ZFC IS INCONSISTENT

A condition by Paul of Venice (1369-1429) solves Russell's paradox, blocks Cantor's diagonal argument, and provides a challenge to ZFC

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Abstract

Paul of Venice (1369-1429) provides a consistency condition that resolves Russell's Paradox in naive set theory without using a theory of types. It allows a set of all sets. It also blocks the (diagonal) general proof of Cantor's Theorem (in Russell's form, for the power set). The Zermelo-Fraenkel-Axiom-of-Choice (ZFC) axioms for set theory appear to be inconsistent. They are still too lax on the notion of a 'well-defined set'. The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics in ZFC for the foundations of set theory.

For amendment of ZFC two alternatives are mentioned: ZFC-PV (amendment of de Axiom of Separation) or BST (Basic Set Theory).

Theorems 2.5 & 2.7 show for the singleton that ZFC is inconsistent. Lemma 3.2 shows that a Cantorian reading of ZFC implies the possibility of the weaker Pauline reading. Theorem 3.3 gives a constructive proof of the existence of a Pauline set. Appendix D deproves Cantor's Theorem.

Keywords: Paul of Venice • Russell's Paradox • Cantor's Theorem • ZFC • naive set theory • well-defined set • set of all sets • diagonal argument • transfinites

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Contents

1. Introduction

1.1. From Epimenides to Paul of Venice, to Cantor, to Russell, to ZFC

This paper deals with self-reference and derives a contradiction. It may thus be difficult to follow. The reader can maintain clarity by holding on to the key notion of freedom of definition. When a restriction on that freedom generates a consistent framework, while release of that restriction generates confusion, then the restriction is to be preferred above too much freedom.

An example are naive set theory with its freedom and Russell's paradoxical set R = $\{x \mid x \notin x\}$. We derive $R \in R \Leftrightarrow R \notin R$ and naive set theory collapses. Axiomatic set theory studies how to regulate the freedom of definition. This paper shows that the ZFC-system (Zermelo, Fraenkel, Axiom of Choice) appears to be inconsistent. ZFC still has too much freedom of definition. The cause lies in some self-reference again.

There is an old way to deal with self-reference. Bochenski (1956, 1970:250) discusses that Paul of Venice (1369-1429) (Paulus Venetus) formulated the following condition w.r.t. the Liar Paradox. A proposition may be extended with *"and this proposition is true".* The Liar *"This proposition is false"* becomes "*This proposition is false and this proposition is true",* whence it shows itself to be inconsistent. An idea is to apply such a condition to Russell's set. Consider S = $\{x \mid x \notin x \ \& \ x \in S\}$ with the small consistency condition inspired by Paul of Venice. We find $S \in S \Leftrightarrow (S \notin S \ \& \ S \in S)$, which reduces to $S \notin S$ without contradiction. The translation of Paul of Venice's idea to set theory is a bit involved, notably with infinite regress, when a test on S on the left causes a test on S on the right, which causes a test on the left again, and so on; but the truthtable of $p \Leftrightarrow (\neg p \ \& \ p)$ allows a formal decision. These issues can be resolved, with the new notation ***p* &| *q*** for such self-referential conditions (see below).

While the first result of this paper is that ZFC is inconsistent, the second result is that the Paul of Venice condition might be used to repair this, like shown above for Russell. The following sections will make the argument formal. Section 2 discusses the singleton and shows that ZFC is inconsistent. Section 3 discusses the general situation. Section 4 concludes. This introduction proceeds with basic definitions and theorems.

1.2. Definition of ZFC

We take our definitions from a matricola course at Leiden and Delft.

Definition (Coplakova et al. (2011:18), I.4.7): Let *A* be a set. The power set of *A* is the set of all subsets of *A.* Notation: *P*[*A*]. Another notation is 2^A, whence its name.

Definition (Coplakova et al. (2011:144-145)): ZFC.

Comment: This includes the axiom that each set has a power set. (POW)

Definition of the *Axiom of Extensionality* (Coplakova et al. (2011:145)):

$$(A = B) \Leftrightarrow ((\forall x)\ (x \in A \Leftrightarrow x \in B)) \qquad \text{(EXT)}$$

Definition of the *Axiom of Separation* (Coplakova et al. (2011:145), inserting here a by-line on freedom): If *A* is a set and $\gamma[x]$ is a formula with variable *x,* while *B* is not free in $\gamma[x]$, then there exists a set *B* that consists of the elements of *A* that satisfy $\gamma[x]$:

$$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \Leftrightarrow ((x \in A) \ \& \ \gamma[x])) \qquad \text{(SEP)}$$

Comments: This is also called an axiom-schema since there is no quantifier on γ. Russell's set cannot be formed in ZFC and thus is no ZFC-set.

1.3. Cantorian sets in ZFC

Definition of a Cantorian set: Let A be a set, $P[A]$ its power set, and let $f: A \to P[A]$. A strictly Cantorian set is $\Psi = \Psi[f] = \{x \in A \mid x \notin f[x]\}$. A generalized Cantorian set has $x \notin f[x]$ as part of its definition. A 'Cantorian set' without qualification can depend upon the context.

Theorem 1.3.A. Existence of a strictly Cantorian set. Let A be a set, $P[A]$ its power set. For every function $f: A \to P[A]$ there is a strictly Cantorian set.

Proof: (a) $P[A]$ exists because of the Axiom of the Powerset. (b) f can be regarded as a subset of A x $P[A]$, and f exists because of Axiom of Pairing. (c) Ψ exists because of the Axiom of Separation. Find $\Psi \subseteq A$, thus $\Psi \in P[A]$. Q.E.D.

Theorem 1.3.B. Weakest Conjecture on strictly Cantorian sets. Let A be a set. For every $f: A \to P[A]$ there is a $\Psi \in P[A]$ such that for all $\alpha \in A$ it holds that $\Psi \neq f[\alpha]$.

Proof: Define $\Psi = \{x \in A \mid x \notin f[x]\}$. Take $\alpha \in A$. Check the two possibilities.

Case 1: $\alpha \in \Psi$. In that case $\alpha \notin f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in \Psi \setminus f[\alpha]$.)

Case 2: $\alpha \notin \Psi$. In that case $\alpha \in f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in f[\alpha] \setminus \Psi$.) Q.E.D.

Comment: This theorem-conjecture combines various issues: the definition of strictly Cantorian sets, the existence proof and an identification of their key property. It is essentially a rewrite of the definition $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$ and application of extensionality (EXT).

Comment: The Cantorian set clearly has some kind of self-reference. This set is important for Cantor's Theorem and the 'diagonal argument', see **Appendix D**.

1.4. An axiom for a solution set

The following is not in ZFC but will help to understand ZFC. It uses notation 'p &| q'.

Definition of an *Axiom of a Solution Set* (this paper):

$$(\forall A)\ (\exists Z)\ (\exists B)\ ((B \in Z) \Leftrightarrow\ (\forall x)\ ((x \in B)\ \Leftrightarrow\ ((x \in A)\ \&\ \gamma[x]\ \&|\ (x \in B)))) \quad \text{(SOL-PV)}$$

This SOL-PV can reduce to the Axiom of Separation (SEP). A way is to eliminate $B \in Z$ as superfluous and self-evident. Another way is to replace $B \in Z$ by $B = Z$. This imposes uniqueness. When $\gamma[x]$ has more solutions then a contradiction arises when SEP requires that a single solution B is also the whole set Z. Below for the singleton $A = \{\alpha\}$, **Theorem 1.3.B** finds $B = \Psi = A$ but we will find $Z = \{\emptyset, A\} = P[A]$. In itself it is true that $\Psi \in P[A]$, but when $Z = P[A]$ then it is erroneous to require $Z \in P[A]$. It is not just an issue of notation.

1.5. Appendices

Colignatus (1981, 2007, 2011) "*A Logic of Exceptions"* (**ALOE**) in 1981 applied the Paul of Venice consistency condition to the Russell set (p129), and applied it in 2007 (p239) also to Cantor's (diagonal) argument (in Russell's version for the power set). ALOE does not develop the formal ZFC system of axioms for set theory. ALOE's discussion may be seen as intermediate between naive set theory and this present paper. **Appendix A** discusses the versions of ALOE, for proper reference.

The new issue in this paper is the challenge to ZFC. The ZFC system may still be too lax on the notion of a *'well-defined set'.* This discussion originated from considering Cantor's diagonal argument. The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics of ZFC w.r.t. the foundations of set theory. Colignatus (2012, 2013) CCPO-PCWA defines a ***bijection by abstraction*** between the natural and real numbers ($\aleph \sim \Re$). This challenges ZFC, that denies a bijection. This present paper puts ZFC central, and Cantor's Theorem is derivative in **Appendix D.**

Appendix B supports this Introduction. The appendix restates the exception switch from ALOE for when $x \neq S$. A new proposed notation with asymmetric ***p* &| *q*** allows for a shorthand of the exception switch. In the body of the paper '&|' is used, but the structure of the argument can be followed when reading this as '&' - with a backup in the appendix.

Appendix C illuminates *Cantorian* versus *Pauline* strategies on ZFC that are relevant for a key point in the reasoning.

Appendix D discusses Cantor's Theorem. There are: (a) A table with the various forms of the theorems, proofs and their refutations. Remember that Cantor presented his theorems before ZFC. (b) The semantics of *deproving a theorem*: refuting an argument by showing that a proof is invalid. (c) The ***common*** proof of Cantor's theorem that uses a bijection and the ***standard*** theorem that uses only a surjection. (Compare with the weakest Theorem 1.3.B.)

Appendix E has more on the genesis of this paper. It must be observed that this author is no expert on ZFC or Cantor, see for background Colignatus (2013). This paper may reject various proofs but perhaps there are other proofs

2. The singleton

2.1. The singleton with a nutshell link between Russell and Cantor

Let A be a set with a single element, $A = \{\alpha\}$. Thus $P[A] = \{Ø, A\}$. Let $f : A \rightarrow P[A]$.

If $f[\alpha] = Ø$ then $\alpha \notin f[\alpha]$. If $f[\alpha] = A$ then $\alpha \in f[\alpha]$. Thus $f[\alpha] = Ø \Leftrightarrow \alpha \notin f[\alpha]$. Consider:

(1) In steps: define $\Psi = \{x \in A \mid x \notin f[x]\}$, find $\Psi \in P[A]$, then try $f[\alpha] = \Psi$.

(2) Directly: $f[\alpha] = \{x \in A \mid x \notin f[x]\}$

(3) Either directly or indirectly via (1) or (2): $\Psi = \{x \in A \mid x \notin \Psi\}$. This is *Russellian* !

With the element of the set obviously in the set A: $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin \Psi)$.

Thus (1) - (3) are only consistent when $\Psi \neq f[\alpha]$. This is an instance of **Theorem 1.3.B**.

Formula (3) is an instance of Russell's paradox. Choosing $f[\alpha] = \Psi$ in (1) assumes freedom that conflicts with the other restrictions.

We have liberty to choose $f[\alpha] = Ø$ or $f[\alpha] = A$. This choice defines f and we should write $\Psi = \Psi[f]$. This shows why (2) with $f[\alpha] = \Psi[f]$ is tricky. If (2) is an implicit definition of f then it doesn't exist. If it exists then this $f[\alpha]$ will not be in its definition.

2.2. A truthtable for the singleton Cantorian set

Consider the definition $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$ (Theorem 1.3.B). It is possible to weaken this by means of the tautology $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$. The truthtable for the singleton Cantorian set is in Table 1. The rows are labeled with a Δ-*case-number* for reference to other tables below. (Δ for 'marginal'.) The truthtable holds for every f while $\Psi = \Psi[f]$.

Theorems 1.3.AB establish the LHS. Modus ponens with the tautology gives the RHS as a *separate* expression - provided that we maintain the original $\Psi = \Psi[f]$:

$$\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \;\&\; \alpha \in \Psi).$$

An idea that $\alpha \in \Psi$ on the LHS covers all cases of $\alpha \notin f[\alpha]$ is false: it doesn't cover Δ1.

