The Eigen-complete Difference Ratio of classes of Graphs- Domination, Asymptotes and Area

Paul August Winter* and Samson Ojako Dickson

Abstract

The energy of a graph is related to the sum of ® -electron energy in a molecule represented by a molecular graph and originated by the HMO (Hückel molecular orbital) theory. Advances to this theory have taken place which includes the difference of the energy of graphs and the energy formation difference between and graph and its decomposable parts. Although the complete graph does not have the highest energy of all graphs, it is significant in terms of its easily accessible graph theoretical properties and has a high level of connectivity and robustness, for example. In this paper we introduce a ratio, the eigen-complete difference ratio, involving the difference in energy between the complete graph and any other connected graph G, which allows for the investigation of the effect of energy of G with respect to the complete graph when a large number of vertices are involved. This is referred to as the eigen-complete difference domination effect. This domination effect is greatest negatively (positively), for a strongly regular graph (star graphs with rays of length one), respectively, and zero for the lollipop graph. When this ratio is a function f(n), of the order of a graph, we attach the average degree of G to the Riemann integral to investigate the eigen-complete difference area aspect of classes of graphs. We applied these eigen-complete aspects to complements of classes of graphs.

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*Corresponding author: email: winterp@ukzn.ac.za

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1. Introduction

In this paper graphs G will be on n vertices. We shall adopt the definitions and notation of Harris, Hirst, and Mossinghoff. It is assumed that G is simple, that is, it does not contain loops or parallel edges.

The energy of a graph is the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph in consideration. This quantity is studied in the context of spectral graph theory. In short, for an n -vertex graph G with adjacency matrix A having eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_n$, the energy $E(G)$ is defined as: $E(G) = \sum_{i=1}^{n} |a_i|$

It is related to the sum of ® -electron energy in a molecule represented by a molecular graph. If we know some chemistry, then we might fully appreciate the origin of graph energy. In a private communication, Gutman (see Gutman) claimed that the HMO (Hückel molecular orbital) theory is nowadays superseded by new theories that provide better explanations and which do not make unnecessary assumptions.

Graph energy became a very popular topic of mathematical research; this is evident in the reviews and recent papers.

In the paper "Energy of Graphs" by Brualdi the difference of the energy of two graphs G and H on the same number n of vertices was presented.

Although the complete graph K_n does not have the maximum energy of all graphs (see Haemers), it is a very important and well-studied class of graphs – for example it has a high degree of connectiveness and robustness. Thus one would like to compare its energy with the energy of any other graph G in terms of how close their energies are, and how the energy of G compares with the energy of K_n where a large number of vertices are involved. This energy idea can be translated to that of molecules made up of atoms with bonds, where we map the atoms to vertices and bonds to edges, and the domination effect will allow for the investigation of how other molecular energies compare with that of a molecule with all possible bonds between atoms.

The eigen-complete difference ratio allowed for the investigation of the *domination effect* of the energy of graphs on the energy of the complete graph when a large number of vertices are involved. We found that this domination effect is the greatest negatively (positively) for a strongly regular graph (star graphs with rays of length one), respectively. and is zero for the lollipop graph.

Ratios and graphs

Ratios have been an important aspect of graph theoretical definitions. Examples of ratios are: expanders, (see Alon and Spencer), the central ratio of a graph (see Buckley), eigen-pair ratio of classes of graphs (see Winter and Jessop), Independence and Hall ratios (see Gábor), tree-cover ratio of graphs (see Winter and Adewusi), eigen-energy formation ratio of graphs (see Winter and Sarvate), t-compete sequence ratio (see Winter, Jessop and Adewusi) and the chromatic-cover ratio of graphs (see Winter).

We now introduce the idea of ratio, asymptotes and areas involving energy difference between the complete graph and G, similar to that of Winter and Adewusi, Winter and Jessop, Winter and Sarvate, Winter, Jessop and Adewusi, and Winter.

2. Eigen-complete difference ratio- asymptotes, domination effect and area

Let K_n be the complete graph on n vertices.

