

## Combined Gravitational Action (V): *Extension*

Mohamed E. Hassani<sup>1</sup>

Institute for Fundamental Research  
BP.197, CTR, Ghardaïa 47800, Algeria

**Abstract:** In a series of papers related to the Combined Gravitational Action (CGA) as an alternative theory of gravity, we have developed the CGA-formalism, which is exclusively based on a new form of velocity-dependent gravitational potential energy. The present paper is an additional exploration, exploitation, and extension of the CGA with the main aim of showing: (i) the Galilean invariance of CGA-equations; (ii) that the CGA is, in a certain manner, a metric gravity theory; (iii) the existence of an important similarity between the CGA and General Relativity Theory; (iv) a new CGA-formula for perigee and perihelion precession is derived and proved to be in excellent agreement with observations; (v) the CGA-effects on orbital motion inside and outside the Solar System are investigated; (vi) the conversion rate of orbital energy into gravitational radiation is calculated, and its effects are studied.

**Keywords:** CGA-effects, orbital motion, perigee, perihelion, periastron, Moon, major asteroids, inner planets, binary pulsars, gravitational radiation

### 1. Introduction

In previous papers [1,2,3,4], we have already investigated the CGA-effects arising from the CGA-formalism as an alternative gravity theory capable of predicting and explaining some old and new gravitational effects. This has allowed us to resolve, within its context, some unexpected and challenging problems that have occurred inside and outside the Solar System (SS), such as the anomalous deceleration of Pioneer 10, the observed secular increase of the Astronomical Unit, and the apsidal motion anomaly of the eclipsing binary star systems AS Camelopardalis and DI Herculis. For example, in paper [3], we investigated the CGA-effects on the orbital motion of planets and noncompact and compact stellar objects. In the CGA-fourth part [4], we showed the existence of CGA-spin-orbit coupling precession and applied CGA to large-scale structure, resolving the problem of galactic rotation curves. Additionally, Milgrom's theory of Modified Newtonian Dynamics (MOND) [5,6,7,8], as an alternative model to the dark matter (DM) hypothesis, was further supported by CGA [4].

The present work is based entirely on CGA-third part [3] with the primary purpose of demonstrating, among other things, the invariance of CGA-equations under Galilean transformations, the CGA's role as a metric gravity, and the significant similarity between the CGA-equations of motion and those of General Relativity Theory (GRT). It is the additional terms that are responsible for the major secular effects, as we will see. In paper [3], we have already calculated and listed the numerical values of CGA-effects on the SS' planets (see Tables 1 and 2 in Ref.[3]). For example, Table 2 illustrates the excellent agreement between the observed perihelion precession of planets and the CGA-predictions. The same appreciation holds for CGA-apsidal motion in eclipsing binary star systems [3].

---

<sup>1</sup> E-mail: hassani641@gmail.com

From all that, we can say this is all the more impressive, particularly when we take into account the fact that the Brans-Dicke and other alternative gravity theories, except GRT, contain some adjustable parameters, which is why they are called ‘adjustable gravity. However, the CGA has no freedom to adjust its predictions because it is simply an inadjustable gravity theory by its proper formalism.

As we can see from the earlier papers [1,2,3,4], asteroids have been completely neglected. For this reason, an important part of the present work is devoted to them. More recently, the author of CGA realized that since some major asteroids are potentially hazardous, such as Apollo and Icarus, which are known to cross Earth's orbit, asteroids are quite interesting celestial bodies. They can serve as a celestial laboratory enabling us to understand physical processes that take place on asteroids as well as on other similar SS-bodies, and also to test more rigorously alternative gravity theories. Hence, the investigation of CGA-effects on the orbital motion of asteroids is very important. Readers interested in the CGA-formalism can refer to Ref.[3].

