THREE PROOFS FOR LEGENDRE’S CONJECTURE

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“They prevented me in the day of my calamity: but the LORD was my stay.” — 2 Samuel 22:19.

Abstract. We write three proves for Legendre’s conjecture: given an integer, \( n > 0 \), there is always one prime, \( p \), such that \( n^2 < p < (n+1)^2 \), using the prime-counting function, the Bertrand’s postulate and the Hardy-Wright’s estimate.

1. Introduction

The Legendre’s conjecture, named after Adrien-Marie Legendre (1752-1833), says that: There is always one prime number between a square number and the next. Algebraically speaking, as originally proposed by G. H. Hardy and E. M. Wright, in his book An Introduction to the Theory of Numbers:

**Theorem 1. (Legendre’s conjecture) [1, p. 23]** There is always one prime between \( n^2 \) and \( (n+1)^2 \).

Put yet another way, \( \pi((n+1)^2) - \pi(n^2) > 0 \), where \( \pi(n) \) denotes the classical prime-counting function.

In Landau’s problems list, this conjecture was considered unproved, in 1912.

Chen Jingrun (1933-1996) demonstrated a weaker version of the conjecture: There is either a prime \( n^2 < p < (n+1)^2 \) or a semiprime \( n^2 < pq < (n+1)^2 \), where \( q \) is one prime unequal to \( p \). [1, p. 594]

H. Laurent [2, p. 427] noted this peculiar relation to prime numbers:

\[
\frac{e^{\frac{2\pi(i(k))}{k}} - 1}{e^{\frac{2\pi(i)}{k}} - 1} = \begin{cases} 
1, & \text{if } k \text{ is prime}, \\
0, & \text{if } k \text{ is composite},
\end{cases}
\]

provided \( k \in \mathbb{N}_{\geq 5} \), where \( \Gamma(k) \) denotes the gamma function.

In 2013, I and the Dr. Raja Rama Gandhi [2, Theorem 2, pp. 5-7] demonstrated the Legendre’s conjecture based in the partial summation for prime-counting function:

\[
\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{\frac{2\pi(i(k))}{k}} - 1}{e^{\frac{2\pi(i)}{k}} - 1},
\]

for \( n \in \mathbb{N}_{\geq 5} \), here \( \Gamma(n) \) is the gamma function.

In section 3, we use the same strategy to demonstrate, again, the Legendre’s conjecture; but, this time, we will utilize other partial summations for prime-counting function.

In section 4, we use a new strategy to prove the Legendre’s conjecture, yet based on lower bound for prime-counting function, which we found in the book of Hardy and Wright [1], the Theorem 20, page 21.

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2. Preliminaries

In present, we will need of the two partial summation for prime-counting function:

**Theorem 2.** For \( n \in \mathbb{N}_{\geq 5} \), then

\[
\pi(n) = 2 + \sum_{k=5}^{n} 1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k) + 2\pi}{k}\right) \tag{1}
\]

and

\[
\pi(n) = 2 + \sum_{k=5}^{n} \frac{\left(1 - e^{-\frac{2\pi\Gamma(k)}{k}}\right)\left(1 - e^{-\frac{2\pi}{k}}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \tag{2}
\]

where \( \Gamma(n) \) denotes the gamma function.

**Proof.** See [3 and 4, Theorem 1, pp.1-4]. \( \square \)

The weak Bertrand’s postulate says:

**Theorem 3.** *(weak Bertrand’s postulate)* For every \( n \geq 1 \) there is always, at least, one prime number, \( p \), such that \( n < p \leq 2n \).

**Proof.** See [5, pp. 208-209]. \( \square \)

Setting another way, \( \pi(2n) - \pi(n) > 0 \), here \( \pi(n) \) denotes the classical prime-counting function.

**Theorem 4.** For \( n \in \mathbb{N}_{\geq 1} \), then

\[
\pi(n) \geq \frac{\log n}{2\log 2}. \tag{3}
\]

**Proof.** See [1, Theorem 20, page 21]. \( \square \)

3. Two proofs for theorem 1

3.1. First Proof.

**Proof.** PART 1. For \( n \in \mathbb{N}_{\geq 5} \). Replace \((n+1)^2\) and \(n^2\), respectively, into \( n \), in Eq. (1), and obtain

\[
\pi((n+1)^2) = 2 + \sum_{k=5}^{(n+1)^2} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k) + 2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \tag{4}
\]

\[
= 2 + \sum_{k=5}^{2n} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k) + 2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}
\]

\[
+ \sum_{k=2n+1}^{(n+1)^2} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k) + 2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}
\]

\[
= \pi(2n) + \sum_{k=2n+1}^{(n+1)^2} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k) + 2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}
\]
\[
\pi(n^2) = 2 + \sum_{k=5}^{n^2} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k) + 2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}
\]

\[
= \left[ 2 + \sum_{k=5}^{n} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k) + 2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right]
\]

\[
\sum_{k=n+1}^{n^2} \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k) + 2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)}
\]