Definition 2.1

The difference between the energy of K_n and a connected graph G on the same number of vertices n is given by:

$$\left\langle D_n^G = E(K_n) - E(G)\right.$$

And is called the *eigen-complete difference* associated *with G.*

If the graph G in belongs to a class $\Im$ of graphs of order n, then the *complete-energy difference associated with* $\Im$ is defined as*:*

$$\left\langle D_n^{\Im} = E(K_n) - E(G); G \in \Im.\right.$$

Dividing the complete-energy difference by the energy of K_n will give an "average" of the complete-energy difference with respect to G. This provides motivation for the following definition:

Definition 2.2

The *eigen-complete difference ratio* with respect to $G(\Im)$, respectively, is defined as:

$$Rat\left\langle D_n^G = \frac{E(K_n) - E(G)}{E(K_n)}\right.; Rat\left\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)}\right.; G \in \Im$$

Definition 2.3

If the eigen-complete difference ratio is a function f(n) of the order of $G \in \Im$, then its horizontal asymptote results in the *eigen-complete difference asymptote*:

$$Asymrat\left\langle D_n^{\Im} = \lim_{n \to \infty}\left[\frac{E(K_n) - E(G)}{E(K_n)}\right]\right.; G \in \Im$$

This asymptote allows for the investigation of the effect of the energy of a graph G on the complete graph when a large number of vertices are involved, referred to as the *domination eigen-complete difference effect.*

Definition 2.4

Attaching the average degree of graph G, with m' edges, to the Riemann integral of $Rat\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)}; G \in \Im$ we obtain the *eigen-complete difference area:*

$$Arat\langle D_n^{\Im} = \frac{2m'}{n}\left|\int [\frac{E(K_n) - E(G)}{E(K_n)}]dn\right|;$$ with $Arat\langle D_k^{\Im} = 0$ where k is the smallest order of $G \in \Im$.

The average degree is referred to as the *length* of the area, while the integral part is the *height* of the area.

Lemma

The eigen-complete difference ratio can take on one of the following:

(1) $$EG) < E(K_n) \Rightarrow Rat\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)} = 1 - \frac{E(G)}{E(K_n)} > 0; G \in \Im$$

(2) $$EG) > E(K_n) \Rightarrow Rat\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)} = 1 - \frac{E(G)}{E(K_n)} < 0; G \in \Im$$

(3) $$EG) = E(K_n) \Rightarrow Rat\langle D_n^{\Im} = \frac{E(K_n) - E(G)}{E(K_n)} = 0; G \in \Im$$

3. Examples

3.1 The complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.

The energy of this graph is n and it has $\frac{n^2}{4}$ edges while that of the complete graph is 2n-2 so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(K_{\frac{n}{2},\frac{n}{2}})}{2n-2}\right] = \frac{(2n-2)-n}{2n-2} = \frac{n-2}{2n-2} = \frac{n-2}{2n-2}$$

$$Asymrat\langle D_n^{\Im} = \underset{n\to\infty}{Lim}[\frac{n-2}{2n-2}] = \frac{1}{2}$$

$$Arat\langle D_n^{\Im} = \frac{n}{2}\int[\frac{n-2}{2n-2}]dn = \frac{n}{4}\int\frac{n-2}{n-1}dn = \frac{n}{4}(n-\ln(n-1)+c)$$ with smallest order 2 we have :

$c = 2$

3.2 The star graph $K_{1,n-1}$ with n-1 rays of length1.

The energy of this star graph is $2\sqrt{n-1}$ so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(K_{1,n-1})}{2n-2}\right] = \frac{(2n-2)-2\sqrt{n-1}}{2(n-1)} = 1-\frac{1}{\sqrt{n-1}}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}[1-\frac{1}{\sqrt{n-1}}]=1$$

$$Arat\langle D_n^{\Im} = \frac{2(n-1)}{n}\int[1-\frac{1}{\sqrt{n-1}}]dn = \frac{2(n-1)}{n}[n-2\sqrt{n-1}+c]$$

With smallest star graph on 2 vertices we have:

$c = 0$.