The real novelty of the present paper is related to the derivations of two new CGA-formulae, one for the perigee and perihelion precession and the other related to the conversion rate of orbital energy into gravitational radiation. To begin, for the convenience of the reader, it is best to briefly recall the basic foundation of the CGA-formalism. The CGA, as an alternative gravity theory, is fundamentally based on the concept of the combined gravitational potential energy (CGPE), which is typically a new form of velocity-dependent-GPE defined by the expression

$$U \equiv U(r, v) = -\frac{k}{r} \left(1 + \frac{v^2}{w^2}\right), \quad (1)$$

where  $k = GMm$ ;  $G$  is the Newton's gravitational constant;  $M$  and  $m$  are the masses of the gravitational source  $A$  and the moving test-body  $B$ ,  $r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$  is the relative distance between  $A$  and  $B$ ,  $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$  is the velocity of the test-body  $B$  relative to the reference frame of the source  $A$ , and  $w$  is a specific kinematical parameter with the dimensions of a constant velocity defined by

$$w = \begin{cases} c & \text{if } B \text{ is in relative motion inside the vicinity of } A \\ v_{\text{esc}} = \sqrt{2GM/R} & \text{if } B \text{ is in relative motion outside the vicinity of } A \end{cases}, \quad (2)$$

where  $c$  is the light speed in local vacuum and  $v_{\text{esc}}$  is the escape velocity at the surface of the gravitational source  $A$ .

In the CGA-context, the velocity-dependent-GPE (1) is simply called CGPE because it is, in fact, a combination of the static-GPE  $V(r) = kr^{-1}$  and the dynamic-GPE  $W(r, v) = kr^{-1}(v/w)^2$ . The main difference between the CGPE (1) as a generalization of classical GPE and the previously well-known velocity-dependent-GPEs is clearly situated in the originality and simplicity of Eq.(1), which may be rewritten in the form  $U \equiv U(r, \dot{r}) = kr^{-1}[1 + (\dot{r}/w)^2]$ , with  $v \equiv \dot{r} = dr/dt$ . The uniqueness of CGPE is evident in the fact that it explicitly depends on  $r$  and  $v$ , but also implicitly depends on  $w$  since the latter is, by definition, a specific kinematical parameter with the physical dimensions of a constant velocity. The implicit dependence of CGPE on  $w$  is expressed in terms of ‘inside the vicinity of  $A$ ’ and ‘outside the vicinity of  $A$ ’ in definition (2). Additionally, the

CGPE may be reduced to the static-GPE when  $v \ll w$  or  $v = 0$ . Thus the main physical reason for the choice of the expression (1) for the CGPE lies in its consequence as a generalization of the static-GPE. Moreover, the CGPE defined by Eq.(1) constitutes a fundamental solution to a system of three second-order PDEs called ‘potential equations’ because  $U \equiv U(r, v)$  is a common solution to these three equations. Indeed, it is easy to show, under some appropriate boundary conditions, that the combined potential field is really a fundamental solution to the following PDEs

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r} \frac{\partial U}{\partial r} = 0, \quad (3)$$

$$\frac{\partial^2 U}{\partial v^2} - \frac{1}{v} \frac{\partial U}{\partial v} = 0, \quad (4)$$

$$\frac{\partial^2 U}{\partial r \partial v} + \frac{1}{r} \frac{\partial U}{\partial v} = 0. \quad (5)$$

Remark: since Eqs.(3-5) are homogeneous and admit the same potential function  $U$  as a fundamental solution, this implies, among other things, that the test-body  $B$  is in a state of motion at a relative velocity  $v$ , sufficiently far from the main gravitational source  $A$ . Furthermore, the same fundamental solution is the *origin* of the CGA-equations of motion and the CGA-field equations because, as we have previously seen in [2,3], the potential function  $U$  is the leading term of the CGA-Lagrangian.