Table 1. Truthtable for a singleton Cantorian set, with $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$

Case	(α ∈ Ψ	⇔	α ∉ *f*[α])	⇒	(α ∈ Ψ	⇔	(α ∉ *f*[α]	&	α ∈ Ψ))
Δ2	1	**1**	1	1	1	**1**	1	1	1
Δ4	1	**0**	0	1	1	**0**	0	0	1
Δ1	0	**0**	1	1	0	**1**	1	0	0
Δ3	0	**1**	0	1	0	**1**	0	0	0

2.3. Possibilities for the singleton LHS of Table 1

Checking all possibilities on the LHS gives Table 2. Check the Δ-*case-numbers*. We again see that an idea that α ∈ Ψ covers all instances of α ∉ *f*[α] is false: it doesn't cover Δ1.

Table 2. Test of the singleton: $\alpha \in \Psi$? **via** $\alpha \in \Psi \Leftrightarrow \alpha \notin f[\alpha]$

For all cases: $\alpha \in A$	Ψ = Ø, α ∉ Ψ	Ψ = *A*, α ∈ Ψ
f[α] = Ø α ∉ *f*[α]	**Δ1**: $\alpha \in \varnothing \Leftrightarrow \alpha \notin \varnothing$ *f*[α] = Ψ, **impossible**	**Δ2**: $\alpha \in A \Leftrightarrow \alpha \notin \varnothing$ *f*[α] ≠ Ψ, possible
f[α] = *A* α ∈ *f*[α]	**Δ3**: $\alpha \in \varnothing \Leftrightarrow \alpha \notin A$ *f*[α] ≠ Ψ, possible	**Δ4**: $\alpha \in A \Leftrightarrow \alpha \notin A$ *f*[α] = Ψ, **impossible**

Table 2 summarizes as follows. The arrows indicate links and no equivalences: in an equivalence we would also test the possibility that $\alpha \in \varnothing$, and that is not relevant here.

$$(f[\alpha] = \varnothing) \leftrightarrow (\alpha \in \Psi) \leftrightarrow (\alpha \notin f[\alpha]) \leftrightarrow (\alpha \notin \varnothing) \leftrightarrow (\alpha \in A) \leftrightarrow (\Psi = A)$$

$$(f[\alpha] = A) \leftrightarrow (\alpha \notin \Psi) \leftrightarrow (\alpha \in f[\alpha]) \leftrightarrow (\alpha \in A) \leftrightarrow (\Psi = \varnothing)$$

2.4. Possibilities for the singleton RHS of Table 1

For the RHS it is useful to write Φ = Ψ, see Table 3. The same Δ-*case-numbers* apply.

Now Δ1 is allowed too: a possible *f*[α] = Φ rather than an impossible *f*[α] = Ψ. The possibility seems relevant when we want to construct a bijection in the infinite.

Table 3. Test of the singleton: $\alpha \in \Phi$? **via** $\alpha \in \Phi \Leftrightarrow (\alpha \notin f[\alpha] \;\&\; \alpha \in \Phi)$

For all cases: $\alpha \in A$	Φ = Ø, α ∉ Φ	Φ = *A*, α ∈ Φ
f[α] = Ø α ∉ *f*[α]	$\alpha \in \varnothing \Leftrightarrow (\alpha \notin \varnothing \;\&\; \alpha \in \varnothing)$ *f*[α] = Φ, **possible**: α ∉ Ø	$\alpha \in A \Leftrightarrow (\alpha \notin \varnothing \;\&\; \alpha \in A)$ *f*[α] ≠ Φ, possible
f[α] = *A* α ∈ *f*[α]	$\alpha \in \varnothing \Leftrightarrow (\alpha \notin A \;\&\; \alpha \in \varnothing)$ *f*[α] ≠ Φ, possible	$\alpha \in A \Leftrightarrow (\alpha \notin A \;\&\; \alpha \in A)$ *f*[α] = Φ, **impossible**: $\alpha \notin A$

Note that *f*[α] = Ø doesn't give a unique Φ. Both Φ = Ø (Figure 1) and Φ = *A* (Figure 2) are possible: the figures use Φ = Ψ. Note that *f* is still a function and no correspondence.

PM 1. If φ runs over *A* and we require *f*[φ] = Φ then we find $\varphi \notin \Phi$. Choosing *f*[φ] = Φ reduces our freedom for choosing *f*: In the singleton the only set in *P*[*A*] that has non-membership is Φ = Ø. For the singleton thus *f*[φ] = Ø.

PM 2. We can also gain access to Δ4 by another relaxing condition but we are interested in the α ∉ *f*[α] case.

Figure 1. Diagram of the Cantorian set for the singleton, case Δ1: $f[\alpha] = Ø = \Psi$

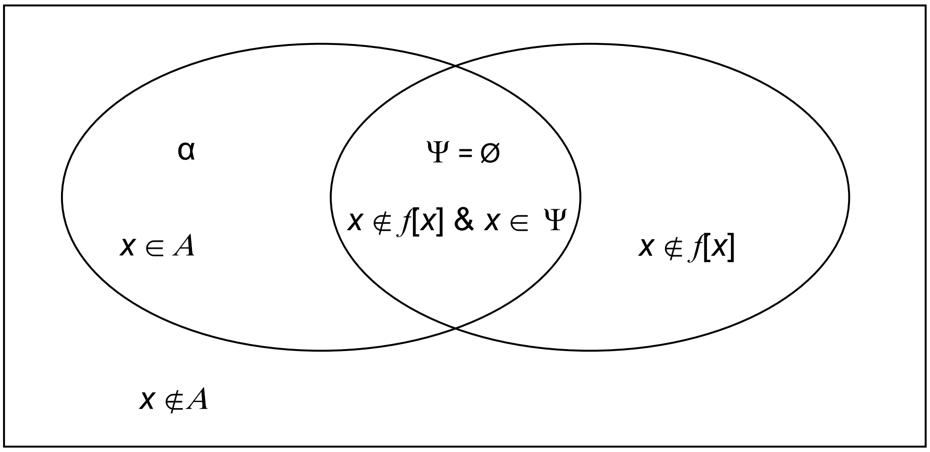
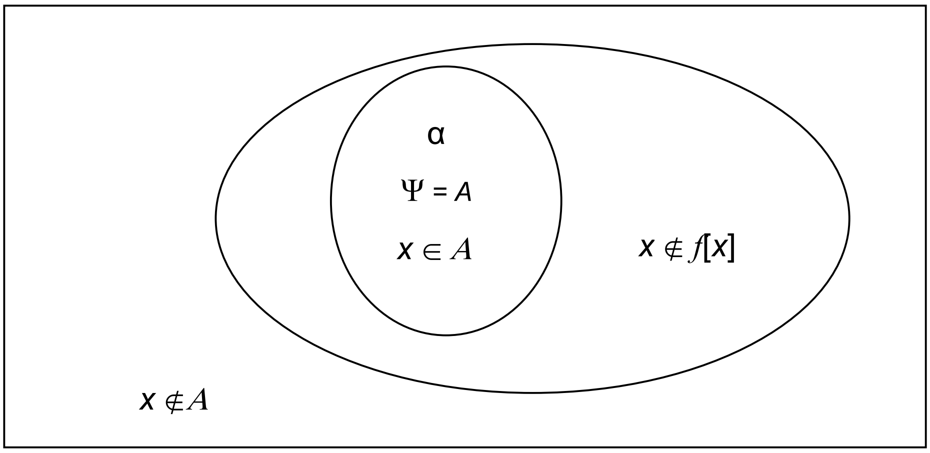


Figure 2. Diagram of the Cantorian set for the singleton, case Δ2: $f[\alpha] = Ø \neq \Psi$



2.5. A counterexample for Theorem 1.3.B for the singleton Cantorian case

The discovery of Δ1 generates this theorem.

Theorem 2.5. For the singleton Cantorian case there are a f and Ψ with $f[\alpha] = \Psi$.

Proof: Let $A = \{\alpha\}$ have a single element. Thus $P[A] = \{Ø, A\}$. Let $f : A \to P[A]$ with $f[\alpha] = Ø$. Then $\alpha \notin Ø$ and $\alpha \notin f[\alpha]$.

(1) *Existence*. Take $q = (\alpha \notin f[\alpha])$ and use $q \Rightarrow (p \Leftrightarrow (q \,\&\, p))$ for any p, see Table 4.

Table 4. Truthtable for a singleton Cantorian set, with $q \Rightarrow (p \Leftrightarrow (q \,\&\, p))$

Case	α ∉ f[α]	⇒	(p	⇔	(α ∉ f[α]	&	p))
Δ2	1	1	1	**1**	1	1	1
Δ4	0	1	1	**0**	0	0	1
Δ1	1	1	0	**1**	1	0	0
Δ3	0	1	0	**1**	0	0	0

We can take $p = (\alpha \in A)$ or not-$p = (\alpha \in Ø)$. Take the latter, apply modus ponens, and find $\alpha \in Ø \Leftrightarrow (\alpha \notin Ø \,\&\, \alpha \in Ø)$ (Δ1) that reduces into $\alpha \notin Ø$. See Figure 1. The case clearly is possible.

(2) *Name.* Consider $\Psi = \{x = \alpha \mid x \notin f[x]\}$ or for convenience $\Psi = \{x \in A \mid x \notin f[x]\}$. Look at Table 1. With f[α] = Ø the equivalence on the LHS only allows a solution Ψ = *A*. Look at row Δ1. On the LHS we have (α ∉ Ψ) & (α ∉ f[α]), and the equivalence would declare this combination Δ1 impossible. However, there is also the relaxed condition. Use ($\alpha \in \Psi$) ⇔ (α ∉ f[α]) and tautology $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$ (Table 1), and apply modus ponens to find $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \;\&\; \alpha \in \Psi)$. In this deduction we have maintained the name Ψ, the modus ponens is independent of the possibility that also Ψ = *A* might be derived via another route, and the formula now *stands as a separate* relation for Ψ. On the RHS of Table 1, we find that f[α] = Ø = Ψ can remain consistent. We have the same case $\alpha \in Ø \Leftrightarrow (\alpha \notin Ø \;\&\; \alpha \in Ø)$ that we saw above, which reduces to α ∉ Ø, or $\alpha \in A$. Thus we have f[α] = Ψ[f] = Ø as a possibility.

Q.E.D.

Comments:

(1) Theorem 2.5 seems to contradict Theorem 1.3.B. The latter holds that every f has this Ψ such that Ψ ≠ f[*α*]. Conventionally, Theorem 1.3.B indeed is read as the *only* possibility, whence this subsection carries the title of 'counterexample'. Theorem 1.3.B however isn't quite untrue since Table 4 would also allow *p* = (α ∈ *A*) and Figure 2. A viewpoint is that the two theorems supplement each other by indicating the two possibilities.

(2) Another view is: A definition of Ψ = Ψ[f] on the LHS results via the tautology into a weaker relation on the RHS that contradicts that definition. Δ1 is a clear conflict zone.

(3) The problem with Theorems 1.3.AB is that they ***impose*** the equivalence. This assumes a freedom of definition, whence it assumes that the truthtable on the LHS is true, whence Δ1 is forbidden. But that freedom of definition does not exist. Something exists, that is infringed upon by the definition. See the counterexample. When Ψ is just another name for the empty set, as in the singleton possibility of Figure 1, then one does no longer have the freedom to switch from Ø to *A,* see Figure 2. See also the discussion in Section 2.1 about the freedom of definition. See the opening paragraph of the Introduction.

(4) A line of argument in modal logic might be to hold that when the impossible (LHS) generates the possible (RHS), while there are only impossibility and possibility, then the possibility remains.