3.3 Star graphs $S_{r,2}$ with r rays of length 2

The energy of this star graph with $r = n-1$ edges is:

$n-3+\sqrt{2}\sqrt{n+1}$ so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(K_{1,n-1})}{2n-2}\right] = \frac{(2n-2)-(n-3+\sqrt{2}\sqrt{n+1})}{2n-2} = \frac{n+1-\sqrt{2}\sqrt{n+1}}{2n-2}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}[\frac{n+1-\sqrt{2}\sqrt{n+1}}{2n-2}] = \frac{1}{2}$$

$$Arat\langle D_n^{\Im} = \frac{2(n-1)}{n}\int[\frac{n+1-\sqrt{2}\sqrt{n+1}}{2n-2}]dn = \frac{2(n-1)}{n}\int[\frac{n-1}{2(n-1)}+\frac{2}{2(n-1)}-\frac{1}{\sqrt{2}}\frac{\sqrt{n+1}}{\sqrt{n-1}}]dn$$

$$= \frac{2(n-1)}{n}[\frac{n}{2}+\ln(n-1)-\frac{1}{\sqrt{2}}.A+c]$$

$$A = \int\frac{\sqrt{n+1}}{\sqrt{n-1}}dn = \text{let } n = u^2+1 \Rightarrow dn = 2udu \Rightarrow A = \int\frac{\sqrt{u^2+2}}{u}2udu\,.$$

Put $u=\sqrt{2}\tan t$ so that $A=4\int \sec^3 t dt = 4[\frac{\sec t\tan t + \ln(\sec t + \tan t)}{2}]$

Thus: $A=2[\frac{\sqrt{n+1}}{\sqrt{2}}\frac{\sqrt{n-1}}{\sqrt{2}}+\ln(\frac{\sqrt{n+1}}{\sqrt{2}}+\frac{\sqrt{n-1}}{\sqrt{2}})]=\sqrt{n^2-1}+2\ln(\frac{\sqrt{n+1}}{\sqrt{2}}+\frac{\sqrt{n-1}}{\sqrt{2}})$

The smallest such star graph is non 3 vertices so that

$$A=\sqrt{8}+2\ln(\frac{\sqrt{10}+\sqrt{8}}{\sqrt{2}}))=\sqrt{8}+2\ln(2+\sqrt{5})$$

Thus:

$$c=2+\sqrt{2}\ln(2+\sqrt{5})-\frac{3}{2}-\ln 2$$

And eigen-complete difference area is:

$$=\frac{2(n-1)}{n}[\frac{n}{2}+\ln(n-1)-\frac{1}{\sqrt{2}}.[\sqrt{n^2-1}+2\ln(\frac{\sqrt{n+1}}{\sqrt{2}}+\frac{\sqrt{n-1}}{\sqrt{2}})]+2+\sqrt{2}\ln(2+\sqrt{5})-\frac{3}{2}-\ln 2]$$

3.4 The line graph of K_n

The line graph $L(K_n)$ of K_n has $p=\frac{n(n-1)}{2}$ vertices and energy $2n^2-6n$ (see Brualdi). The number q of edges is the sum of the square of the degrees minus the number of edges of K_n:

$$q=n\frac{(n-1)^2}{2}-\frac{n(n-1)}{2}=\frac{n(n-1)}{2}[n-1-1]=\frac{n(n-1)(n-2)}{2}$$

$$2n^2-6n=4\frac{n(n-1)}{2}-4n=4p-4n$$

$$n^2-n-2p=0\Rightarrow n=\frac{1\pm\sqrt{1+8p}}{2}=\frac{1+\sqrt{1+8p}}{2}$$

Thus:

$$E(L(K_n)) = 2n^2 - 6n = 4\frac{n(n-1)}{2} - 4n = 4p - 2 - 2\sqrt{1+8p}$$

$$Rat\langle D_p^{\Im} = \left[\frac{E(K_p) - E(L(K_n))}{E(K_p)}\right] = \frac{2p - 2 - 4p + 2 + \sqrt{1+8p}}{2p-2} = \frac{-2p + \sqrt{1+8p}}{2p-2}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}\left[\frac{-2p - \sqrt{1+8p}}{2p-2}\right] = -1$$

$$Arat\langle D_n^{\Im} = \frac{2q}{p}\left|\int \frac{-2p + \sqrt{1+8p}}{2p-2}\,dp\right| = omit\ \ absolute\ \ sign$$

$$n\ \ \frac{2q}{p}\int\left[-\frac{2p-2}{2p-2} + \frac{-2+\sqrt{1+8p}}{2p-2}\right]dp; u^2 = 1+8p \Rightarrow dp = \frac{udu}{4}; p = \frac{u^2-1}{8}$$

