the value of the previous term. So, for example, the derivative of the third term in the e^x series is equal to the value of the second term.

$$\frac{d}{dx}\frac{x^3}{3!} = \frac{3x^2}{3!} = \frac{x^2}{2!} \tag{5}$$

It is interesting that for every term in any of the power series, the factorial value of the denominator is always equal to the exponential value of the numerator. Even when you take the derivative of a term, these two values change but still remain equal.

The derivative of the e^x function is calculated by taking the derivative of each of the terms in the series. Note that the derivative of e^x is simply e^x.

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \tag{6}$$

$$\frac{d}{dx}e^x = 0 + \frac{1x^0}{1!} + \frac{2x^1}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \ldots \tag{7}$$

$$\frac{d}{dx}e^x = 0 + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{4!} + \frac{x^4}{4!} + \cdots = e^x \tag{8}$$

$$\frac{d}{dx}e^x = e^x \tag{9}$$

Similarly, the derivatives of the cos(x) function are calculated by taking the derivative of each of the individual terms.

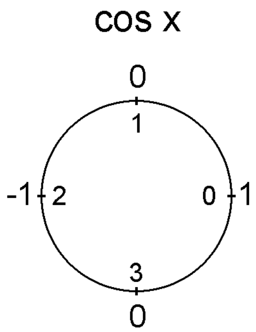
$$\begin{aligned}
\cos x &= \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots & (10)\\
\frac{d}{dx}\cos x &= 0 - \frac{2x^1}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!} - \frac{10x^9}{10!} + \ldots & (11)\\
&= 0 - \frac{x^1}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots = -\sin x & (12)\\
\frac{d^2}{dx^2}\cos x &= 0 - \frac{1x^0}{1!} + \frac{3x^2}{3!} - \frac{5x^4}{5!} + \frac{7x^6}{7!} - \frac{9x^8}{9!} \cdots & (13)\\
&= 0 - \frac{x^0}{0!} + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = -\cos x & (14)\\
\frac{d^3}{dx^3}\cos x &= 0 - 0 + \frac{2x^1}{2!} - \frac{4x^3}{4!} + \frac{6x^5}{6!} - \frac{8x^7}{8!} \cdots & (15)\\
&= 0 - 0 + \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x & (16)\\
\frac{d^4}{dx^4}\cos x &= 0 - 0 + \frac{1x^0}{1!} - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} \cdots & (17)\\
&= 0 - 0 + \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots = \cos x & (18)
\end{aligned}$$

The first derivative of the cos function is equal to the -sin. The second, third, and fourth derivatives are the -cos, sin, and cos respectively. This illustrates how these derivatives loop endlessly (spin).

3.3 Cosine and Sine in Spin Equations

The cosine and sine functions are the other real functions used with spin. They are very similar to the function e^x. They both start at the origin and step counterclockwise around the circle. What is different is that they use a mask for interpreting each term in the e^x series. The mask value used to compute the cosine series is simply $\cos x$, so the value for any term a in the series is equal to:

$$Term_a = \cos(x) \times \frac{x^a}{a!} \tag{19}$$



n	x	$\cos x$	$(i^n + i^{-n})/2$	$x^n/n!$
0	0	1	1	$x^0/0!$
1	$\pi/2$	0	0	$x^1/1!$
2	π	-1	-1	$x^2/2!$
3	$3\pi/2$	0	0	$x^3/3!$
4	2π	1	1	$x^4/4!$
5	$5\pi/2$	0	0	$x^5/5!$

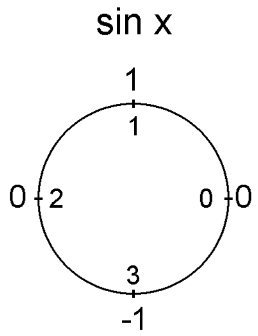
$$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots \tag{20}$$

An interesting point about the cosine function is that it has an underlying connection to imaginary numbers. In the table above, the fourth column (with imaginary numbers in the heading), has the same values as the third ($\cos x$) column. So

$$\cos(x) = (i^n + i^{-n})/2 \tag{21}$$

Similarly, determining each value of the $\sin x$ series also involves using a mask, but this time we use what is in the $\sin x$ column as the mask. So to determine the value of any term a in the $\sin x$ series:

$$Term_a = \sin(x) \times \frac{x^a}{a!} \tag{22}$$



n	x	$\sin x$	$x^n/n!$
0	0	0	$x^0/0!$
1	$\pi/2$	1	$x^1/1!$
2	π	0	$x^2/2!$
3	$3\pi/2$	-1	$x^3/3!$
4	2π	0	$x^4/4!$
5	$5\pi/2$	1	$x^5/5!$

$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots \tag{23}$$

All three power series (e^x, $\cos x$, and $\sin x$) are generated by spin equations.