(5) As said, this paper is about self-reference and identifies a contradiction. It may well be that these steps are difficult to follow. The cause is the framing by ZFC.

2.6. The crucial observations

The following observations are crucial:

(a) To repeat: an idea that Ψ in Theorems 1.3.AB or Table 2 covers all α ∉ f[α] appears to be false: it doesn't cover Δ1. Thus when α ∉ Ψ, there still exists a case of α ∉ f[α]. Now, isn't Ψ *supposed* to cover *all* such latter cases ? The conclusion is: Theorems 1.3.AB do not cover the *intended interpretation* (DeLong (1971)). However: since Theorem 2.5 deducts these neglected truths, and still is in ZFC, we cannot hold that ZFC is *deductively incomplete*.

(b) We can look at the tables 2 and 3 in horizontal or vertical direction. This reflects the schism in philosophy between *nominalism* and *realism*. (See William of Ockham.)

(b1) The horizontal view gives the **realists** who take predicates as 'real': α ∉ f[α] versus α ∈ f[α]. They are also sequentialist: Δ1 & Δ2 versus Δ3 & Δ4.

(b2) The vertical view gives the **nominalists** (Occam) who regard the horizontal properties as mere stickers, and who more realistically look at $\Psi = \Phi = Ø$ versus $\Psi = \Phi = A$. They see the table in *even* versus *uneven* fashion: Δ2 & Δ4 versus Δ1 & Δ3.

(c) The change from LHS and Table 2 to RHS and Table 3 was supported for clarity by a switch of the name Ψ to Φ. The name change is justified by that the LHS allows two cases and the RHS allows three cases: and one should know what one is discussing. That John is known as Charley in Amsterdam should not be too confusing. The different names Ψ and Φ should not confuse us either. A nominalist will point to the phenomenon that there are only two sets that we are discussing here: Ø and *A*.

(d) The nominalist reasoning is: The sets Ø and *A* exist, as above tables show. We are merely discussing how they are referred to. The expressions for the sets Ψ and Φ are not *defining* statements but *derivative* observations. Once the functions have been mapped out, the criteria can be used to see whether the underlying sets may get also another sticker Ψ or Φ. We are discussing *'consistent referring'* and not existence. If we would choose to say that $\Psi \neq \Phi$, as another way to express that ZFC would be inconsistent or incomplete, then this may arise as a property of the logical phrases but this doesn't change anything about the properties and existence of the underlying sets and the underlying ability to define functions.

(e) At issue is now whether ZFC has sufficient logical strength to block nonsensical situations. ZFC has a realist bend. It translates predicates into sets (their extensions). Instead it can be better to *only test* whether a predicate is useful. Merely cataloguing differently what already exists should not be confused with existence itself. The freedom of definition can be a mere illusion (see $f[\varphi] = \Phi$ in 2.4) and then should not be abused to create nonsense.

2.7. Inconsistency and diagnosis

Let us observe inconsistency and make the right diagnosis.

Theorem 2.7. ZFC is inconsistent.

Proof: Theorems 1.3.AB generate for the singleton that $\Psi = A$. **Theorem 2.5** generates for the singleton the possibility that $\Psi = Ø$. Thus it is possible that $A = Ø$. This is a clear contradiction. Q.E.D.

Comments:

(1) The diagnosis is that Ψ is rather a variable (name) than a constant set. There is a solution set $\Psi^* = \{Ø, A\}$, and Ψ is a variable that runs over Ψ^*. Compare to algebra, when one uses a variable x with value $x = 2$ in one case and $x = 4$ in another case: then one might derive $2 = x = 4$, but this goes against the notion of a variable. The inconsistency in ZFC is caused by that it does not allow for that Ψ is such a variable. See section 1.4 for a proposed *Axiom of a Solution Set,* that is not how ZFC operates.

(2) Since **Theorems 1.3.AB** are well accepted in the literature and **Theorem 2.5** is new, there is great inducement to find error in **Theorem 2.5.** Indeed, **Theorem 2.5** allows the deduction of a contradiction in **Theorem 2.7,** and thus one might hold that it should go. However, its steps are correct. It is more productive for the reader to accept the inconsistency of ZFC. As said, this paper is about self-reference and identifies a contradiction, and it may well be that these steps are difficult to follow. The problem lies not in the paper but in the inconsistency of ZFC. Potentially the distinction between constant and variable in comment (1) has most effect for clarity. It is not sufficient to suggest to read **Theorem 1.3.B** now as generating a value for the variable, rather than restricting the solution set to that value: since then one reads into it something which it does not do, for it really restricts that solution set.

(3) One might hold that **Theorems 1.3.AB** are not necessarily wrong, since one can find for any f a $\Psi[f]$ such that for all α: $f[\alpha] \neq \Psi$. (For the singleton $f[\alpha] = \emptyset$ gives $\Psi = A$.) But the formula of that Ψ allows **Theorem 2.5** to also find another $f[\alpha] = \Psi$. (For the singleton $f[\alpha] = \emptyset$ gives $\Psi = \emptyset$.) One can conceive that the two options co-exist, but ZFC does not allow for that. Thus **Theorem 2.5** is a real counterexample for **Theoreom 1.3.B.** The freedom of definition used in **Theorem 1.3.B** depends upon the existence **Theorem 1.3.A**. Then something is wrong with Theorem 1.3.A, that proved the existence of what 1.3.B uses. The theorems were derived in ZFC. Thus ZFC has a counterexample and thus is inconsistent.

(4) Colignatus (2014b, 2015) (PV-RP-CDA-ZFC) version June 12, its appendix C, deproves **Theorem 1.3.B**, by showing that its assumption "every f", neglects to distinguish between surjections and other. **Theorem 3.3** below indicates the possibility of a surjection for denumerability. The crux remains in the self-reference shown by the singleton Δ1.

2.8. In summary, in relation to Table 1

Let us look back at what we have analysed, and it may be instructive to repeat some observations while referring to Table 1.

(1) Row Δ1 indicates a case of $\alpha \notin f[\alpha]$ that is not covered by $\alpha \in \Psi$. Since $\alpha \notin \Psi$ is ambiguous, i.e. allows two possibilities, a new name Φ can be used for the relaxed condition on the RHS that also catches this obscured case that $\alpha \notin f[\alpha]$. This new name Φ generates the idea of possibilities that $\Psi = \Phi$ and $\Psi \neq \Phi$. But because of Table 1 we still have $\Psi = \Phi$.

(2) Table 1 is an example of lemma 3.2 (Section 3). It shows (in the columns in bold) that the LHS (Table 2) only allows Δ2 or Δ3, while the RHS also allows Δ1. For the RHS it is useful to write Φ for Ψ, see Table 3. The same Δ-case-numbers apply.

(3) Again, consider $f[\alpha] = \emptyset = \Phi$ (Δ1). This is consistent, but cannot be seen directly by Ψ, even though it is covered in Table 1 by the falsehood of $\alpha \in \Psi$. In a realist mode of thought, we deduce from $f[\alpha] = \emptyset$ that $\Psi = A$, which is the only possibility on the LHS for $\alpha \notin f[\alpha]$ that Ψ recognises (row Δ2). This is not necessarily the proper response. The problem with ZFC is that it focuses on the LHS and neglects the RHS. We can derive a relaxed condition and then Theorem 2.5 allows to recover Δ1. The latter deductions are actually within ZFC and thus there is scope to argue that Theorem 1.3.B presents only part of the picture. However, that part is formulated in such manner that it causes the contradiction in **Theorem 2.7.** We must switch to a better axiomatic system that covers the *intended interpretation* and that blocks the paradoxical Ψ. The better system, ZFC adapted to Paul of Venice (ZFC-PV), blocks the LHS and allows only the RHS.

(4) A switch to ZFC-PV, with the RHS with the two solutions for $f[\alpha] = \emptyset$, will eliminate the transfinites, even though one of the solutions will still be $f[\alpha] = \emptyset \neq \Psi = A$. Cantor's Theorem in **Appendix D** uses the latter, but as a necessary condition, and not as a mere option. When a bijection has been given using $f[\varphi] = \Phi$ then the other option is not relevant.

(5) While the singleton establishes inconsistency, there still arise some questions about the general situation, notably also for when we look for solutions.

3. The general case

3.1 The notation 'p &| q'

The Paul of Venice consistency condition for the singleton still allows the use of '&'. The general case might cause infinite regress, and **Appendix B.3** introduces the notation with asymmetric 'p &| q' to handle this. The condition tests on a logical inconsistency of p & q, and if there is no conflict, then q is discarded. One can still follow the argument in outline by reading this as '&'. The following generalized Cantorian set is called *Pauline*:

$\Phi = \Phi[f] = \{x \in A \mid x \notin f[x] \ \&| \ x \in \Phi\}$

We compare this with $\Psi = \Psi[f] = \{x \in A \mid x \notin f[x]\}$. Does ZFC allow for Φ ? We saw this for the singleton but what is the situation in general ?

3.2. The Axiom of Separation in ZFC, Pauline and Cantorian readings

The discussion now focuses on the Axiom of Separation (SEP), stated in Section 1.2, and the condition "B is not free in $\gamma[x]$". There are a *Pauline* reading (supported by Lemma 3.2 below) and a defensive *Cantorian* reading of this condition.

Definition: The **Pauline** reading of SEP takes the axiomatic formula, $(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \ \Leftrightarrow\ ((x \in A)\ \&\ \gamma[x]))$, as basic, so that $\gamma'[x] = (\gamma[x]\ \&|\ (x \in \Phi))$ is allowed with the Paul of Venice consistency condition, since $B = \Phi$ is not free but bound by the existential quantifier $(\exists B)$. Thus the formation of Φ in 3.1 is allowed in ZFC. The expression $\gamma'[x]$ is created within the formula and can be taken out for inspection, but one should be careful about conclusions once it has been taken out. The explanatory text in the axioma around the formula is about the whole formula, and not necessarily about expressions once they have been taken out.

Definition: The **Cantorian** reading is that the expression $\gamma[x]$ is created outside of the axiomatic formula, is judged on its own properties (though unstated *why* and *how*), and only afterwards substituted into the formula. It holds: $\gamma'[x] = (\gamma[x]\ \&|\ (x \in \Phi))$ is not allowed because $B = \Phi$ would be a free variable *before* the substitution (and not a constant).

Appendix C discusses these two interpretations. There exist not only different forms of ZFC but also different readings. Lemma 3.2 however proves that the Cantorian reading implies the possibility of the Pauline reading, for membership to the B which is being defined.

Lemma 3.2. For "$(x \in B)$": (The Cantorian reading) $\Rightarrow$ (A possible Pauline reading).

Proof: There is the tautology in propositional logic (see Table 1):

$$(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$$

With $p = (x \in B)$ and $q = (x \in A \,\&\, \gamma[x])$ we get:

$$\big((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x])\big) \Rightarrow \ \big((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x] \,\&\, (x \in B))\big)$$

Take the Cantorian reading that $(\forall A)\ (\exists B)\ (\forall x)\ ((x \in B) \Leftrightarrow (x \in A\ \&\ \gamma[x]))$ and eliminate the existential quantifier by some constant set, say $C = C[A, \gamma]$:

$$(\forall A)\ (\forall x)\ \ ((x \in C)\ \Leftrightarrow\ (x \in A\ \&\ \gamma[x]))$$

Given that the tautology holds for all p and q, then apply Modus Ponens.

$$(\forall A)\ (\forall x)\ \ \big((x \in C) \Leftrightarrow (x \in A \,\&\, \gamma[x] \,\&\, (x \in C))\big) \qquad \text{(key)}$$

Then abstract to an existential quantifier again. There are two ways to do this. The first way is set-preserving, in which the constant C is kept on the RHS.

(∀*A*) (∃*B*) (∀*x*) $((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x] \,\&\, (x \in C)))$.