$$= \frac{2q}{p}\left[(-p) + \int \frac{(-2+u)}{(\frac{u^2-1}{4} - 2)}\frac{udu}{4}\right] = \frac{2q}{p}\left[(-p) + \int \frac{(-2+u)}{\frac{u^2-9}{4}}\frac{udu}{4}\right] = \frac{2q}{p}\left[(-p) + \int \frac{u^2-2}{u^2-9}du\right]$$

$$= \frac{2q}{p}\left[(-p) + \int du + \int \frac{7}{u^2-9}du\right] = replacing\ \ absolute\ \ sign$$

$$\frac{2q}{p}\left[p - \sqrt{1+8p} - \frac{7}{6}(\ln(\sqrt{1+8p} - 3) + \frac{7}{6}\ln(\sqrt{1+8p} + 3)\right] + c$$

$p = 3$ yields:

$$c = -3 + 5 + \frac{7}{6}(\ln(2) - \frac{7}{6}\ln(8) = 2 + \frac{7}{6}(\ln(2) - \frac{7}{6}\ln(8)$$

So that the eigen-complete area of the line graph of K_n on p vertices is:

$$\frac{2q}{p}\left[p - \sqrt{1+8p} - \frac{7}{6}(\ln(\sqrt{1+8p} - 3) + \frac{7}{6}\ln(\sqrt{1+8p} + 3) + 2 + \frac{7}{6}(\ln(2) - \frac{7}{6}\ln(8)\right]$$

3.5 Strongly regular graphs

Koolen and Moulton have proved that the energy of a graph on n vertices is at most n(1+$\sqrt{n}$)/2, and that equality holds if and only if the graph is strongly regular with parameters (n,(n+$\sqrt{n}$)/2,(n+2$\sqrt{n}$)/4,(n+2$\sqrt{n}$)/4). Such graphs are equivalent to a certain type of Hadamard matrices. Here we survey constructions of these **Hadamard matrices** and the related strongly regular graphs (see Haemers).

Its energy is $\frac{n(1+\sqrt{n})}{2}$ and to find the number of edges m' we use:

$$\sum_{1}^{n} d(v) = 2m' \Rightarrow n\frac{(n+\sqrt{n})}{2} = 2m' \Rightarrow m' = \frac{n(n+\sqrt{n})}{4}$$

Thus $Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(SR(G))}{2n-2}\right] = \frac{(2n-2)-\frac{n(1+\sqrt{n})}{2}}{2n-2} = \frac{3n-n\sqrt{n}-4}{4n-4}$

.

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}[\frac{3n-n\sqrt{n}-4}{4n-4}] = -\infty$$

$$Arat\langle D_n^{\Im} = \frac{2m'}{n}\left|\int[\frac{3n-n\sqrt{n}-4}{4n-4}]dn\right| = \frac{(n+\sqrt{n})}{2}\left|\int[\frac{3n-n\sqrt{n}-4}{4n-4}]dn\right| = removing\ \ absolute$$

$$\frac{(n+\sqrt{n})}{2}\int[\frac{3}{4}(\frac{4n-4}{4n-4})-(\frac{1+n\sqrt{n}}{4n-4})]dn = \frac{(n+\sqrt{n})}{2}[\frac{3}{4}n-\int\frac{1+n\sqrt{n}}{4n-4}dn]$$

Now

$$\int \frac{1+n\sqrt{n}}{4n-4}dn = \frac{1}{4}\ln(n-1)+\frac{1}{4}\int \frac{n\sqrt{n}}{n-1}dn = \ put\ \ n=u^2 : \frac{1}{4}\ln(n-1)+\frac{1}{4}\int \frac{u^2u}{u^2-1}2udu$$

But $\frac{1}{2}\int \frac{u^4}{u^2-1}du = \frac{1}{2}\int u^2 \frac{u^2-1}{u^2-1}+\frac{u^2}{u^2-1}du = \frac{1}{6}u^3+\frac{1}{2}\int \frac{u^2-1}{u^2-1}+\frac{1}{u^2-1}du$

$$=\frac{1}{6}u^3+\frac{1}{2}u+\frac{1}{2}\ln\frac{u-1}{u+1} = \frac{1}{6}n^{\frac{3}{2}}+\frac{1}{2}\sqrt{n}+\frac{1}{2}\ln\frac{\sqrt{n}-1}{\sqrt{n}+1}$$