4 Integrating Real and Imaginary Spin

It's not obvious how imaginary spin (based on i) relates to real spin (based on e^x, $\cos x$, and $\sin x$). However, Euler wrote an equation - now referred to as Euler's Formula - that integrates these two types of spin:

$$e^{ix} = \cos x + i \sin x \tag{24}$$

Below is his interesting derivation of this equation, which is based on the power series for e^x. (the explanation follows the derivation)

$$\begin{aligned}
e^x &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots && (25) \\
e^{ix} &= \frac{i^0x^0}{0!} + \frac{i^1x^1}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \dots && (26) \\
e^{ix} &= \frac{x^0}{0!} + \frac{ix^1}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \dots && (27) \\
e^{ix} &= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + \left(\frac{ix^1}{1!} - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \dots\right) && (28) \\
e^{ix} &= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) && (29) \\
e^{ix} &= \cos x + i \sin x && (30)
\end{aligned}$$

Formula 26 shows x replaced with ix. Formula 27 is a simplification of the imaginary terms. Formula 28 groups the terms with even and odd exponents. Formula 29 factors out an i from the the odd exponents group.

Remarkably, the terms grouped within parentheses in formula 29, are the power series for the $\cos x$ and $\sin x$ respectively. Formula 30 simply substitutes the terms "$\cos x$" and "$\sin x$" for the respective series.

Since e^{ix} is a spin equation, we can graph it and look at the table values to better understand its properties.

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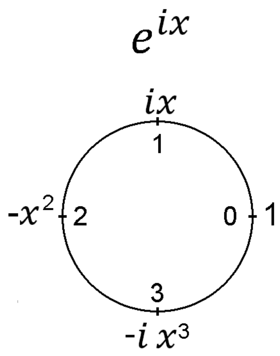

n	x	$\sin x$	$i \sin x$	$\cos x$	e^{ix}
0	0	0	0	1	1
1	$\frac{1}{2}\pi$	1	i	0	i
2	π	0	0	-1	-1
3	$\frac{3}{2}\pi$	-1	$-i$	0	$-i$
4	2π	0	0	1	1
5	$\frac{5}{2}\pi$	1	i	0	i

$$e^{ix} = \cos x + i \sin x \tag{31}$$

The table for e^{ix} looks complicated but it is very similar to previous tables. There is a new "$i \sin x$" column which is i times the "$\sin x$" column. The last column (e^{ix}), is simply "$i \sin x + \cos x$".

The equation e^{ix} starts at the origin ($n = 0$) and steps counterclockwise around the circle.

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4.1 Euler's' Identity

Starting with the complex Euler equation (formula 32), if you substitute π for x and simplify, you get an equation referred to as Euler's identity (shown in formula 35).

$$e^{ix} = \cos x + i \sin x \quad (32)$$
$$e^{i\pi} = \cos \pi + i \sin \pi \quad (33)$$
$$e^{i\pi} = -1 + 0 \quad (34)$$
$$e^{i\pi} + 1 = 0 \quad (35)$$

It is a famous identity because it integrates the three main mathematical constants: e, i, and π. The physicist Richard Feynman called Euler's identity "the most remarkable formula in mathematics". In 1988, a survey by the Mathematical Intelligencer reported that its readers voted this equation the "Most beautiful mathematical formula ever".

Mathematicians appreciated the elegance of this identity, even though its meaning was unclear. Now we know that this identity refers to the second step of a complex spin equation. We can look back at the e^{ix} table (used with formula 31) to see that e^{ix} is equal to -1 when $x = \pi$. So the identity basically says $1 + -1 = 0$.

4.2 Deriving the Complex Conjugate of e^{ix}

The odd powered terms generated by the equation e^{ix} contain imaginary numbers. This is a problem since imaginary terms in the answer can not be resolved. However, if we pair the e^{ix} equation with its complex conjugate e^{-ix}, then the complex terms cancel out, leaving only the real terms.