## 2. Galilean Invariance of CGA-Equations

It is important to remember that the CGA, as an alternative gravity theory, is fully developed within the framework of the Galilean relativity principle and Euclidean geometry. This is due to the fact that in nature, major physical phenomena occur at subrelativistic velocities, which are significantly smaller than the speed of light in a local vacuum. Therefore, if we focus on cases where velocities are subrelativistic, we can demonstrate the invariance of CGA-equations under Galilean transformation (GT). To achieve this, we can utilize a shortcut: since the CGA-formalism is primarily based on the CGPE (1), the Galilean invariance of CGA-equations implies the invariance of CGPE under GT. Consider two inertial reference frames (IRFs)  $S$  and  $S'$ , which are moving uniformly relative to each other at a subrelativistic velocity  $u$ . For simplicity, assume that the vector velocity  $\mathbf{u}$  of the IRFs is along their common  $x | x'$ -axis with corresponding parallel planes. Additionally, let the origins  $O$  and  $O'$  coincide at the moment  $t = t' = 0$ . The GT that facilitates the transition from one IRF to another can be expressed in vector form:

$$S \rightarrow S' : \begin{cases} \mathbf{r}' = \mathbf{r} - \mathbf{u}t \\ t' = t \end{cases}, \quad (6)$$

where  $\mathbf{r}$  and  $\mathbf{r}'$  are position vectors as observed from  $S$  and  $S'$  at any later time  $t' = t$ . Since the quantities  $k$  and  $w$  in (1) are constant, the CGPE may be written in the form:

$$U \equiv U(\mathbf{r}, \mathbf{v}) = -\frac{k}{\|\mathbf{r}\|} \left( 1 + \frac{\|\mathbf{v}\|^2}{w^2} \right). \quad (7)$$

Now, suppose there are two material points  $P_1$  and  $P_2$  whose position vectors are, respectively,  $(\mathbf{r}_1, \mathbf{r}_2)$  relative to  $S$  and  $(\mathbf{r}'_1, \mathbf{r}'_2)$  relative to  $S'$ , and their masses are  $m = m_1$  and  $M = m_2$  as observed

from the two IRFs.  $S'$  is in uniform relative motion at a subrelativistic velocity  $u$  with respect to  $S$ . Therefore, since under GT (6), we have

$$\mathbf{r}'_{21} = \mathbf{r}'_2 - \mathbf{r}'_1 = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}_{21} \quad \text{and} \quad \mathbf{v}'_{21} = \mathbf{v}'_2 - \mathbf{v}'_1 = \mathbf{v}_2 - \mathbf{v}_1 = \mathbf{v}_{21}.$$

Hence, we obtain the following for the CGPE (7):

$$U(\mathbf{r}'_{21}, \mathbf{v}'_{21}) = U(\mathbf{r}_{21}, \mathbf{v}_{21}). \quad (8)$$

This clearly illustrates the invariance of the CGPE under GT.

### 3. Is CGA a metric gravity theory?

After having proved the Galilean invariance of CGPE, which led to the invariance of CGA-equations under GT, we shall now attempt to answer the above question in the CGA-context without concerning ourselves with the conceptual details. For our theory, metricity only has heuristic importance. To this end, let us rewrite the CGPE (1) as follows:

$$r^2 = \left(\frac{k}{U}\right)^2 \left(1 + \frac{v^2}{w^2}\right)^2, \quad r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}. \quad (9)$$

Or equivalently

$$dx^2 + dy^2 + dz^2 - \left(\frac{k}{U}\right)^2 = \left(\frac{k}{U}\right)^2 \left[2\left(\frac{v}{w}\right)^2 + \left(\frac{v}{w}\right)^4\right], \quad (10)$$

where

$$dx = x - x_0, \quad dy = y - y_0, \quad dz = z - z_0.$$

Since the quantity  $(k/U)^2$  has the geometrical dimensions of squared length, let us define it as  $(k/U)^2 = w^2 dt^2$  and  $(k/U)^2 [2(v/w)^2 + (v/w)^4] = ds^2$  for convenience. After substitution into (10), we get

$$ds^2 = dx^2 + dy^2 + dz^2 - w^2 dt^2. \quad (11)$$

The expression of the quadratic form or equally the space-time interval (11) allows us to say that, in a certain sense, the test-body may be gravitationally evolved in flat space-time, called here, CGA-space-time. Let us prove the invariance of (11) under some spatio-temporal transformation more general than GT. With this aim, considering again two IRFs  $S$  and  $S'$ , which are in uniform relative motion at a velocity  $\mathbf{u} = u\mathbf{e}_x$  such that  $u < w$ . Also, the two origins  $O$  and  $O'$  coincide at the moment  $t = t' = 0$ . The two IRFs  $S$  and  $S'$  are linked by the following spatio-temporal transformation given in differential form as follows:

$$S \rightarrow S': \begin{cases} dx' = \eta(dx - udt) \\ dy' = dy \\ dz' = dz \\ dt' = \eta\left(dt - \frac{udx}{w^2}\right) \end{cases}, \quad \eta = \left(1 - \frac{u^2}{w^2}\right)^{-1/2}, \quad u < w. \quad (12)$$