So the eigen-complete area (without absolute sign) is:

$$\frac{(n+\sqrt{n})}{2}[\frac{3}{4}n-\frac{1}{4}\ln(n-1)-\frac{1}{6}n^{\frac{3}{2}}-\frac{1}{2}\sqrt{n}-\frac{1}{2}\ln\frac{\sqrt{n}-1}{\sqrt{n}+1}+c]$$

The term $n^{\frac{3}{2}}$ dominates for large n, so introducing absolute sign:

$$\frac{(n+\sqrt{n})}{2}[\frac{1}{6}n^{\frac{3}{2}}-\frac{3}{4}n+\frac{1}{4}\ln(n-1)+\frac{1}{2}\sqrt{n}+\frac{1}{2}\ln\frac{\sqrt{n}-1}{\sqrt{n}+1}+c]$$

3.6 Lollipop graph

The proof of the following theorem can be found in Haemers, Liu and Zhang:

Theorem 1

Let G be a graph with an end vertex x_1 adjacent to vertex x_2, and let G' be the subgraph of G induced by removing the vertex x_1. and let G'' be the subgraph of G induced by removing the vertex x_2. Then:

$$P_{A(G)}(a) = aP_{A(G')}(a) - P_{A(G'')}(a)$$

Where $P_{A(G)}(\lambda)$ is the characteristic polynomial $\det(A(G)-\lambda I)$, and $A(G)$ the adjacency matrix of G.

Example with complete graph joined to end vertex

So if LP(G) is the complete graph on n-1vertices (the base of the lollipop graph) joined to a single end vertex x_2 by an edge x_1x_2, we have:

$$P_{A(G)}(\lambda)=\lambda P_{A(G')}(\lambda)-P_{A(G'')}(\lambda)=\lambda(\lambda+1)^{n-2}(\lambda-(n-2))-\lambda(\lambda+1)^{n-3}(\lambda-(n-3))$$

$$\lambda(\lambda+1)^{n-3}[(\lambda+1)(\lambda-(n-2))-(\lambda-(n-3)]$$

$$\lambda(\lambda+1)^{n-3}[\lambda^2-\lambda(n-2)+\lambda-(n-2)-\lambda+(n-3)]$$

$$\lambda(\lambda+1)^{n-3}[(\lambda^2-\lambda(n-2)-1]$$

Roots of quadratic are:

$$\lambda=\frac{(n-2)\pm\sqrt{n^2-4n+4+4}}{2}$$; we have roots:

$$\lambda=0;\lambda=-1(multiplicity\ \ n-3);\lambda=\frac{(n-2)+\sqrt{n^2-4n+8}}{2};\lambda=\frac{(n-2)-\sqrt{n^2-4n+8}}{2}$$

Energy of this graph is therefore:

$$0+1(n-3)+\frac{(n-2)+\sqrt{n^2-4n+8}}{2}+\frac{\sqrt{n^2-4n+8}-(n-2)}{2}\quad \text{since } n\ge 4$$

$$=(n-3)+\sqrt{n^2-4n+8}$$

Theorem 2

The energy of the lollipop graph with base the complete graph on n-1 vertices is:

$$(n-3)+\sqrt{n^2-4n+8}$$

To find $\sum dv$ for this graph we have:

$$\sum dv=(n-2)(n-2)+(n-1)+1=(n-2)(n-2)+n=n^2-3n+4=2m'$$

Thus:

$$Rat\langle D_n^{\Im}=\left[\frac{E(K_n)-E(LP(G))}{2n-2}\right]=\frac{(2n-2)-[(n-3)+\sqrt{n^2-4n+8}]}{2n-2}=\frac{n+1-\sqrt{n^2-4n+8}}{2n-2}$$

$$Asymrat\langle D_n^{\Im}=\lim_{n\to\infty}\left[\frac{n+1-\sqrt{n^2-4n+8}}{2n-2}\right]$$