The other abstraction considers the whole expression and gives the Pauline reading:

(∀*A*) (∃*B*) (∀*x*) $((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x] \,\&\, (x \in B)))$.

Propositional logic also accomodates the exception switch '&|'. (**Appendix B.3**.)

Q.E.D.

The lemma shows that *there is no necessity* to reject the Pauline interpretation.

This paper now faces a choice of composition. The "(key)" relation above can be used to copy the analysis on the singleton, now for the general situation, with proper choice of γ and *C.* However, as it has already been shown that ZFC is inconsistent, there is little advantage in such copying of the argument. It is more interesting to investigate what consequences there would be for the Pauline interpretation and for a development of a ZFC-PV. We proceed on that second course. Above proof generates the following consequence.

Corollary 3.2. The set created by the Pauline reading is not necessarily equal to the set created by the Cantorian reading.

Proof: The proof of lemma 3.2 used the method of eliminating the existential quantifier by subsitution of a constant set *C* = *C*[*A*, γ]. Thus the constant set *C* that satisfies the Cantorian reading also satisfies a Pauline reading. This conclusion is not affected by the later step of abstraction to (∃*B*). However, abstraction over the whole expression may introduce new solutions. Q.E.D.

Comments:

(1) The above holds for any set and expression, not just the paradoxical ones.

(2) The inconsistency of ZFC in Section 2 was derived under the Cantorian reading. The Cantorian reading allows Ψ = Φ and subsequent Theorems 2.5 and 2.7. The inconsistency does *not* depend upon the Pauline reading but upon direct propositional logic.

(3) The tautologies in the proof for Theorem 2.5 apply always. Still, Section 2 gives an example with Table 2 for Ψ and Table 3 for Φ. While that Section has Ψ = Φ, there also arises the idea to differ. While it was not clear there under what circumstances an independent application of Φ was allowed, this clarity has now been given. The Cantorian interpretation always generates a Pauline form for the same sets (Section 2 with Ψ = Φ). There might be a difference however in the normal situation, when one would allow existential abstraction from a proposition too.

(4) It might be useful - e.g. for the expansion to '&|' - to write out some relations for the paradoxical Φ and Ψ. Given the deduction in 3.2 we find for Ψ:

$$((x \in \Psi) \Leftrightarrow (x \in A \,\&\, (x \notin f[x]))) \;\Rightarrow\; ((x \in \Psi) \Leftrightarrow (x \in A \,\&\, (x \notin f[x]) \,\&\, (x \in \Psi)))$$

Given that the antecedens holds (∀*A*) (∀*x*), also the consequence holds (∀*A*) (∀*x*).

For Φ there is the consequence term only:

(∀*A*) (∀*x*) $((x \in \Phi) \Leftrightarrow (x \in A \,\&\, (x \notin f[x]) \,\&\, (x \in \Phi)))$

Thus we have the same expression for both now:

For *K* = Φ, Ψ: (∀*A*) (∀*x*) $((x \in K) \Leftrightarrow (x \in A \,\&\, (x \notin f[x]) \,\&\, (x \in K)))$

With only the latter information it is doubtful whether ZFC is strong enough to derive whether these in ZFC are just different names for the same set Φ = Ψ or not. However, lemma 3.2 and corollary 3.2 make it certain, via another route, that there is ambiguity indeed.

The following question then becomes more acute.

3.3. What is the difference between Ψ and Φ?

The deduction in Section 2 poses a challenge to ZFC. Sets R and S in the Introduction were in naive set theory, so it has relatively little meaning - for now - to ask about the difference between R and S. However, Ψ and Φ belong to ZFC - see 3.1 & 3.2 and **Appendix C** - and thus the question is (more) meaningful. Users of ZFC will have a hard time trying to clarify:

(a) that the Pauline consistency condition should have no effect,

(b) but actually can have an effect.

I have considered this question only to some limited extent since I have no vested interest in ZFC. I leave it to users of ZFC to clarify this issue. The following are useful clarifications based upon the little that I could do.

(A) My solution of this issue is that Ψ is badly defined and that Φ is well-defined. I am interested in a convincing argument to the contrary but haven't seen it yet. Note that Ψ at first seems to work for finite sets, see for example the singleton in Section 2. However, there we identified that Ψ does not cover the intended interpretation and causes inconsistency. While Φ allows more solutions and thus clearly works as a variable, Ψ must be a variable too but gives the suggestion as if it were a single solution and a constant.

(B) The lemma and corollary in 3.2 allow us to pose the question about Ψ and Φ a bit more acutely. Let us neglect the inconsistency in the singleton and focus (more didactically) on why it may not be seen.

A Cantorian is likely to insists on $\Psi = \Phi$ for Table 1 but on $\Phi \neq \Psi$ for Table 3. Then Hilton's Hotel allows the construction of a function f such that a Cantorian may sooner accept that both contradictory conditions hold. Let us rework a function at the margin (whence Δ's).

Theorem 3.3 (removing impossibility). Let A be denumerable, $P[A]$ the power set. (i) For any arbitrary non-trivial $h: A \to P[A]$ there are a $f: A \to P[A]$ and a $\varphi \in A$ with $f[\varphi] = \Phi = \Phi[A] = \{x \in A \mid x \notin f[x] \ \ \& \mid x \in \Phi\}$. (When Φ is written without brackets: $\Phi = \Phi[A]$.) (ii) The direct test has $\varphi \notin \Phi$ without direct contradiction.

Proof:

(i) When there is a $\varphi \in A$ such that $h[\varphi] = \Phi = \{x \in A \mid x \notin h[x] \ \ \& \mid x \in \Phi\}$ then the proof ends with $f = h$. (To block repeat application of the proof.)

Otherwise: consider an ordering of $A = \{a[1], ...\}$ and let be $\varphi = a[1]$.

Let $B = A \setminus \{ \varphi \}$ and $g: B \to P[A]$ as in Hilbert's Hotel: $g[a[n]] = h[a[n-1]]$ for $n > 1$.

$\Phi^*[B] = \{x \in B \mid x \notin g[x] \ \ \& \mid x \in \Phi^*[B]\}$.

(PM 1. $\Phi^*[B]$ exists in ZFC, see Theorem 1.3.A: (a) Note that g's domain is B and its range is $P[A]$. Note that g can be regarded as a subset of B x $P[A]$. Then g exists because of the Axiom of Pairing. (b) Because of $P[A]$, instead of $P[B]$, there is a different $\Phi^*[B]$. This $\Phi^*[B]$ exists because of the Axiom of Separation applied to the part without '&|', and then applying Lemma 3.2. N.B. There are the set-preserving $\Phi 1^*[B]$ and the free $\Phi 2^*[B]$ versions.)

Define $f: A \to P[A]$ as:

(a) $x \in B$: $f[x] = g[x]$. Rewrite: $\Phi^*[B] = \{x \in B \mid x \notin f[x] \ \ \& \mid x \in \Phi^*[B]\}$

(b) $x = \varphi$: $f[\varphi] = \Phi^*[B]$

We need to prove that $\Phi = \Phi[A] = \Phi^*[B]$.

Since $(\varphi \notin B)$ also $(\varphi \notin \Phi^*[B])$.

Define: $M = \{x = \varphi \mid x \notin f[x] \ \& (x \in \Phi^*[B])\}$ = Ø. (Margin, '&', not '&|')

(PM 2. M = Ø is in ZFC. Or, Separation of {φ} with $\gamma^*[x] = (x \notin f[x]) \& (x \in \Phi^*[B])$, in which $\Phi^*[B]$ is not a free variable but a constant given from the above.)

A union with M allows a rewrite from B to A:

$$\Phi^*[B] = \Phi^*[B] \cup M = \{x \in A \mid x \notin f[x] \ \&| \ x \in \Phi^*[B]\}$$ (Substeps)

Rewrite: $K = \{x \in A \mid x \notin f[x] \ \&| \ x \in K\}$ for $K = \Phi^*[B]$.

This is using another name for Φ, so that $K = \Phi^*[B] = \Phi$. (Substep 3)

(ii) The direct test on consistency is:

$$(\varphi \in \Phi) \Leftrightarrow (\varphi \notin f[\varphi] \ \& \ \varphi \in \Phi)$$

Whence it follows without direct contradiction that $\varphi \notin \Phi$.

Q.E.D.

Substeps:

(1) $(\{x \in A \mid \gamma[x]\} \cup \{x \in B \mid \gamma[x]\}) \Leftrightarrow \{x \in A \cup B \mid \gamma[x]\}$

For the elements of A and B, also allowing for infinity, LHS and RHS mean, also when some or all subsets reduce to the empty set:

$\{a_1 \mid \gamma[a_1]\} \cup ... \cup \{b_1 \mid \gamma[b_1]\} \cup ...$

(2) Joining the two sets into K and directly introducing the '&|' notation may be tricky. However, work in the opposite direction. Test $(\varphi \in K) \Leftrightarrow ((\varphi \notin f[\varphi]) \ \& (\varphi \in \Phi^*[B]))$ since the contradiction on the RHS takes &|-precedence. The RHS *falsum* gives non-membership. Then $(x \in A)$ reduces to $(x \in B)$ so that the original definition of $\Phi^*[B]$ is retrieved.

(3) These are just names. If one allows the rewrite to K then this label K can also be used in the Theorem itself rather than Φ. Who requires a difficult route first defines $\Psi[A]$ then derives $\Phi[A]$ and then embarks on proving $\Phi[A] = K$, using (2). $\Psi[A]$ gives q for Table 4.

Comments:

(1) Let $Y = \{x \in A \mid x \notin h[x]\}$. With Theorem 1.3.B there is no x in A such that $h[x] = Y$. Thus $h[\varphi]$ is unequal to Y and this would not give a bijection. The theorem shows that it is not precluded that one can construct a bijection (by abstraction) by some f however.

(2) Some constructive methods still allow for fixed points (Brouwer). The proposition *"This proposition is true"* is self-referential without much problem.

Hodges (1998) also reviewed criticism w.r.t. Cantor's Theorem but confirmed in a personal communication August 10 2012 that he allows me to quote from (though w.r.t. the different paper CCPO-PCWA): "You are coming at Cantor's proof from a constructivist point of view. That's something that I didn't consider in my paper, because all of the critics that I was reviewing there seemed to be attacking Cantor from the point of view of classical mathematics; I don't think they knew about constructivist approaches. Since then some other people have written to me with constructivist criticisms of Cantor. There is not much I can say in general about this kind of approach, because constructivist mathematicians don't always agree with each other about what is constructivist and what isn't."

Theorem 3.3 can be used for the question how Φ and Ψ compare.

Lemma 3.3. Let A be denumerable, $P[A]$ the power set. For any arbitrary non-trivial $h: A \to P[A]$ there are a f: $A \to P[A]$ and a $\varphi \in A$ with $f[\varphi] = \Phi$ such that $\Phi \neq \Psi$.

Proof: Theorem 3.3 generates f with $f[\varphi] = \Phi = \{x \in A \mid x \notin f[x] \ \ \&| \ x \in \Phi\}$.

Theorem 1.3.B generates $\Psi[f] = \{x \in A \mid x \notin f[x]\}$ such that for all $\alpha \in A$ it holds that $\Psi \neq f[\alpha]$. Thus also $\Psi \neq f[\varphi] = \Phi$. Q.E.D.

Comments:

(1) Another way: Find $\varphi \notin f[\varphi]$, thus $\varphi \notin \Phi$ and $\varphi \in \Psi$, so that $\Phi \neq \Psi$.

(2) There is inconsistency if also $\Phi = \Psi$. Section 2 already derived inconsistency for the singleton. Who has doubts on the singleton may now check on denumerability.

Corollary 3.3. ZFC is inconsistent for the countable infinite (denumerability).

Proof:

(i) The Cantorian reading generates the possibility of a Pauline reading, see Lemma 3.2. Select the set-preserving case such that $\Psi = \Phi$.