Observe that by using the spatio-temporal transformation (12), we obtain the following invariance

$$ds'^2 = dx'^2 + dy'^2 + dz'^2 - w^2 dt'^2 = dx^2 + dy^2 + dz^2 - w^2 dt^2 = dt^2. \quad (13)$$

This means, among other things, that the CGA is, in a certain manner, a metric gravity theory. In passing, and without going into detail about the process of derivation of (12), it is easy to prove that this transformation forms an orthogonal-orthochronous group. Also, the same transformation may be reduced to Lorentz transformation for the case when  $w = c$ . Finally, the reader can observe that the classical (Galilean) notion of absolute time – second equation in (6) – agreeing for all the IRFs is not always valid because the fourth-equation in (12) may be reduced to  $t' = t$  only if the ratio  $(u/w)$  is sufficiently smaller than unity.

#### 4. Some similarity between CGA and GRT

In the CGA-second part [2], we derived Eq.(30), which is the general equation of motion in the combined gravitational field. From this equation, we derived Eq.(32), which is actually a generalization of Binet's orbital equation:

$$\frac{d^2u}{d\varphi^2} + u - \left[ \frac{3GM}{c^2} \right] u^2 = \frac{GM}{\kappa^2}. \quad (14)$$

Eq.(14) has the exact form derived under GRT. Further, Eq.(14) allows us, of course, to study the perihelion advance of Mercury and other planets, and the angular deflection of light in the combined gravitational field [2]. In the CGA-third part [3], we have derived from the CGA-Lagrangian, the CGA-equations of motion in compact form, *i.e.*, Eqs.(10) in Ref.[3]:

$$\frac{dv}{dt} + \left( 1 + \frac{v^2}{w^2} \right) \left( 1 + \frac{2GM}{w^2 r} \right)^{-1} \nabla \phi = 0, \quad (15)$$

where  $\phi$  is the Newtonian gravitational potential defined by

$$\phi \equiv \phi(r) = -\frac{GM}{r}. \quad (16)$$

Now, considering the case when the test-body  $B$  is evolving in the vicinity of the gravitational source  $A$ , hence according to definition (2), Eq.(15) becomes:

$$\frac{dv}{dt} + \left( 1 + \frac{v^2}{c^2} \right) \left( 1 + \frac{2GM}{c^2 r} \right)^{-1} \nabla \phi = 0, \quad (17)$$

Moreover, if the quantity  $(2GM/c^2 r)$  is sufficiently small compared to unity, Eq.(17) may be written in the form:

$$\frac{dv}{dt} = -\frac{GM}{r^3} \mathbf{r} + \frac{GM}{r^3} \left[ \frac{2GM}{c^2 r} \left( 1 + \frac{v^2}{c^2} \right) \mathbf{r} - \frac{(\mathbf{r} \cdot \mathbf{v})}{c^2} \mathbf{v} \right]. \quad (18)$$

We can see that the first term on the right-hand side of Eq.(18) is the well-known Newtonian gravitational acceleration. The remaining terms are the CGA-correction which, in part, gives rise to the CGA-effects like, for example, the orbital precession (perigee, perihelion, and periastron precession) as we will see later. Furthermore, many authors in the field of relativistic gravitational physics have, derived in the context of the post-Newtonian approximation, certain equations of motion similar to Eq.(18). Among them, we can, for instance, refer to Ref.[5] for the following one:

$$\frac{d^2\mathbf{r}}{c^2 dt^2} = -\frac{\chi}{r^3}\mathbf{r} + \frac{\chi}{r^3}\left[\left(4\frac{\chi}{r} - \frac{v^2}{c^2}\right)\mathbf{r} + 4\frac{(\mathbf{r}\cdot\mathbf{v})}{c^2}\mathbf{v}\right], \quad \chi = Gm_{\odot}/c^2. \quad (19)$$

Why should CGA arrive at the same results as GRT?