Multiply top and bottom of $\frac{n+1-\sqrt{n^2-4n+8}}{2n-2}$ by:

$$n+1+\sqrt{n^2-4n+8}:\quad\frac{(n+1)^2-(n^2-4n+8)}{(2n-2)(n+1+\sqrt{n^2-4n+8})}$$

For large n, the numerator is of order 6n and the denominator is of order $4n^2$ so that $Asymrat\langle D_n^{\Im}=\lim_{n\to\infty}\left[\frac{6n}{4n^2}\right]=0$.

$$Arat\langle D_n^{\mathfrak{I}} = \frac{2m'}{n}\int\left[\frac{n+1-\sqrt{n^2-4n+8}}{2n-2}\right]dn = \frac{m'}{n}\int\frac{n+1-\sqrt{n^2-4n+8}}{n-1}dn$$

$$= \frac{m'}{n}\int[\frac{n-1}{n-1}+\frac{2-\sqrt{n^2-4n+8}}{n-1}]dn = \frac{m'}{n}[n+\int\frac{2-\sqrt{(n-2)^2+4}}{n-1}dn$$

Theorem 3

$Rat\langle D_n^{\mathfrak{I}}$; $Asymrat\langle D_n^{\mathfrak{I}}$ and $Arat\langle D_n^{\mathfrak{I}}$ for the following classes of graphs are, respectively:

$$\mathfrak{I} = K_{\frac{n}{2},\frac{n}{2}} : \frac{n-2}{2n-2};\ \frac{1}{2};\frac{n}{4}(n-\ln(n-1)+2)$$

$$\mathfrak{I} = K_{1,n-1} : 1-\frac{1}{\sqrt{n-1}};1;\frac{2(n-1)}{n}[n-2\sqrt{n-1}]$$

$$\mathfrak{I} = K_{2,r} : \frac{n+1-\sqrt{2}\sqrt{n+1}}{2n-2};\frac{1}{2};$$

$$\frac{2(n-1)}{n}[\frac{n}{2}+\ln(n-1)-\frac{1}{\sqrt{2}}.[\sqrt{n^2-1}+2\ln(\frac{\sqrt{n+1}}{\sqrt{2}}+\frac{\sqrt{n-1}}{\sqrt{2}})]+2+\sqrt{2}\ln(2+\sqrt{5})-\frac{3}{2}-\ln 2]$$

$$\mathfrak{I} = L(K_n) :. \frac{-2p+\sqrt{1+8p}}{2p-2};-1;$$

$$\frac{2q}{p}[p-\sqrt{1+8p}+\frac{7}{6}(\ln(\sqrt{1+8p}-3)+\frac{7}{6}\ln(\sqrt{1+8p}+3)+2+\frac{7}{6}(\ln(2)-\frac{7}{6}\ln(8)]$$

Where $q = \frac{n(n-1)(n-2)}{2}$

$$\mathfrak{I} = SR(G) : \frac{3n-n\sqrt{n}-4}{4n-4};-\infty;$$

$$\frac{(n+\sqrt{n})}{2}[\frac{1}{6}n^{\frac{3}{2}}-\frac{3}{4}n+\frac{1}{4}\ln(n-1)+\frac{1}{2}\sqrt{n}+\frac{1}{2}\ln\frac{\sqrt{n}-1}{\sqrt{n}+1}+c]$$

$$\Im = LP(G): \frac{n+1-\sqrt{n^2-4n+8}}{2n-2}; 0; \; Arat\langle D_n^{\Im} = 2\frac{m'}{n}\int\left[\frac{n+1-\sqrt{n^2-4n+8}}{2n-2}\right]dn$$

Lemma 1

$$\frac{3n-n\sqrt{n}-4}{4n-4} \le Rat\langle D_n^{\Im} \le 1$$

Proof

Since $E(G) \ge 0$ for any graph G we get the right hand inequality:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] \le \left[\frac{E(K_n)}{2n-2}\right] = 1$$

And since $E(G) \le E(SR(G)) \Rightarrow E(K_n) - E(SR(G)) \ge 2n-2-\frac{n(1+\sqrt{n})}{2}$ we get the left hand inequality.

Lemma 2

The domination eigen-complete effect is at most one and is greatest negatively for the strongly regular graph examined in the above example.