Subtracting (4) from (5), it follows that

\[
\pi((n + 1)^2) - \pi(n^2) = \pi(2n) - \pi(n)
\]
We easily noted the inequality
\[
0 = \min_{k \in \mathbb{N}_{> 3}} \left( \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi \Gamma(k)}{k}\right) + \cos\left(\frac{2\pi \Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right) \tag{7}
\]
\[
\leq \max_{k \in \mathbb{N}_{> 3}} \left( \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi \Gamma(k)}{k}\right) + \cos\left(\frac{2\pi \Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right) = 1.
\]

From (6) and (7), it follows that
\[
\pi(2n) - \pi(n)
+ \sum_{k=n^2+1}^{(n+1)^2} \min_{k \in \mathbb{N}_{> 3}} \left( \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi \Gamma(k)}{k}\right) + \cos\left(\frac{2\pi \Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right)
- \sum_{k=n+1}^{2n} \min_{k \in \mathbb{N}_{> 3}} \left( \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi \Gamma(k)}{k}\right) + \cos\left(\frac{2\pi \Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right)
\leq \pi((n+1)^2) - \pi(n^2) < \pi(2n) - \pi(n)
+ \sum_{k=n^2+1}^{(n+1)^2} \max_{k \in \mathbb{N}_{> 3}} \left( \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi \Gamma(k)}{k}\right) + \cos\left(\frac{2\pi \Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right)
- \sum_{k=n+1}^{2n} \max_{k \in \mathbb{N}_{> 3}} \left( \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi \Gamma(k)}{k}\right) + \cos\left(\frac{2\pi \Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right).
\]

Thereafter, we encounter
\[
\pi(2n) - \pi(n) + \sum_{k=n^2+1}^{(n+1)^2} 0 - \sum_{k=n+1}^{2n} 0 \leq \pi((n+1)^2) - \pi(n^2)
\leq \pi(2n) - \pi(n) + \sum_{k=n^2+1}^{(n+1)^2} 1 - \sum_{k=n+1}^{2n} 1,
\]
consequently,
\[
\pi(2n) - \pi(n) \leq \pi((n+1)^2) - \pi(n^2) < \pi(2n) - \pi(n) + 2n + 1 - n,
\]
that is,
\[
\pi(2n) - \pi(n) \leq \pi((n+1)^2) - \pi(n^2) < \pi(2n) - \pi(n) + n + 1. \tag{8}
\]

From Theorem 3 (weak Bertrand’s postulate) and Eq. (8), obviously, I have
\[
0 < \pi(2n) - \pi(n) \leq \pi((n+1)^2) - \pi(n^2) < \pi(2n) - \pi(n) + n + 1.
\]

In other words, \(\pi((n+1)^2) - \pi(n^2) > 0\), and this prove the first part, for \(n \in \mathbb{N}_{> 5}\).

PART 2. We calculate explicitly from \(n = 1\) at \(n = 4\). For \(n = 1\), thus \(\pi(2^2) - \pi(1^2) = 2 - 0 = 2 > 0\); for \(n = 2\), thus \(\pi(3^2) - \pi(2^2) = 4 - 2 = 2 > 0\); for \(n = 3\), thus \(\pi(4^2) - \pi(3^2) = 6 - 4 = 2 > 0\); for \(n = 4\), thus \(\pi(5^2) - \pi(4^2) = 9 - 6 = 3 > 0\). This completes the proof. \(\square\)
3.2. Second Proof.

Proof. PART 1. For \( n \in N_{\geq 5} \). Replace \((n + 1)^2\) and \(n^2\), respectively, into \( n \), in Eq. (2), and obtain

\[
\pi((n + 1)^2) = 2 + \sum_{k=5}^{(n+1)^2} \frac{1 - e^{-\frac{2\pi(i)(k)}{n}}}{2 - 2\cos(\frac{2\pi}{n})}
\]

(9)

\[
\pi(n^2) = 2 + \sum_{k=5}^{n^2} \frac{1 - e^{-\frac{2\pi(i)(k)}{n}}}{2 - 2\cos(\frac{2\pi}{n})}
\]

(10)

Subtracting (9) to (10), it follows that

\[
\pi((n + 1)^2) - \pi(n^2) = \pi(2n) - \pi(n)
\]