(ii) Do Theorem 3.3 for the latter case. Thus $\Phi 1^*[B] = \Psi^*[B]$ in Pauline format. The Pauline format is important to make this work: to create $M = \varnothing$ and then have the union with $\Phi 1^*[B] = \Psi^*[B]$. Because only $M = \varnothing$ is included, the result must be equal to $\Psi[A]$.

(iii) While $\Psi = \Phi$, lemma 3.3 shows that also $\Phi \neq \Psi$.

Q.E.D.

Comment:

That ZFC is inconsistent is not the major insight in this paper. It are the considerations that count. It seems likely that **Appendix D.4** is the major insight.

3.4. Amendments to the Axiom of Separation in ZFC

To meet the challenge in 3.3 we would require the PV-condition in general.

Possibility 3.4.1: Amendment by Paul of Venice to the Axiom of Separation:

$$(\forall A)\ (\exists B)\ (\forall x)\ ((x \in B) \ \Leftrightarrow \ ((x \in A)\ \&\ \gamma[x]\ \&|\ (x \in B))) \qquad \text{(SEP-PV)}$$

Lemma 3.2 proves that SEP implies SEP-PV, but there arises a new system when SEP is replaced by SEP-PV. In this case, Ψ is no longer possible, the proof for Cantor's theorem collapses, and question 3.3 disappears since Ψ becomes ill-formed and nonsensical. My suggestion is to call this the *neat* solution, and use the abbreviation **ZFC-PV**.

SEP-PV might also support a confusion that $\gamma[x]$ generates a unique B, whence SOL-PV in Section 1.4 might be even more preferable. However, as shown, application of SEP-PV generates multiple solutions, see Table 3, and hence B clearly is a variable.

Another possibility is to move from ZFC closer to naive set theory, discard the axiom of separation, and adopt an axiom that allows greater freedom to create sets from formulas.

Possibility 3.4.2: Discard the separation axiom and have extensionality of formula's, a.k.a. comprehension:

$$(\forall \varphi)\ (\exists B)\ (\forall x)\ ((x \in B) \ \Leftrightarrow \ (\gamma[x]\ \&|\ (x \in B))) \qquad \text{(EFC-PV)}$$

This axiom protects against Russell's paradox and destroys the standard proof of Cantor's theorem. This resulting system might be called ZFC-SEP+PV.

The Axiom of Regularity (REG) forbids that sets are member of themselves. Instead, it is useful to be able to speak about the set of all sets. Though it is another discussion, my suggestion is to drop this axiom too, then to call this the 'basic' solution, and use the abbreviation BST (basic set theory), thus **BST** = ZFC-SEP+PV-REG. I would also propose a rule that the PV-condition could be dropped in particular applications if it could be shown to be superfluous. However, for paradoxical γ[*x*] it would not be superfluous.

I am not aware of a contradiction yet. I have not looked intensively for such a contradiction, since my presumption is that others are better versed in (axiomatic) set theory and that the problem only is that those authors aren't aware of the potential relevance of the consistency condition by Paul of Venice. A question for historians is: Zermelo (1871-1953) and Fraenkel (1891-1965) might have embraced the Paul of Venice's condition if they had been aware of it.

4. Conclusion

1. ZFC is inconsistent. Useful alternatives are in ZFC-PV or BST (Section 3.4).

2. Users of ZFC who do not accept this should give an answer to Sections 2 and 3, and clarify why they accept Ψ and not Φ that has a better definition of a well-defined set. Theorems 2.5 and 2.7, lemma 3.2 and theorem 3.3 would require deproof.

3. If one holds that ZFC is consistent, against all logic, then one also accepts the construction of a 'proof' for 'Cantor's Impression' that generates the transfinites - and the curious "continuum hypothesis" - which makes one wonder what this system is a model for. We can agree with Cantor that the essence of mathematics lies in its freedom, but the freedom to create nonsense somehow would no longer be mathematics proper. See **Appendix D** and its conclusion.

4. It becomes feasible to speak again about the *'set of all sets'.* This has the advantage that we do not need to distinguish (i) sets versus classes, (ii) *all* versus *any*.

5. The prime importance of this discussion lies in education, see Colignatus (2011). Mathematics education should respect that education itself is an empirical issue. In teaching, there is the logic that students can grasp and the idea to challenge them with more; and there is the wish for good history and and still not burden students with the confusions of the past. My suggestion is that Cantor's transfinites can hardly be grasped, are not challenging, and are burdening rather than enlightening. CCPO-PCWA clarifies that highschool education and matricola for non-math majors could be served well with a theory of the infinite that consistently develops both the natural and real numbers, without requiring more than denumerability ($\aleph \sim \Re$), using the notion of *bijection by abstraction*. See Colignatus (2015af) for a discussion on abstraction. A major problem in schools would be when mathematics teachers think that 'Cantor's Theorem' and its transfinites would be a great result and that they would feel frustrated when they would not be in a position to explain it properly - while such frustration would only be based upon a mirage and still show up in behaviour.

Acknowledgements

Let me repeat my gratitude stated in the other paper CCPO-PCWA. For this paper, I thank Richard Gill (Leiden) for various discussions, and Klaas Pieter Hart (Delft) over 2011-2015 and Bas Edixhoven (Leiden) in 2014 for some comments and for causing me to look closer at ZFC. Hart and Edixhoven apparently have missed the full argument of this paper and take the Cantorian position in **Appendix C**. I am sorry to have to report a breach in scientific integrity, see **Appendix E** and Colignatus (2015e). All errors remain mine.

Appendix A: Versions of ALOE

The following comments are relevant for accurate reference.

(1) Colignatus (1981, 2007, 2011) (**ALOE**) existed first unpublished in 1981 as *In memoriam Philetas of Cos,* then in 2007 rebaptised and self-published. It was both retyped and programmed in the computer-algebra environment of *Mathematica* to allow ease of use of three-valued logic. In 2011 it was marginally adapted with a new version of *Mathematica.* At that moment it could also refer to a new rejection of Cantor's particular argument for the natural and real numbers, using the notion of *bijection by abstraction* - in 2011 still called *bijection in the limit* but now developed in Colignatus (2012, 2013), and see Colignatus (2015af) on abstraction.

(2) Gill (2008) reviewed the 1st edition of ALOE of 2007. That edition refers to Cantor's standard set-theoretic argument and rejects it.

(3) Gill (2008) did not review the 2nd edition of ALOE of 2011. That edition also refers to Cantor's original argument on the natural and real numbers in particular. That edition of ALOE mentions the suggestion that $\aleph \sim \Re$. The discussion itself is not in ALOE but is now in Colignatus (2012, 2013) (CCPO-PCWA), using the notion of *bijection by abstraction.*

(4) ALOE is a book on logic and not a book on set theory. It presents the standard notions of naive set theory (membership, intersection, union) and the standard axioms for first order predicate logic that of course are relevant for set theory. But I have always felt that discussing *axiomatic* set theory (with ZFC) was beyond the scope of the book and my actual interest and developed expertise. This present paper is in my sentiment rather exploratory.

Appendix B: Support on the Introduction

B.1. History and dynamic if-switch

Aristotle gave the first formalisation of the notions of *none*, *some* and *all*, of which an origin can be found in the Greek language. This developed into modern set theory, in which the notion of a *set* provides for the *all*. There is a parallel between constants in propositional logic and set theory: ***and*** giving *intersection*, ***or*** giving *union,* ***implication*** giving *subset.* Still, different axioms give different systems. A common contrast is between the formal ZFC system (from Zermelo, Fraenkel and the Axiom of Choice) and *naive set theory* (not quite defined, but perhaps Frege's system, and not to be confused with Halmos's verbal description of ZFC). There is a plethora - perhaps an infinity - of models for properties of sets.

In naive set theory, Russell's set is $R = \{x \mid x \notin x\}$. Subsequently $R \in R \Leftrightarrow R \notin R$ and naive set theory collapses. Russell's problem was a blow to Frege's system, and researchers spoke about a crisis in the foundations of logic and mathematics. The idea of a crisis was eventually put to rest by the ZFC system. A consequence of ZFC is a *'theory of types',* so that a set cannot be member of itself, and with the impossibility of a *'set of all sets'.*

Define however $S = \{x \mid (x \notin x) \,\&\, (\mathrm{If}\,(x = S)\,\mathrm{then}\,(x \in S))\}$ i.e. with the small consistency condition inspired by the discussion by Bochenski (1956, 1970:250) of Paulus Venetus or Paul of Venice (1369-1429). The *If*-switch gives a dynamic process of going through the steps, and it is not a mere static implication. Without contradiction we find $S \notin S$.

The consistency condition with the exception switch was presented in Colignatus (1981, 2007, 2011:129) *"A Logic of Exceptions"* (**ALOE**).

It is not clear what Russell's set would be, since it is inconsistent; but who wants to work sensibly with a related notion can use S without problem.

B.2. Some aspects of the exception switch

PM 1. The dynamic *If*-switch may be replaced by static S = $\{x \mid (x \notin x) \ \& \ ((x = S) \Rightarrow (x \in S))\}$ but then the truthtable is a bit more involved.

PM 2. Obviously S = $\{x \neq S \mid x \notin x \}$ has the same effect, but this has the suggestion of choice, while the point is that one must show that the property $x \neq S$ is necessary.

PM 3. In some texts like the Introduction above I have used the shorthand form S = $\{x \mid x \notin x \ \& \ x \in S\}$, as shorthand only. This allows students an introductory focus on S. Experts however do not regard themselves as students who need education; they quickly recognise that this shorthand form causes infinite regress when $x \neq S$, and then they put this analysis aside, disappointed that it contains such an elementary confusion. However, the shorthand only indicates the intuition by Paul of Venice on the Liar paradox, that must be developed into modern consistency for sets. It is rather curious that this intuition doesn't inspire the experts on set theory.

PM 4. See **Appendix D.3** on *ad hoc* solutions for Cantorian sets.

B.3. A shorthand notation with asymmetric '*p* &| *q*'

The use of a shorthand form remains useful, and thus I propose the following notation.

Notation: V = $\{x \mid f[x] \ \&| \ x \in V\}$, with non-symmetric '*p* &| *q*', stands for the longer V = $\{x \mid f[x]$ unless $(f[x] \ \& \ x \in V)$ is contradictory (also formally, preventing infinite regress)}. Alternatively V = $\{x \mid \mathrm{If}(f[x] \ \& \ x \in V) \Leftrightarrow \mathit{falsum}$ then falsum else $f[x]\}$ in which the first test can be formal again without infinite regress. In static logic this reduces to V = $\{x \mid f[x] \ \& \ x \in V\}$ but the idea is the dynamic switch, in which it is tested first whether the *Unless*-condition reduces to a falsehood, formally without infinite regress, and if not, then the unprotected original rule *f*[*x*] is applied.

Also: V = $\{\backslash x \mid f[x] /\}$ means V = $\{x \mid f[x] \ \&| \ x \in V\}$.

Example: In the above we could write S = $\{\backslash x \mid (x \notin x) /\}$ - and compare this with R.

B.4. Relation to three-valued logic

There is no reason for a crisis in the foundations of logic and mathematics and there is no need for a theory of types - though you can use them if needed.

Observe: (1) A theory of types - like in ZFC - forbids the set of all sets while it is a useful concept. (2) A theory of types has R in the category *'may not be formed'* and thus implies a *third category* next to truth and falsehood. It would be illogical to reject that third category. It is logical instead to generalize that third category into the general notion of 'nonsense'. This gives a three-valued logic with values *true, false, nonsense.* Thus we do well to formally develop a three-valued logic to determine that R is nonsense - though it has meaning (that allows us to see that it is nonsense). It remains an issue that three-valued logic is not without its paradoxes, but ALOE holds that these can be solved too.