Proof

The strongly regular graph in the above example has the greatest energy of all graphs so that:

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}[\frac{3n-n\sqrt{n}-4}{4n-2}] = -\infty$$

The above lemmas can be used to verify the following theorem:

Theorem 4

$Asymrat\langle D_n^{\Im} \in (-\infty,1]$ with end points attained for the strongly regular graph and star graphs with rays of length 1, respectively, and $Asymrat\langle D_n^{\Im} = 0$ for the lollipop graph .

Corollary 1

The eigen-complete difference height of the strongly regular graph above is the greatest of all eigen-complete heights.

Proof

Since $E(SR(G)) \geq E(G')$ for any graph $G' \in \Im$ we have:

$$\left|\int [\frac{E(K_n) - E((SR(G)}{E(K_n)}]dn\right| = \int [\frac{E((SR(G) - E(K_n)}{E(K_n)}]dn \geq \int [\frac{E(G') - E(K_n)}{E(K_n)}]dn$$

Conjecture 1

Except for strongly regular graphs, the eigen-complete difference asymptote lies on the interval [-1,1].

4. Eigen-complete different ratios of complements of classes of graphs

4.1 The complete-split bipartite graph

The complement of $K_{\frac{n}{2},\frac{n}{2}}$ consists of two disjoint copies of $K_{\frac{n}{2},}$. It energy is therefore:

$2n-4$ so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{2n-2-(2n-4)}{2n-2} = \frac{2}{2n-2} = \frac{1}{n-1}$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}[\frac{1}{n-1}] = 0$$

$$Arat\langle D_n^{\Im} = \frac{n-2}{2n}\int\left[\frac{1}{n-1}\right]dn = \frac{n-2}{n}[\ln(n-1)+c]$$

Smallest such graph occurs for n=4 so that:

$c = -\ln 3$

The eigen-complete difference ratio for the complement of the complete-split bipartite graph is $f(n) = \frac{1}{n-1}$

The eigen-complete difference ratio of the original graph is:

$$g(n) = Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{2n-2-n}{2n-2} = \frac{n-2}{2n-2} = \frac{n}{2n-2} - \frac{2}{2n-2}$$

$$= \frac{n}{2}f(n) - f(n) = f(n)(\frac{n}{2}-1) \Rightarrow g'(n) = \frac{f(n)}{2} + \frac{nf'(n)}{2} - f'(n) = f'(n)[\frac{n}{2}-1] + \frac{f(n)}{2}$$

$$g'(n) = \frac{(2n-2)-2(n-2)}{(2n-2)^2} = \frac{2}{(2n-2)^2} = f'(n)[\frac{n-2}{2}] + \frac{f(n)}{2} \Rightarrow f'(n) + \frac{1}{n-2}f(n)$$

$$= \frac{2g'(n)}{(n-2)} = \frac{4}{(n-2)(2n-2)^2} = \frac{1}{(n-2)(n-1)^2}$$

$$\frac{d(f(n))}{dn} + \frac{1}{n-2}f(n) = \frac{1}{(n-2)(n-1)^2}; IF = (n-2) \Rightarrow (n-2)f(n) = \int\frac{1}{(n-1)^2}dn$$

$$\Rightarrow (n-2)f(n) = -\frac{1}{n-1} + c \Rightarrow f(n) = -\frac{1}{(n-2)(n-1)} + \frac{c}{(n-2)}$$

$$\Rightarrow \frac{1}{n-1} = -\frac{1}{(n-2)(n-1)} + \frac{c}{(n-2)} \Rightarrow 1 = -\frac{1}{n-2} + \frac{c(n-1)}{n-2}$$

$\Rightarrow n-2=-1+c(n-1)\Rightarrow c=1$

So $\Rightarrow f(n)=-\dfrac{1}{(n-2)(n-1)}+\dfrac{1}{(n-2)}$

Thus with f(n) and g(n) we associate the quadratic

$h(n)=n^2-3n+2$

$h(n+1)-h(n)=n^2+2n+1-3n-3+2-n^2+3n-2=2n-2$ (1)

$h(n+2)-h(n+1)=n^2+4n+4-3n-6+2-(n^2+2n+1-3n-3+2)=2n$ (2)

$h(n+3)-h(n+2)=n^2+6n+9-3n-9+2-(n^2+4n+4-3n-6+2)=2n+2$ (3)