Because if we break down the concept of the curvature of space-time, we find that contrary to Newton's theory of gravitation, CGA and GRT (under the post-Newtonian formalism) take into full consideration the relative motion of the test-body and the speed of light in local vacuum. In CGA, the speed of light plays a triple role: (i) a specific kinematical parameter of normalization; (ii) the propagation speed of changes in the combined gravitational potential  $\Phi \equiv m^{-1}U$ ; (iii) the speed of gravitational radiation. In GRT, 'c' is supposed to be the speed of gravity propagation. As previously mentioned in [3], the principal result of the CGA-formalism [1,2,3,4] is the existence of the dynamic gravitational (acceleration) field (DGF)  $\Lambda$  derived from Eq.(28) of Ref.[3] in the following form:

$$\Lambda: \begin{cases} \Lambda_x = -\frac{GM}{r^2} \frac{x-x_0}{r} \left(\frac{v}{w}\right)^2 \\ \Lambda_y = -\frac{GM}{r^2} \frac{y-y_0}{r} \left(\frac{v}{w}\right)^2 \\ \Lambda_z = -\frac{GM}{r^2} \frac{z-z_0}{r} \left(\frac{v}{w}\right)^2 \end{cases}. \quad (20)$$

The DGF is actually an induced field; it is more precisely a type of gravitational induction due to the relative motion of material bodies in the vicinity of the gravitational source. Certainly, the static gravitational field  $\mathbf{y} = \nabla\phi$  is generally always stronger than DGF, but  $\Lambda$  has its proper role and effects. For example, as an additional field,  $\Lambda$  is responsible for the perihelion advance of Mercury and other planets [1,2,3,4]. In his 1912 argument [6], Einstein himself noted that “*gravitation acts more strongly on a moving body than on the same body if it is at rest.*”<sup>2</sup> It seems Einstein's remark reflects very well the expression of the combined gravitational field [1,2,3,4]

$$\mathbf{g} = \mathbf{y} + \Lambda. \quad (21)$$

It is clear from (21) that the combined gravitational field,  $\mathbf{g}$ , may be reduced to the static gravitational field,  $\mathbf{y}$ , only for the case  $\Lambda = 0$ , that is, when the material test-body under the action of the field is at relative rest with respect to the main gravitational source. Moreover, as we know from [2,3], the combined gravitational field is derived from the CGPE defined by the expression (1). Concerning the second result of CGA, namely, the dynamic gravitational force  $\mathbf{F}_D$  is defined as the product of mass  $m$  of the moving test-body  $B$  and  $\Lambda$ , that is

$$\mathbf{F}_D = m\Lambda. \quad (22)$$

Curiously, Lorentz had already arrived at a conclusion very comparable to that of Einstein, but more than a decade before him. In his influential work [7] entitled ‘Considerations on Gravitation’ published in 1900, Lorentz wrote “*Every theory of gravitation has to deal with the problem of the influence, exerted on this force by the motion of the heavenly bodies.*” To my great surprise, when I read Lorentz's article [7] for the first time, after having written the CGA, I found, among other

<sup>2</sup> However, Einstein's claim was made in 1912, before the publication of the final version of GRT in 1915, in which the realistic concept of gravitational force is abandoned and replaced by the concept of the curvature of space-time.

interesting things, that Lorentz had arrived at an extra-gravitational force (Eqs.(24) in Ref.[7]) whose components are very similar to those of  $\mathbf{F}_D$ . Again, Lorentz's claim and findings reinforce the fact that the couple  $(\mathbf{\Lambda}, \mathbf{F}_D)$  is really induced by the relative motion of the massive test-body  $B$  in the gravitational field of central body  $A$ . Additionally, Broginsky, Caves, and Thorn, in their seminal paper [8] entitled 'Laboratory experiments to test relativistic gravity' published in 1977, found an extra-gravitational acceleration called post-Newtonian gravitational acceleration (Eq.(2.1) in Ref.[8]) whose magnitude is also comparable to that of  $\mathbf{\Lambda}$ .