This paper uses *'well-defined'* rather than *'well-formed'.* The context of this paper is ALOE that presents three-valued logic, to the effect that logic allows to determine whether an expression reduces to nonsense. This allows leisure on form, so that the Russell set can be said to be of acceptable form - so that it is meaningful for deductive steps - but it turns out to be nonsense and thus not-well-defined. In the same way the Cantorian set would not be rejected merely because of form, i.e. in three-valued logic. In two-valued logic, both Russell and Cantor do not satisfy criteria on well-formed-ness.

For formalisation of an alternative to ZFC there are at least two approaches. One approach is to forbid the formation of R by always requiring the Paul of Venice consistency condition. (This is ZFC-PV.) Alternatively we can allow that R is formally acceptable: then we need a three-valued logic to determine that R is nonsense. It has meaning, that allows us to see that it is nonsense. (This is BST.)

Appendix C: Pauline versus Cantorian ZFC

Section 2 shows that ZFC is inconsistent, using only the singleton. The Pauline interpretation is not required, since the tautology uses only propositional logic. The following applies when one does not know about Section 2 or rejects its deduction.

This paper discusses the choice of various possible axiomatic systems for set theory. The chosen system defines what is well-defined according to that system. ZFC provides for ZFC-sets. BST (to be developed) provides for BST-sets. A common idea is that developers of ZFC have succeeded in capturing the notion of well-defined-ness best. This assumption is often not mentioned in proofs (e.g. **Appendix D.2**, the *addendum*). This paper challenges that idea. It may be that ZFC-PV or BST capture it better.

Let us now consider ***Pauline*** versus ***Cantorian*** approaches to ZFC. The following arises when we look at these approaches from the angles of didactics or research strategy.

Section 3.2 in the paper defines the Pauline versus Cantorian readings of the Axiom of Separation. Lemma 3.2 shows that the Cantorian reading implies the possibility of the Pauline one. We take account of two cases:

(α) Lemma 3.2 is known and accepted. What are the consequences ? (α1) Does one hold that the Pauline reading does not affect ZFC ? Or is some effect accepted as well, and which one ? (α2) Does one also accept Theorem 3.3 ? (α3) Does one accept the *bijection by abstraction* ?

(β) Lemma 3.2 is not known or not accepted. In this case the different positions are regarded as deriving from a 'difference in personal view' only. How does this affect the perception of the challenge to ZFC ? The Pauline view will be ***tolerant,*** the Cantorian view will be ***defensive.*** What are the reasons for tolerance or defence ?

C.1. The Pauline cq. tolerant interpretation

Section 3.2 presents the view that the condition "B is not free in $\gamma[x]$" is satisfied when B is bound by the existential quantifier. This is a **Pauline** interpretation and approach. It causes that Ψ and Φ belong to ZFC, so that the question about their differences (and the anomaly) can be discussed *within* ZFC. This paper shows that this leads to a rejection of ZFC as a proper axiomatic development for set theory.

This Pauline approach acknowledges the existence of the condition "B is not free in $\gamma[x]$" since this highlights the challenge to ZFC. However, ZFC is ambiguous. There are versions available without this condition. We rely on Coplakova et al. (2011:145), but we had to insert the by-line in Section 1.2. A notable example is also Weisstein (2015) of MathWorld.

3. Axiom of Subsets: If φ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : \varphi(u, p)\}$ that contains all those $u \in X$ that have the property φ. (also called Axiom of Separation or Axiom of Comprehension)

$$\forall X \, \forall p \, \exists Y \, \forall u \, (u \in Y \equiv (u \in X \wedge \varphi(u, p))).$$

In Hart (2013:29) we find the following formulation that can be judged to be at least *ambiguous*. Its formulation allows the *Pauline* interpretation, i.e. that the test on the free variables of $\gamma[x]$ happens under the existential quantifier. (i) *B* can be regarded as a given or constant, and not a free variable for $\gamma[x]$, and then one finds $y = B$. (ii) Is it really precluded that one starts out with an independent *y* but then deduces that $y = w_1$ would be a solution ? It would be an additional assumption ('clarification of how to read the axiom') to adopt the Cantorian defence below.

AXIOMA 3. HET AFSCHEIDINGSSCHEMA. Als ϕ een (welgevormde) formule is met zijn vrije variabelen in de rij x, z, w_1, $\ldots$, w_n (allen ongelijk aan y) dan bestaat bij elke verzameling x een verzameling die bestaat uit precies die elementen van x die aan ϕ voldoen:

$$(\forall x)(\forall w_1)\cdots(\forall w_n)(\exists y)(\forall z)\big(z \in y \leftrightarrow (z \in x \wedge \phi)\big)$$

C.2. The Cantorian cq. defensive interpretation

A **Cantorian** approach to ZFC would be to maintain that authors in the world are free, but that only the interpretation of ZFC is acceptable that blocks Φ . This approach is to reject the Pauline interpretation, and deny the challenge.

The defence is: $\gamma[x]$ *originates as an independent expression. It is **put into** and is not **lifted out** from the Axiom of Separation and its existential quantifier.*

The Cantorian view remains: the condition "*B* is not free in $\gamma[x]$" allows the creation of Ψ but blocks Φ . This also presumes that Φ cannot be a constant. (Some systems allow that 'variable' might indicate symbols: constants and proper variables.)

This Cantorian approach seems to have the appeal of solidity, i.e. that ZFC exists now for some time, that some researchers find fransfinites attractive and a work of art, and that the view might be maintained even in the face of the Paul of Venice challenge. But it comes at the price of some questions that are not answered except by dogma.

C.3. Questions for the Cantorian interpretation

(1) It does not matter if something is "put into" or "lifted out" while the relevant issue concerns the scope of the existential quantifier. Yes, there is predicate logic that would allow the creation of $\gamma[x]$. But the idea is that set theory harnesses predicate logic.

(1a) A defence is that $\gamma[x]$ is not lifted out from the formula, but already existed as an independent expression, say as a predicate in predicate logic. This merely shifts the problem to predicate logic, and without looking into that problem there. One idea for set theory was to harness predicate logic, but if one drops this idea then how is one to establish well-definedness for predicates ?

(1b) To look at $\gamma[x]$ within the Axiom of Separation is the easiest way to see the scope of the existential quantifier. Why would one lift it out ? Why judge it separately ? Axioms are not posed out of thin air, but we generally look for a reason.

(1c) The transfinites are no reason, when they can be diagnosed as an illusion based upon not-well-defined sets. They don't exist in reality, only in fancy because of ZFC. Sticking to ZFC-Cantorian merely because of the transfinites is begging the question.

(2) Why deny the freedom for researchers to adopt the Pauline interpretation ? Didn't Cantor himself argue that mathematics allows for freedom ? Why could ZFC-PV or BST not be fine axiomatic systems, that deserve mention in an introduction course on set theory ?

(3) Beware of theology and the dispute between Gomarus (predestination) and Arminius (freedom of choice). A former version of PV-RP-CDA-ZFC met with criticism that the Pauline approach was based upon a 'misconception' and 'elementary error', and that the use of the consistency condition 'was not allowed'. While the paper only posed a problem, clearly formulated so that others could understand it, if only they opened their minds to it: but the reader did not see the problem but only error and sin. This reader apparently was so married to ZFC in its Cantorian interpretation, that he did not see alternatives, and he was no longer aware that set theory was about studying axioms for sets and not just ZFC. Instead, in reality, there are alternatives to ZFC, also alternatives in interpreting ZFC. (It is fortunate that there now is lemma 3.2, so that the power of the *rational argument in mathematics* can be relied upon to help resolve an issue. In a way though this is too simple, since arguments (1) and (2) were already convincing mathematically. This also holds for the next points.)

(4) While (2) emphasizes freedom, there is also necessity. While the Cantorian approach blocks the question on Ψ and Φ *within* ZFC-Cantorian, ZFC-defenders must acknowledge that the question exists *within* ZFC-Pauline.

(4a) Thus, if they reject to answer the question *within* ZFC-Pauline, they must answer it *across* variants of ZFC.

(4b) Thus, please, explain why Ψ in ZFC-Cantorian generates transfinites and Φ in ZFC-Pauline does not ? What causes the difference, while the cause is a consistency condition that should have no effect ?

(4c) Rather than neglecting the issue, and getting lost in the illusion of the transfinites, ZFC-defenders might feel obliged to explain why that difference arises. It is not only the consequence of the evaluation $\gamma[x]$, but also the impact of the consistency condition. Is it a *consistency* condition or not ? How can it be that the insertion of consistency can cause the collapse of Cantor's Theorem and the transfinites ?

(4d) While questions (4a) - (4c) allow for that lemma 3.2 and theorem 3.3 are not known, they become more acute when those and their proofs are accepted. One must now answer such questions across systems, say between ZFC and ZFC-PV (Section 3.4).

(5) It would help to establish whether this challenge to ZFC, based upon the Paul of Venice condition, is *new* to researchers of set theory or not. I have no knowledge on this.

(5a) If it is new, then perhaps the tradition of ZFC has been based upon an illusion.

(5b) If it is old, then perhaps it was not properly evaluated in the past.

(5c) There is also the variety of formulations of ZFC that needs explanation, compare e.g. Coplakova et al. (2011), Hart (2013) and Weisstein (2015).

(6) How is it with naive notions like the *'set of all sets'* ? Hart (2015) describes the incongruity of using formal ZFC-sets and informal 'classes'. Would it not be mathematically attractive when these could be brought in line within one consistent system ? The objective is not to limit the freedom in mathematics but to find an adequate system for education, see CCPO-PCWA, Colignatus (2011)(2015cd) and the conclusion above (Section 4).

(7) We cannot persistently neglect Section 2.

C.4. Diagram of views when the propositions in Section 3 are known

Let us assume that the propositions in Section 3 are known and accepted.

Figure 3 shows how the views on ZFC relate to each other and to the overall *'Intended Interpretation'.* The figure indicates only the expressions that are allowed. (In the Cantorian reading of ZFC, the bijection by abstraction *Б* is rejected, and its Ψ[*Б*] disappears in terms of 'content', leaving only its grin. But a non-surjective *f* would generate its Ψ[*f*].) The advantage of this situation & figure is that it shows that the issues can be discussed *across* systems.

Notions are:

(1) The *roman letters* indicate subareas, and the *text labels* their unions.

(2) ZFC-Pauline = *M* U *P* U *R*. ZFC-Cantorian = *M*. ZFC-PV = *R* U *S*.

(3) The Intented Interpretation might be *K* U *M* U *P* U *R* U *S* but perhaps parts drop out. Ideally, ZFC and ZFC-PV are a model for the Intended Interpretation. Then at least *K* = Ø.

(4) Ideally ZFC and ZFC-PV have the same Intended Interpretation, so researchers have to determine which has to give way. (An important case is Δ1 in Section 2.)

(5) Lemma 3.2 is that ZFC-Cantorian implies ZFC-Pauline, and thus the former is a subset of the latter.