Below is a way to derive an equation that pairs e^{ix} (formula 36) with its complex conjugate. The first step is to determine the complex conjugate of e^{ix} which is e^{-ix}. We can get this by starting with the formula for e^{ix} and then substituting $-x$ for x (formula 37).

$$e^{ix} = \cos x + i \sin x \tag{36}$$

$$e^{i(-x)} = \cos(-x) + i \sin(-x) \tag{37}$$

Then with the help of the following two identities (formulas 38 and 39), $e^{i(-x)}$ can be simplified to formula 40.

$$\cos(-x) = \cos(x) \tag{38}$$

$$\sin(-x) = -\sin(x) \tag{39}$$

$$e^{-ix} = \cos(x) - i \sin(x) \tag{40}$$

At this point we have a formula for both e^{ix} (formula 36) and and its complex conjugate e^{-ix} (formula 40). All we have to do is add those two equations together and we get an equation (formula 41) that always yields real answers.

$$e^{ix} + e^{-ix} = 2 \cos x \tag{41}$$

(For mathematical consistency, there needs tp be a new form of brackets placed around $e^{ix} + e^{-ix}$ to indicate that the two paired expressions must be treated as a single mathematical entity.)

4.3 Elements of the Spin Equation

If we add the basic spin equation (Ψ) to its complex conjugate ($\Psi^\star$), we get a wave equation (ψ). The basic spin equation takes the form:

$$\Psi = e^{ix} \tag{42}$$

The complex conjugate of Ψ ($\Psi^\star$) is the same equation spinning backwards:

$$\Psi^\star = e^{-ix} \tag{43}$$

Ψ added to $\Psi^\star$ generates a wave (ψ) with the characteristics of $2\cos x$

$$e^{ix} + e^{-ix} = 2\cos x \tag{44}$$

$$\psi = 2\cos(x) \tag{45}$$

5 Computing Energy

5.1 Single Variable Solution (t)

If you multiply Planck's reduced constant ($\hbar$) by the rate-of-spin you get the energy (E).

$$\hbar \times \text{Rate-of-spin} = E \tag{46}$$

Which means

$$\text{Rate-of-spin} = \frac{E}{\hbar} \tag{47}$$

Since Ψ is a spin equation with a constant velocity, the derivative of Ψ (with respect to time) is the rate of spin.

$$\frac{\partial \Psi}{\partial t} = \frac{\Delta \Psi}{\Delta t} = \text{Rate-of-spin} \tag{48}$$

If we put this rate into our spin equation, then the first derivative of our spin equation give us an energy formula.

$$\Psi(t) = e^{i\frac{E}{\hbar}t} \tag{49}$$

$$\frac{\partial \Psi}{\partial t} = i\frac{E}{\hbar}\Psi \tag{50}$$

$$-i\hbar\frac{\partial \Psi}{\partial t} = E\Psi \tag{51}$$

Formula 51 says that $\hbar$ times the rate of spin ($\frac{\partial \Psi}{\partial t}$) equals the energy.

5.2 Two Variable Solution (x, t)

Just as we used the derivative with respect to time to build an energy formula, the derivative with respect to x can be used to create a momentum formula. We know the rate used for a time equation is $E/\hbar$, similarly the rate to use for computing momentum is $p/\hbar$. So a solution for Ψ that varies with both x and t could be:

$$\Psi = e^{i(\frac{p}{\hbar}x - \frac{E}{\hbar}t)} \tag{52}$$

With this one spin equation, we can take derivatives to create the momentum and energy formulas. The energy formula comes by taking the first derivative of Ψ (t) and then multiplying both sides of the equation by $i\hbar$.

$$\frac{\partial \Psi}{\partial t} = -i\frac{E}{\hbar}\Psi \tag{53}$$

$$i\hbar\frac{\partial \Psi}{\partial t} = E\Psi \tag{54}$$

Formula 54 says that multiplying $\hbar$ by the derivative of Ψ (rate of spin) equals the energy.

We can also create a momentum formula by taking the derivative of Ψ (x), and then multiplying through by $-i\hbar$

$$\frac{\partial \Psi}{\partial x} = i\frac{p}{\hbar}\Psi \tag{55}$$

$$-i\hbar\frac{\partial^2 \Psi}{\partial x^2} = p\Psi \tag{56}$$

Even though the momentum formula can not directly be used to compute energy, it can be used indirectly. The momentum (mv) can be converted to kinetic energy ($\frac{1}{2}mv^2$). It is technically easier to convert energy from p^2 (than p) so we can use the second derivative of Ψ(x).