## 5. Effects of $(\mathbf{\Lambda}, \mathbf{F}_D)$ on orbital motion

Now, we arrive at the heart of our subject, namely, the CGA-effects on orbital motion. As we know from the papers [1,2,3,4], the CGA-effects consist of average changes in orbital period, orbital velocity, radial distance, and secular perigee precession of satellites, secular perihelion precession of planets, angular deflection of light, secular apsidal motion in eclipsing binary star systems, periastron advance of pulsars and CGA-spin-orbit precession. All these CGA-effects arise from the coupling of  $(\mathbf{\Lambda}, \mathbf{F}_D)$ .

In order to familiarize the reader with the CGA-effects, it is best to start by showing that CGA-effects are in fact contained in Eq.(18), *i.e.*, the terms on the right-hand side of (18). Let us prove the above affirmation, namely, that the CGA-corrections to the Newtonian equations of motion are of the form of Eq.(18) by comparing this equation with Newton's

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{r^3}\mathbf{r}. \quad (23)$$

Hence, a simple comparison with Eq.(18) shows us that if we apply Eq.(23) instead of Eq.(18) to, for example, the Earth's orbital motion, we should have an apparently correct result. This is partly due to the fact that the Earth's orbital velocity is too slow compared to the speed of light, and the Sun's gravitational field is much weaker compared to that of, for instance, a neutron star or pulsar. However, in terms of accuracy, the omitted terms on the right-hand side of Eq.(18) would still contribute at least an important amount on average. Thus, if we take into account these neglected terms, we get the CGA-correction to the Earth's semi-major axis  $a$  by an amount of

$$\frac{\delta a}{a} = \left[ \frac{2GM_\odot}{c^2 r} \left( 1 + \frac{v_{\text{orb}}^2}{c^2} \right) - \left( \frac{v_{\text{orb}}}{c} \right)^2 \right] \cong 10^{-8}. \quad (24)$$

In Eq.(24), we used  $M = M_\odot$  instead of  $M = M_\odot + m_\oplus$  because the mass-ratio,  $q = m_\oplus/M_\odot$ , of the system {Sun, Earth} is close to zero. The above illustrative example highlights the CGA-formalism and urges the astronomy community to examine the CGA to evaluate its role and significance as an alternative theory of gravity.

Returning to our main subject and demonstrating the effects of a couple  $(\mathbf{\Lambda}, \mathbf{F}_D)$  on the orbital motion of a material test-body. Firstly, we begin by showing that the orbital eccentricity depends, in part, on the average magnitude of the dynamic gravitational force,  $\mathbf{F}_D$ . Secondly, we should prove that the perigee (if the test-body is a satellite), the perihelion (if the test-body is a planet), and periastron (if the test-body is a star) of any moving material test-body, in a combined gravitational field (21), should exclusively depend on the average magnitude of the dynamic gravitational field  $\mathbf{\Lambda}$ .

### 5.1. Effect of $F_D$ on orbital eccentricity

Let us demonstrate that the orbital eccentricity,  $e$ , partly depends on the magnitude of the dynamic gravitational force  $F_D$ . To this end, suppose the material test-body  $B$  of mass  $m$  is orbiting the gravitational source  $A$  of mass  $M$  at the mean radial distance  $r_{\text{mean}}$  with mean orbital velocity  $v_{\text{orb}} = 2\pi r_{\text{mean}} p^{-1}$ , where  $p$  is the mean orbital period. Since the test-body  $B$  is moving inside the vicinity of the gravitational source  $A$ , this allows us to use the definition (2) and Eqs.(20). Thus, the magnitude of the dynamic gravitational field  $\Lambda$  –when it plays the role of an extra-gravitational acceleration– should be of the form

$$\Lambda = \frac{GM}{r^2} \left( \frac{v_{\text{orb}}}{c} \right)^2 = \frac{4\pi^2 GM}{c^2 p^2} \left( \frac{r_{\text{mean}}}{r} \right)^2. \quad (25)$$