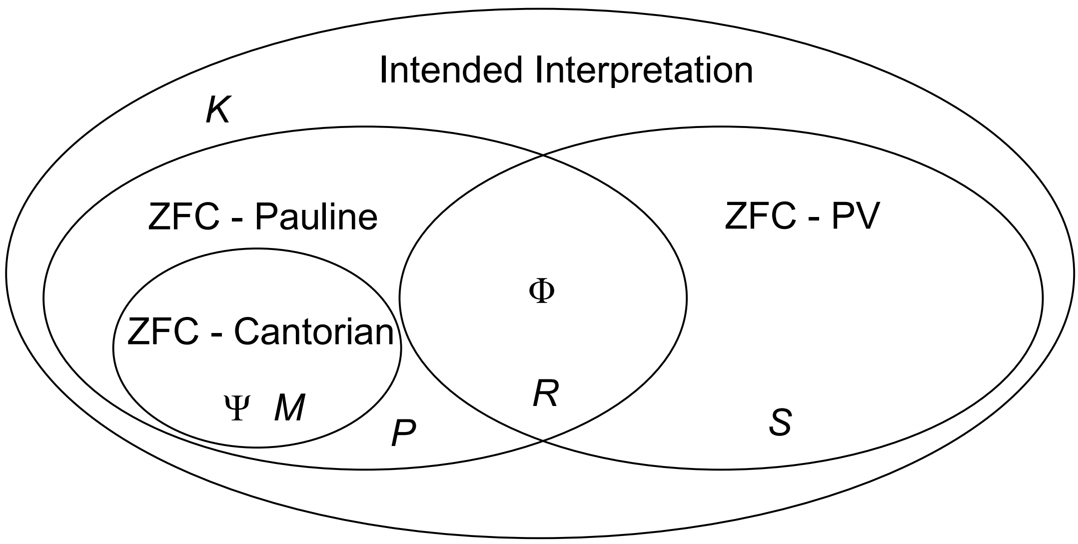
(6) Corollary 3.2. holds that Φ may differ from Ψ. The symbols are distinct in the figure because it looks at the formulas. The sets might disappear e.g. if the bijection by abstraction is rejected. ZFC-Cantorian might hold that on content Φ = Ψ.

(7) Transfinites do not exist out of ZFC-Cantorian (unless there are *really* valid proofs), and thus would not be part of the Intended Interpretation. Thus we should cut out a part of *M* that depends upon Ψ. Thus ZFC-Cantorian has to explain why it includes a part that would not belong to the Intended Interpretation.

(8) ZFC-Cantorian still must explain the difference between Ψ and Φ. One would tend to hold, as in (7), that Φ falls under the Intended Interpretation, so that Ψ has a problem.

(9) This paper only looked at *M* and *R*, and didn't look at other areas. Since ZFC is restrictive, it is not unlikely that *S* = Ø and *P* = Ø. The relevant question is: what can ZFC-Cantorian achieve in *M* that would fall under the Intended Interpretation, but which cannot be achieved by ZFC-PV in *R* ? (The transfinites are excluded because of (7).)

Figure 3: Venn-diagram of ZFC versus ZFC-PV, when the propositions in Section 3 are known and accepted: allowed formulas only (e.g. neglecting that Ψ does not exist when its bijection (by abstraction) doesn't exist)



C.5. Diagram of views when the propositions in Section 3 aren't known

Let us assume that the propositions in Section 3 are not known or accepted. (This is at least the situation before the June 4 & 12 2015 version of PV-RP-CDA-ZFC.)

Figure 4 shows how the views on ZFC relate to each other and to the overall *'Intended Interpretation'.* The figure indicates only the expressions that are allowed. (In the Cantorian reading of ZFC, bijection by abstraction *Б* is rejected, and its Ψ[*Б*] disappears in terms of 'content', leaving only its grin. But a non-surjective *f* would generate its Ψ[*f*].) The advantage of this situation & figure is that it shows that the issues can be discussed *within* ZFC-Pauline.

Notions are:

(1) The *roman letters* indicate subareas, and the *text labels* their unions.

(2) ZFC-Pauline = *R* U *M* (assuming above *P* = Ø). ZFC-Cantorian = *L* U *M*. (Lemma 3.2 found that *L* = Ø, but one does not know or accept this.)

(3) The *Intented Interpretation* might be *K* U *L* U *M* U *R* but perhaps parts drop out. Ideally, ZFC is a model for the Intended Interpretation, and at least *K* = Ø.

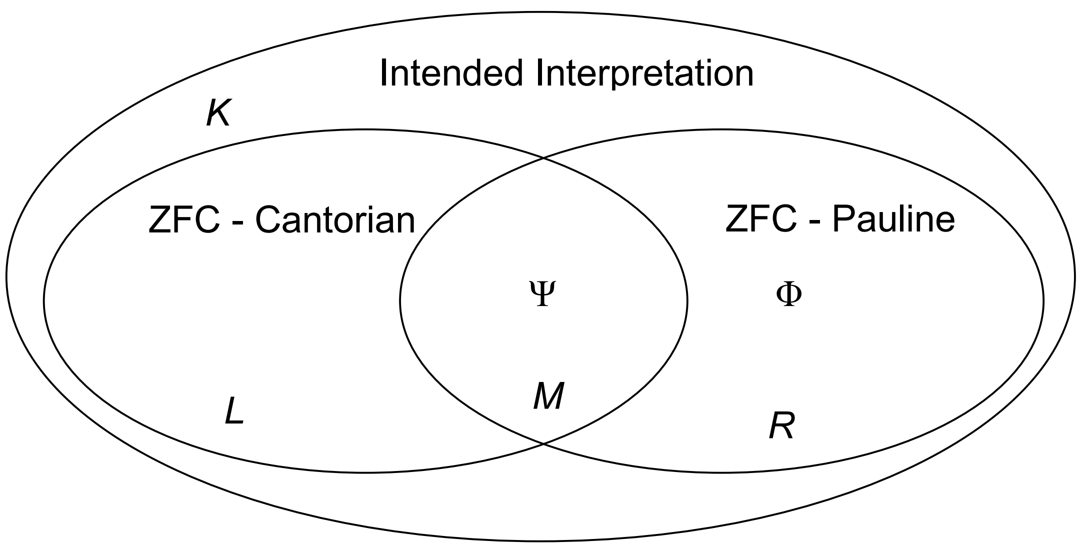
(4) Ideally the two ZFC readings have the same Intended Interpretation, so researchers have to determine which has to give way. (An important case is Δ1 in Section 2.)

(5) Transfinites do not exist out of ZFC-Cantorian (unless there are *really* valid proofs), and thus would not be part of the Intended Interpretation. Thus we should cut out a part of *M* that depends upon Ψ. Thus ZFC-Cantorian has to explain why it includes a part that would not belong to the Intended Interpretation.

(6) ZFC-Cantorian holds that *R* would be nonsense: since lemma 3.2 is unknown. But then ZFC-Cantorian must first explain the difference between between Ψ and Φ. One would tend to hold that Φ falls under the Intended Interpretation, so that Ψ has a problem.

(7) This paper only looked at *M* and *R*, and didn't look at *L*. (Lemma 3.2 found that *L* = Ø.) The relevant question here is: what can ZFC-Cantorian achieve in *L* that would fall under the Intended Interpretation, but which cannot be achieved by ZFC-Pauline ?

Figure 4: Venn-diagram of the Pauline and Cantorian readings of ZFC, when the propositions in Section 3 are not known or accepted: allowed formulas only (e.g. neglecting that Ψ does not exist when its bijection (by abstraction) doesn't exist)



Appendix D: Cantor's theorem on the power set

D.1. Introduction

Sets *A* and *B* have 'the same size' when there is a bijection or one-to-one function between them. Cantor's Theorem holds that a set is always 'smaller' than its power set. For finite sets of arbitrary size - the potential infinite - this can be proven by *numerical succession* - a.k.a. *mathematical induction:* but this is no *induction*, see Colignatus (2015f). The method of numerical succession does not work for the *actual* infinite. Cantor's 'diagonal argument' is supposed to work here.

The reason to look into Cantor's Theorem thus lies *only* in infinity. The infinite is special and requires special care. The 'diagonal argument' uses a self-reference that strongly reminds of Russell's paradox (deconstructed in the Introduction). Hart (2015:42) recalls that this version was actually created by Russell in 1907.

ALOE 2007 showed that Russell's paradox and Cantor's Theorem - common form - were blocked by the condition of Paul of Venice. Colignatus (2012, 2013) (CCPO-PCWA) subsequently developed the notion of ***bijection by abstraction**,* that generates an abstract bijection *Б* between the natural and real numbers, without requiring more than denumerability ($\aleph \sim \mathfrak{R}$). This is a counterexample to Cantor's Theorem in ZFC, and thus causes questions about ZFC. The point may also be rephrased in this manner:

> The logical construction $x \notin f[x]$ and only a single problematic element, in badly understood self-reference, should not be abused to draw conclusions on the infinite. There are ample reasons to look for ways how this can be avoided.

We cannot base mathematical conclusions upon an improper way of expressing our statements. E.g. if our form-conventions allow a substitution of "a" and "oo" so that our conclusions about "man" and "moon" are the same, then we can create art, but not necessarily something that we would want to teach as serious mathematics. Discovering what a good notation for well-defined sets is, has been studied even before the discovery of Russell's paradox.

When a paper challenges a widely accepted theorem then the reader may require a substantial argument and a detailed reconstruction of the proof. Conventionally it would be necessary to go to the source too. For the latter see CCPO-PCWA. Cantor presented his theorem before ZFC existed. This paper focuses on the inconsistency of ZFC. Cantor's Theorem has been delegated to this Appendix.

D.2. Cantor's theorem with bijection (common) or surjection (standard)

The Cantorian set in ZFC plays a key role in the proof of Cantor's theorem on the power set.

Cantor's Theorem (for the power set, Russell's version, with the bijection): Let *A* be a set. There is no bijective function $f : A \to P[A]$.

Proof: Regard an arbitrary set *A*. Let $f: A \to P[A]$ be the hypothetical bijection. Let $\Psi = \{x \in A \mid x \notin f[x]\}$. (*Addendum*, see below.) Clearly Ψ is a subset of *A* and thus there is a $\psi = f^{-1}[\Psi]$ so that $f[\psi] = \Psi$. The question now arises whether $\psi \in \Psi$ itself. We find that $\psi \in \Psi \Leftrightarrow \psi \notin f[\psi] \Leftrightarrow \psi \notin \Psi$ which is a contradiction. Ergo, there is no such f. Q.E.D.

The following is from a matricola course at the universities of Leiden and Delft for students majoring in mathematics. The online syllabus is by Coplakova et al. (2011), and the issue concerns theorem I.4.9, pages 18-19. We translate Dutch into English.

Cantor's Theorem (for the power set) (Coplakova et al. (2011:18), I.4.9): Let *A* be a set. There is no surjective function $f : A \rightarrow P[A]$.

Proof (Coplakova et al. (2011:19), replacing their *B* by Ψ: Assume that there is a surjective function $f : A \rightarrow P[A]$. Now consider the set $\Psi = \{x \in A \mid x \notin f[x]\}$. (*Addendum*, see below.) Since $\Psi \subseteq A$ we also have $\Psi \in P[A]$. Because of the assumption that f is surjective, there is a $\psi \in A$ with $f[\psi] = \Psi$. There are two possibilities: (i) $\psi \in \Psi$ or (ii) $\psi \notin \Psi$.

If (i) then $\psi \in \Psi$. From the definition of Ψ it follows $\psi \notin f[\psi]$ or $\psi \notin \Psi$. Thus (i) gives a contradiction.

If (ii) then we know $\psi \notin \Psi$ and thus also $\psi \notin f[\psi]$. With the definition of Ψ it follows that $\psi \in \Psi$. Thus (ii) gives a contradiction too.

Both cases (i) and (ii) cannot apply, and hence we find a contradiction. Q.E.D.

Comments:

(1) From the contradiction in these proofs, the proper conclusion is not that Cantor's Theorem is proven, but only that it is proven in ZFC. Either Cantor's Theorem is true *or* ZFC doesn't yet provide for well-defined sets.

(2) The *addendum* is: This proof relies on existence Theorem 1.3.A and on the notion that ZFC provides for well-defined sets anyway. Given this addendum, it now should be clearer that if ZFC allows a paradoxical construct then one may feel that ZFC needs amendment.