The second difference involving (1), (2) and (3) is an arithmetic sequence with common difference 2. Thus we have the following theorem:

Theorem 5

The eigen-difference ratio g(n)of the complete-split bipartite and the eigen-difference ratio f(n) its complement are related by the following equation:

$$f'(n)+\frac{1}{n-2}f(n)=\frac{2g'(n)}{(n-2)}$$

This results in the differential equation:

$$\frac{d(f(n))}{dn}+\frac{1}{n-2}f(n)=\frac{1}{(n-2)(n-1)^2}$$

With general solution:

$$\Rightarrow f(n)=-\frac{1}{(n-2)(n-1)}+\frac{c}{(n-2)}$$

Corollary 2

The equation in the above theorem yields the following quadratic sequence:

$0,2,6,\ldots,n^2-3n+2,\ldots$

With second difference sequence with common difference 2:

2,4,6,….,2n-2,…

4.2 Star graphs with rays of length 1

The compliment of the star graph with rays of length one (on at least three vertices) is a complete graph on n-1 vertices together with an isolated vertex. Its energy is therefore:

$2n-4$ so that:

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{2n-2-2n+4}{2n-2} = \frac{2}{2n-2} = \frac{1}{n-1}.$$

$$Asymrat\langle D_n^{\Im} = \underset{n\to\infty}{Lim}\left[\frac{1}{n-1}\right] = 0$$

$$Arat\langle D_n^{\Im} = \frac{m'}{n}\int\left[\frac{1}{n-1}\right]dn = \frac{(n-1)(n-2)}{2n}[\ln(n-1)+c].$$

For n=3 we get $c=-\ln 2$.

4.3 The lollipop graph with complete graph on n-1 vertices as base

The compliment of the lollipop graph consists of a star graphs on n-1 vertices and an isolated vertex. Its energy is therefore $2\sqrt{n-2}$

$$Rat\langle D_n^{\Im} = \left[\frac{E(K_n)-E(G)}{2n-2}\right] = \frac{2n-2-2\sqrt{n-2}}{2n-2} = \frac{n-1-\sqrt{n-2}}{n-1} = g(n)$$

$$Asymrat\langle D_n^{\Im} = \underset{n\to\infty}{Lim}\left[\frac{n-1-\sqrt{n-2}}{n-1}\right] = 1.$$

$$Arat\langle D_n^{\Im} = \frac{m'}{n}\int \frac{n-1-\sqrt{n-2}}{n-1}\,dn = \frac{m'}{n}[n - \int \frac{\sqrt{n-2}}{n-1}dn];\ put\ u^2 = n-2 \Rightarrow 2udu = dn$$

$$\int \frac{\sqrt{n-2}}{n-1}dn = \int \frac{2u^2}{u^2+1}\,du;\ pit\ u = \tan v \Rightarrow 2\int \frac{\tan^2 v}{\sec^2 v}\sec^2 v dv = 2\int (\sec^2 v - 1)dv = 2\tan v - v$$
$$= 2\sqrt{n-2} - \arctan\sqrt{n-2}.$$

Thus:

$Arat\langle D_n^{\Im} = \frac{m'}{n}[n - 2\sqrt{n-2} - \arctan\sqrt{n-2} + c$. Taking n=3 we get:

$$c = \frac{®}{4} - 1.$$

5. Conclusion

In this paper we used the idea of energy difference between two graphs and the significance of the complete graph to formulate the eigen-complete difference ratio which allowed for the investigation of the domination effect that the energy of graphs have with respect to the complete graph when a large number of vertices are involved. This idea can be adopted by molecules with a large number of atoms where the need to examine molecules whose energy may dominate the molecule that is very well bonded. We found that a strongly regular graph dominated in the largest negative way, while the star graph with rays of length one had a domination effect of one- the largest possible positive domination effect. The lollipop graph with base the complete graph had domination effect of zero.

We attached the average degree to the Riemann integral of this eigen-complete ratio to determine eigen-complete areas associated with classes of graphs and applied the above ideas to the complement of classes of graphs. We showed that the eigen-complete difference ratios of the complete-split bipartite graph and its complement are related by a differential equation with an associated quadratic sequence with second difference being a sequence with common difference of two.

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