Further, we have the following well-known relation

$$e = \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}}, \quad (26)$$

and

$$r_{\text{mean}} = \frac{r_{\text{max}} + r_{\text{min}}}{2}. \quad (27)$$

Hence, from (25), (26) and (27), we get

$$\Lambda = \frac{\pi^2 GM}{c^2 p^2} \left( \frac{\delta r}{er} \right)^2, \quad (28)$$

With  $\delta r = r_{\text{max}} - r_{\text{min}}$ . Therefore, by multiplying the both sides of Eq.(28) by the mass,  $m$ , of the moving test-body  $B$ , we find

$$F_D = m\Lambda = \frac{\pi^2 k}{c^2 p^2} \left( \frac{\delta r}{er} \right)^2, \quad k = GMm. \quad (29)$$

Finally, from Eq.(29), we obtain the expected equation that expresses the dependence of  $e$  on  $F_D$

$$e = \left( \frac{\pi}{cp} \cdot \frac{\delta r}{r} \right) \sqrt{\frac{k}{F_D}}. \quad (30)$$

Eq.(30) shows us more clearly the following possible cases:

i) The orbital eccentricity,  $e$ , should always be different from zero, that is,  $e \neq 0 \Rightarrow \delta r \neq 0$ . therefore, the orbit is not circular, and as a direct result, we have:

$$e \neq 0 \Rightarrow \begin{cases} e < 1, \text{ the orbit is an ellipse} \\ e = 1, \text{ the orbit is a parabola} \\ e > 1, \text{ the orbit is a hyperbola} \end{cases}.$$

ii) Eq.(30) is exclusively valid for the case  $e < 1$ .

iii) The orbital eccentricity is extremely high if  $F_D$  is extremely weak.

iv) The orbital eccentricity is extremely low if  $F_D$  is extremely strong.

## 5.2. Effect of $\Lambda$ on orbital precession

As previously reported, we have already derived the formula (32.10) from Eq.(32) of Ref.[2]:

$$\Delta\varphi = \frac{6\pi GM}{c^2 a(1-e^2)} \quad (\text{rad/rev}), \quad (31)$$

For orbital precession, which is identical to that derived from GRT<sup>3</sup>. Our current goal is to demonstrate the exclusive dependence of orbital precession on the average magnitude of the dynamic gravitational field  $\Lambda$ . To achieve this, we need to use (31) solely as a target that should be explicitly deduced from (28) combined with the well-known equation of an ellipse in polar coordinates

$$r = \frac{a(1-e^2)}{1-e\cos\varphi}, \quad (32)$$

All that can be done through some simple algebraic calculations, as we will see immediately. Let us rewrite (28) as follows:

$$r\Lambda = \frac{GM}{r} \frac{6\pi}{c^2} \frac{\omega^2 \delta r^2}{24\pi e^2}, \quad \omega = 2\pi/p. \quad (33)$$

By substituting (32) into the right-hand side of (33), we get:

$$r\Lambda = \frac{(1-e\cos\varphi)GM}{a(1-e^2)} \frac{6\pi}{c^2} \frac{\omega^2 \delta r^2}{24\pi e^2}. \quad (34)$$

And from (34), we immediately deduce the formula (31):

$$\frac{6\pi GM}{c^2 a(1-e^2)} = \frac{6e^2 r \Lambda p^2}{\pi(1-e\cos\varphi) \delta r^2}. \quad (35)$$

Or equivalently

$$\Delta\varphi = \frac{6e^2}{\pi(1-e\cos\varphi)} \frac{r \Lambda p^2}{\delta r^2}. \quad (36)$$

Further, we have from (32)

$$1 - e\cos\varphi = \frac{a(1-e^2)}{r}. \quad (37)$$

Thus, by substituting (37) into (36), we obtain

$$\Delta\varphi = \left[ \frac{6e^2 p^2}{\pi a(1-e^2)} \frac{r^2}{\delta r^2} \right] \Lambda. \quad (38)$$