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= i\frac{p}{\hbar}\Psi && (57) \\ \frac{\partial^2 \Psi}{\partial x^2} &= i^2\frac{p^2}{\hbar^2}\Psi && (58) \\ \frac{\partial^2 \Psi}{\partial x^2} &= -\frac{p^2}{\hbar^2}\Psi && (59) \\ -\hbar^2\frac{\partial^2 \Psi}{\partial x^2} &= p^2\Psi && (60) \end{aligned}$$

Now we need to know how to convert p^2 to kinetic energy.

$$\begin{aligned} p &= mv && (61) \\ p^2 &= m^2v^2 && (62) \\ \frac{p^2}{2m} &= \frac{1}{2}mv^2 && (63) \\ \frac{p^2}{2m} &= \text{kinetic energy} && (64) \end{aligned}$$

We also know that the total energy (E) is equal to the kinetic energy (KE) plus the potential energy (PE).

$$\text{E} = \text{KE} + \text{PE} \qquad (65)$$

Formula 64 tells us that dividing p^2 by $2m$ gives us the kinetic energy. So we can substitute in the kinetic energy, and then let υ represent the potential energy.

$$E = \frac{p^2}{2m} + \upsilon \qquad (66)$$

Next we multiply each term by Ψ

$$E\Psi = \frac{p^2\Psi}{2m} + \upsilon\Psi \qquad (67)$$

Then if we replace $p^2\Psi$ with our momentum equation (formula 60), we have the time independent Schrodinger equation (for one spatial dimension).

$$E\Psi = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}\Psi + \upsilon\Psi \qquad (68)$$

Finally, if we substitute in our energy equation (formula 54) for $E\Psi$ we get the time dependent Schrodinger equation.

$$i\hbar\frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}\Psi + \upsilon\Psi \qquad (69)$$

6 New Definition of i

In order to introduce spin, a solid foundation has been built between imaginary and real numbers. However the definition of i still needs to be updated to reflect these new concepts. The new definition of i is:

The step unit in the imaginary unit circle;
equal in value to the fourth root of 1.

$$i = \sqrt[4]{1}$$

Summary

Imaginary numbers were introduced into our real number system without a mathematical foundation. With no explicit guide showing how to relate real and imaginary numbers, arbitrary assumptions had to be made. While those assumptions allowed us to successfully work with imaginary numbers, some distortions in our concepts occurred.

Spin equations look like waves when you plot them on a Cartesian (x, y) graph. When you plot them on the imaginary unit circle, they spin. They step around in a counterclockwise direction driven by derivatives.

There were hints of spin in our number system already. The power series (for e^x, $\cos x$, and $\sin x$) were all generated by spin equations. The Euler formula (really a half of a formula) is a spin equation.

The format for a basic real spin equation is

$$e^{ix} \tag{70}$$

If we pair this equation with its complex conjugate ($\Psi^\star$) we obtain a complex spin equation that generates a real wave (ψ).

$$\Psi = e^{ix} \tag{71}$$

$$\Psi^\star = e^{-ix} \tag{72}$$

$$e^{ix} + e^{-ix} = 2\cos(x) \tag{73}$$

$$\psi = 2\cos(x) \tag{74}$$

If we want to form energy and momentum equations, then we have to choose the coefficients for x and t appropriately. The coefficients needed to work with the derivatives of Ψ to form the momentum and energy equations are $(\frac{p}{\hbar})$ and $(\frac{E}{\hbar})$ respectively:

$$e^{ix} = e^{i(\frac{p}{\hbar}x - \frac{E}{\hbar}t)} \tag{75}$$

$$\Psi = e^{i(\frac{p}{\hbar}x - \frac{E}{\hbar}t)} \tag{76}$$

To create an energy formula, we took the derivative of Ψ (with respect to time).

$$i\hbar \frac{\partial \Psi}{\partial t} = E\Psi \tag{77}$$

To create a formula for momentum we took the second derivative of Ψ (with respect to x),

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial x^2} = p^2 \Psi \tag{78}$$

The total energy (E) of Ψ is the kinetic energy (KE) plus potential energy (PE). We can substitute in our new version of the kinetic energy and υ.

$$E\Psi = KE\Psi + PE\Psi \tag{79}$$

$$E\Psi = \frac{p^2\Psi}{2m} + \upsilon\Psi \tag{80}$$

Now we just substitute our momentum equation (formula 78) and we get the time independent Schrodinger equation (in one spatial dimension).

$$E\Psi = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial t^2} \Psi + \upsilon\Psi \tag{81}$$

Finally we can replace $E\Psi$ with our energy equation (formula 77 to obtain the time dependent Schrodinger equation.

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial t^2} \Psi + \upsilon\Psi \tag{82}$$