(3) Finding an example in reality for a transfinite object (other than the real numbers) would be sufficient but might not be necessary. DeLong (1971) explains that an axiomatic system tends to have an 'intended interpretation', so that it is a model for that interpretation. Overall, with an axiomatic system *AS,* the system defines well-defined-ness in its realm. When there is an anomaly *a* for *AS*, so that *AS* & *a* cause a contradiction, then adherents to *AS* will reject *a*, but one must always keep in mind that it is also possible to reject *AS.*

D.3. Rejection of this proof (ALOE)

We might hold that above Ψ is badly defined since it is self-contradictory under the hypothesis of a bijection or surjection. A badly defined 'something' may just be a weird expression and need not represent a true set. A test on this line of reasoning is to insert a small consistency condition, giving us $\Phi = \{x \in A \mid x \notin f[x]$ &| $x \in \Phi\}$. See the Introduction and **Appendix B.3** for the notation on '&|'. Reading it as '&' still gives an idea of the argument. The surjection gives that there is a φ such that $f[\varphi] = \Phi$. Now we get:

$$\varphi \in \Phi \Leftrightarrow (\varphi \notin f[\varphi] \;\&\; \varphi \in \Phi)) \Leftrightarrow (\varphi \notin \Phi \;\&\; \varphi \in \Phi) \Leftrightarrow \textit{falsum.}$$

We find $\varphi \notin \Phi$ without contradiction. This closes the argument against the proof.

Comments:

(1) ALOE:239 holds, for infinite sets: Puristically speaking, the Ψ differs lexically from Φ, with the first expression being nonsensical and the present one consistent. Ψ is part of a lexical description but does not meaningfully refer to a set. Using this, define $\Phi^* = \Phi \cup \{\varphi\}$ and we can express consistently that $\varphi \in \Phi^*$. So the 'proof' can be seen as using a confused mixture of Φ and Φ^*. (The confusion affects infinity and is obscured for finite sets. For example, the $x \notin f[x]$ rule *seems* to work in Section 2 for the singleton. Check what it *does.*)

(2) In Section 2 there is no surjection and thus the above does not quite compare. With only one element, $\alpha \notin \Phi$ implies $\Phi = \emptyset$ and $\Phi^* = A$. We however have freedom to take $f[\alpha] = \Phi$ or $f[\alpha] \neq \Phi$. Instead we have always $f[\alpha] \neq \Psi$, so it is no useful building block for a surjection.

(3) The additional condition in Φ only enhances consistency, but Ψ and Φ still have a different effect. Users of ZFC should explain this, see **Appendix C**.

(4) In writing CCPO-PCWA in 2012 I considered using the more general &|-construction, but still preferred $\Phi = \{(x \in A)$ & $(x \neq f^{1}[\Phi]) \mid x \notin f[x]\ \}$ for the hypothesis of a bijection. Now, looking at the challenge to ZFC, it seems better not to linger in *ad hoc* solutions but to emphasize the general idea. If one feels uncomfortable with the &|-switch then it is useful to know that there may be an *ad hoc* definition for Φ. Another *ad hoc* format is:

$$\Phi = \{x \in A \ \& \ f[x] \neq \Phi \ \mid x \notin f[x]\} \ \bigcup \ \{x \in A \ \& \ f[x] = \Phi \mid x \notin f[x] \ \& \ x \in \Phi\} \ .$$

(5) ALOE deals with logic and inference and thus keeps some distance from number theory and issues of the infinite. Historically, logic developed parallel to geometry and theories of the infinite (Zeno's paradoxes). Aristotle's syllogisms with *none, some* and *all* helped to discuss the infinite. Yet, to develop logic and inference proper, it appeared that ALOE could skip the tricky bits of number theory, non-Euclidean geometry, the development of limits, and Cantor's development of the transfinite. Though it is close to impossible to discuss logic without mentioning the subject matter that logic is applied to, ALOE originally kept and keeps some distance from those subjects themselves. But, if logic uses the notion of *all,* it seems fair to ask whether there are limitations to the use of this *all*. Thus it is explained why ALOE 2007 said something about Cantor's Theorem and why this present paper came about. Later CCPO-PCWA developed the notion of *bijection by abstraction* which is a counterexample to the idea that there would be no bijection between infinite sets.

For orientation on the infinite, apart from DeLong (1971) and the syllabus Coplakova et al. (2011), there are popular science Wallace (2003) and its review by Harris (2008).

(6) It may be added here that I did not enjoy the idea of rejecting the proof for Cantor's Theorem, that I originally accepted in 1981-2006, and that has such an acceptance in mathematics since Hilbert, even in constructivism. But, when one studies logic, one learns to respect necessity. See **Appendix E** for more on the genesis of this paper.

D.4. The abuse of realism versus well-defined-ness

Section 2.6 gives the crucial observations on realism versus nominalism.

For a function f from singleton A to $P[A]$ we find that $f[\alpha] = \emptyset$ or $f[\alpha] = A$. There is no doubt that such a function can exist, or that subsets Ø or *A* exist. Whether $f[\alpha] = \Psi$ or $f[\alpha] \neq \Psi$ is merely a derivative property, it is not material to the existence of f, and it gives no definition for Ø or *A.* When we use the self-referential property of Ψ to say something about the existence of f, Ø or *A,* then this is an *abuse of a derivative and contingent property*. This Ψ does not necessarily have to support a conclusion on existence, even though the conclusion is true, and the reasoning superficially accurate. Perhaps the key point in this paper:

> This paper has paid a lot of attention to the Paul of Venice condition and the creation of Φ, partly in order to show that Ψ cannot really block a bijection in infinite. It is ironic - if that is the proper word - that this analysis is actually superfluous. It is namely sufficient to say that Ψ *is irrelevant anyway*. However, adherents to Cantor's Theorem would not believe Occam.

Consider e.g. a theorem "there cannot be a rational army", and a proof with the Catch 22: *If a pilot is insane then he should not fly; if he applies for a test on insanity then this is proof of sanity. An army that lets insane pilots fly is irrational.* Q.E.D. This abuses a *potential* rule, i.e. no necessary rule. The conclusion seems true for many cases and the reasoning is superficially accurate. This 'proof' makes for a good novel but not for mathematics.

Indeed, for many cases there is no surjection, namely for finite sets. This can be proven by numerical succession. The proof in finite cases by means of the Cantorian set does not use a *necessary* condition, and thus is dubious. The real issue is infinity. The infinite does not turn the Cantorian set into a necessary condition. One should ask: is there really some set in infinity, that can *only* be described by the Cantorian rule, so that we *really need* that rule to say something about a surjection ? There is no proof for that. The Cantorian 'set' remains derivative and contingent, and not something that can be used for such a proof on the supposed absence of a surjection.

The derivative and contingent property however can be used to show that ZFC is inconsistent. ZFC namely allows for the formulation of the Cantorian set, with a self-reference that generates an internal contradiction. This hold for the singleton but also for infinity.

The infinite generates inconsistency of ZFC also in the following manner. In a first reading, the deduction may seem too simple, but there is a logical framework now.

Theorem D.6: ZFC is inconsistent.

Proof: Let A be denumerable, $P[A]$ the power set.

CCPO-PCWA shows that there is a bijection by abstraction. Let this be f: $A \rightarrow P[A]$.

ZFC allows the creation of $\Psi = \{x \in A \mid x \notin f[x]\}$.

Because of the bijection there is a ψ such that $f[\psi] = \Psi$.

The direct check on consistency gives: $(\psi \in \Psi) \Leftrightarrow (\psi \notin \Psi)$. Q.E.D.

For ZFC, the problem however is not really caused by infinity. The real question is how ZFC deals with self-reference.

D.5. Overview of various theorems and the refutations of their proofs

Since this paper refers to various forms of 'Cantor's Theorem' it will be useful to collect them in a table. See also Hart (2015) and its evaluation in Colignatus (2014b, 2015) (PV-RP-CDA-ZFC, version June 12 2015, its appendix B) and Colignatus (2015cde).

Author & date	*Theorem*	*Refutation*
Cantor 1874	Reals are nondenumerable, via intervals	CCPO-PCWA 2012
Cantor 1890/91	Diagonal argument, binary, bijection	CCPO version 2007j
Russell 1907	Power set theorem, using bijection ("common")	ALOE & here D.3 & 4
Coplakova et al. 2011	Power set theorem, using surjection ("standard")	Here, D.3 & 4
Hart 2012	Weakest theorem underlying Cantor's Theorem	Here Theorems 2.5-7

D.6. Semantics of deproving a theorem

There is an issue on terminology w.r.t. the term 'refutation' in the table. A theorem is refuted by a counterexample. For the natural numbers and the reals we can find $\aleph \sim \Re$ via the notion of *'bijection by abstraction' Б*, Colignatus (2012, 2013). This uses constructive methods that some might not agree with, and this is not the topic of the present discussion. What is refuted in this paper is Cantor's 'diagonal argument' in the power set form. Perhaps a better term might be 'deproven', in the sense that the theorem is stripped from its proof and no longer can count as a theorem. It may be that the theorem would still hold, but via a different proof. The refutation of the diagonal argument is done by showing that the proof relies on logically improper constructs so that the proof can be rejected as invalid. Saying that 'the proof is rejected' would be too simple because in this realm of discussion - axiomatics - this might suggest that it is a mere act of volition to reject one of the axioms. One might say that the proof is 'invalidated' but this seems uncommon. A proper phrase is that *'it is refuted that the 'proof' would be valid'.* The latter becomes short: *'the proof is refuted'.*

D.7. Conclusion on Cantor's Theorem

We can restate some earlier conclusions and supplement these from looking at ZFC:

1. The common cq. standard proof for Cantor's Theorem on any set is based upon a badly defined and problematic self-referential construct. The proof and the variants evaporate once the problematic construct is outlawed and a sound construct is used.

2. The theorem is proven for finite sets by means of *numerical succession* ('mathematical induction') but is still unproven for (vaguely defined) infinite sets: that is, this author is not aware of other proofs. We would better speak about *'Cantor's Impression'* or *'Cantor's Supposed Theorem'*. It is not quite a conjecture since Cantor might not have done such a conjecture (without proof) if he would have known about above refutation.

3. The transfinites that are defined by using 'Cantor's Theorem' evaporate with it.

4. The distinction w.r.t. the natural and the real numbers now rests (only) upon the specific interval (Cantor 1874) or specific diagonal argument (Cantor 1890/91) - that differ from the common cq. standard proof. See CCPO-PCWA for the conclusion that these original proofs by Cantor for the natural and real numbers evaporate too, specifically for a convenient *'degree of constructivism'*. See also PV-RP-CDA-ZFC, version June 12, its appendix B, on Hart (2015) on these original proofs.

5. It was Cantor himself who emphasized the freedom in mathematics, but that freedom is limited when alternatives are not mentioned. Even a university course like Coplakova et al. (2011) currently presents matricola students only with 'Cantor's Theorem' without mentioning the alternative analysis in ALOE, potentially seducing some students to waste their lives on transfinites.

Appendix E: A bit more on the genesis of this paper

Colignatus (2013) explains my background and **Appendix A** explains about ALOE.

Theorem 1.3.A is a reformulation of the addendum in **Appendix D.2,** provided by B. Edixhoven, statement in Colignatus (2014a), its appendix D.

Theorem 1.3.B was given by K.P. Hart (TU Delft), 2012, in Colignatus (2015b).

A visit to a restaurant in October 27 2014 and subsequent e-mail exchange with Edixhoven (Leiden), co-author of Coplakova et al. (2011), led to the memos Colignatus (2014ab), and the inspiration to write about ZFC. Originally I asked Edixhoven the question in Section 3.3 on the relation between Ψ and Φ. Edixhoven agreed that the Pauline consistency condition should have no effect, and I asked him to explain that it could have an effect. Since November 2014, see Colignatus (2014ab), I have not received a response even though the question was clear and articulate. Hart (Delft), who has invested deeply into the transfinites, simply rejects that Φ belongs to ZFC, thus neglects the ramifications in **Appendix C** and the logic now put into lemma 3.2. Having seen ZFC more often in the course of these exchanges, I decided on the morning of Wednesday May 27 2015 to provide for the answers myself, and established the singleton case, Table 1 and lemma 3.2 before noon. The rest may be seen as didactics. Overall, advised reading is Colignatus (2015e).

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