(11)

We easily noted the inequality

\[
0 = \min_{k \in N_{\geq 5}} \left( \frac{1 - e^{-\frac{2\pi(i)(k)}{n}}}{2 - 2\cos(\frac{2\pi}{n})} \right) \leq \left( \frac{1 - e^{-\frac{2\pi(i)(k)}{n}}}{2 - 2\cos(\frac{2\pi}{n})} \right) \leq \max_{k \in N_{\geq 5}} \left( \frac{1 - e^{-\frac{2\pi(i)(k)}{n}}}{2 - 2\cos(\frac{2\pi}{n})} \right) = 1.
\]

(12)
From (11) and (12), it follows that

$$\pi(2n) - \pi(n) + \sum_{k=n^2+1}^{(n+1)^2} \min_{k \in \mathbb{N}_{\geq 5}} \left( \frac{1 - e^{-2\pi/(k)}}{2 - 2\cos \left( \frac{2\pi}{k} \right)} \right)$$

$$= - \sum_{k=n+1}^{2n} \min_{k \in \mathbb{N}_{\geq 5}} \left( \frac{1 - e^{-2\pi/(k)}}{2 - 2\cos \left( \frac{2\pi}{k} \right)} \right) \leq \pi((n+1)^2) - \pi(n^2) < \pi(2n) - \pi(n)$$

$$+ \sum_{k=n^2+1}^{(n+1)^2} \max_{k \in \mathbb{N}_{\geq 5}} \left( \frac{1 - e^{-2\pi/(k)}}{2 - 2\cos \left( \frac{2\pi}{k} \right)} \right) - \sum_{k=n+1}^{2n} \max_{k \in \mathbb{N}_{\geq 5}} \left( \frac{1 - e^{-2\pi/(k)}}{2 - 2\cos \left( \frac{2\pi}{k} \right)} \right).$$

Therefore, we find

$$\pi(2n) - \pi(n) + \sum_{k=n^2+1}^{(n+1)^2} 0 - \sum_{k=n+1}^{2n} 0 \leq \pi((n+1)^2) - \pi(n^2)$$

$$< \pi(2n) - \pi(n) + \sum_{k=n^2+1}^{(n+1)^2} 1 - \sum_{k=n+1}^{2n} 1,$$

consequently,

$$\pi(2n) - \pi(n) \leq \pi((n+1)^2) - \pi(n^2) < \pi(2n) - \pi(n) + 2n + 1 - n,$$

that is,

$$\pi(2n) - \pi(n) \leq \pi((n+1)^2) - \pi(n^2) < \pi(2n) - \pi(n) + n + 1. \quad (13)$$

From Theorem 3 (weak Bertrand’s postulate) and Eq. (13), obviously, I have

$$0 < \pi(2n) - \pi(n) \leq \pi((n+1)^2) - \pi(n^2) < \pi(2n) - \pi(n) + n + 1.$$

In other words, $\pi((n+1)^2) - \pi(n^2) > 0$, and this prove the first part, for $n \in \mathbb{N}_{\geq 5}$.

PART 2. We calculate explicitly from $n = 1$ at $n = 4$. For $n = 1$, thus $\pi(2^2) - \pi(1^2) = 2 - 0 = 2 > 0$; for $n = 2$, thus $\pi(3^2) - \pi(2^2) = 4 - 2 = 2 > 0$; for $n = 3$, thus $\pi(4^2) - \pi(3^2) = 6 - 4 = 2 > 0$; for $n = 4$, thus $\pi(5^2) - \pi(4^2) = 9 - 6 = 3 > 0$. This completes the proof. \qed

4. A NEDITED SHORT PROOF FOR THEOREM 1

Now, we present an unpublished proof for Legendre’s conjecture.

**Proof.** For $n \in \mathbb{N}_{\geq 1}$, we set $(n+1)^2$ and $n^2$, respectively, into $n$, in Eq. (3), and obtain

$$\pi((n+1)^2) > \frac{2 \log (n+1)}{2 \log 2} = \frac{\log(n+1)}{\log 2} \quad (14)$$

and

$$\pi(n^2) \geq \frac{2 \log n}{2 \log 2} = \frac{\log n}{\log 2} \quad (15)$$

Subtracting (14) to (15), we meet

$$\pi((n+1)^2) - \pi(n^2) > \frac{\log (n+1) - \log n}{\log 2} > 0,$$

since $\log (n+1) > \log n$. Hence, $\pi((n+1)^2) - \pi(n^2) > 0$. This gives us the desired result. \qed

**References**


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