Observe, since according to (26) and (27), we have  $\delta r = r_{\max} - r_{\min} = 2e r_{\text{mean}}$ , therefore, (38) becomes for the case  $r = r_{\text{mean}}$

$$\Delta\varphi = \left[ \frac{3p^2}{2\pi a(1-e^2)} \right] \Lambda. \quad (39)$$

---

<sup>3</sup> Historically, the formula was first established by the German physics teacher Paul Gerbert many years before the appearance of GRT (1915). The reader can refer to Ref. [P. Gerbert, 'The Spatial and Temporal Propagation of Gravity', Journal of Mathematics and Physics (English Translation) **43**, 93 (1898)]. It is quite fair to call the formula (31) Gerbert's formula.

Expression (39) is the expected formula that shows the dependence of  $\Delta\varphi$  on  $\Lambda$ . Moreover, since the perihelion advance is  $\Delta\varphi$  per revolution, the resultant equation for the elliptical orbit in this case should be

$$r = \frac{a(1-e^2)}{1-ec\cos(\varphi+\Delta\varphi)}. \quad (40)$$

Eq.(40) may be regarded as a generalization of (32) when the test-body is evolving in the combined gravitational field (21). Concerning  $\Lambda$  in (39), it is defined according to Ref.[1,2,3,4] by the expression:

$$\Lambda = \frac{1}{a} \left[ \frac{GM}{ca} \right]^2, \quad (41)$$

It is worth noting that when deriving formulas (39) and (40), the mass of the orbiting test-body  $B(m = m_B)$  was implicitly assumed to be smaller than that of the gravitational source  $A(M = m_A)$ . In other words, the mass-ratio  $q = m_B/m_A$  of the system  $\{A, B\}$  is close to zero, and the relative distance between  $A$  and  $B$  is considered to be sufficiently large. However, if  $q$  is not close to zero and the separation is not large enough, we should consider  $M = m_A + m_B$  as the total mass of the system  $\{A, B\}$ . This can be summarized as follows:

$$M = \begin{cases} m_A & \text{if } q \cong 0 \\ m_A(1+q) & \text{if } q > 0 \end{cases}, \quad q = m_B/m_A. \quad (42)$$

Before investigating the CGA-effects of on the orbital motion of major asteroids, it is best to familiarize the reader with the computation process when applying the CGA-formalism. For this pedagogical purpose, we have selected the Moon and asteroid Ceres to study the CGA-effects of on the orbital motion of these two important celestial bodies.

*CGA-effects on the orbital motion of the Moon:* In the system  $\{\text{Earth, Moon}\}$ , the Earth plays the role of the principal gravitational source  $A$ , and the Moon has the role of test-body  $B$ . According to [3], the following orbital and physical parameters of the Moon are  $e=0.0549$ ,  $a=3.844 \times 10^8$  m,  $p=27.322$  d =  $2.360620 \times 10^6$  s,  $m_B=7.3477 \times 10^{22}$  kg. For the values of the Earth's mass and the physical constants, we take  $m_A=5.9722 \times 10^{24}$  kg,  $G=6.67384 \times 10^{-11}$  m<sup>3</sup> kg<sup>-1</sup>s<sup>-2</sup>,  $c=299\,792\,458$  ms<sup>-1</sup> and the mass-ratio:  $q = m_B/m_A = 0.0123$ . As we can see,  $q$  is not sufficiently close to zero. Therefore, according to (42), we must consider the total mass  $M = m_A + m_B$  for the average magnitude of the dynamic gravitational field. After substituting all the above parameters into (41):

$$\Lambda = 3.188946 \times 10^{-14} \text{ ms}^{-2}. \quad (43)$$

Hence, in terms of the field, the Earth –as the main gravitational source in the system  $\{\text{Earth, Moon}\}$ – exerts on the Moon as a test-body an extra-gravitational field  $\Lambda$  of average magnitude (43). Or according to (22), the Earth exerts on the Moon an extra-gravitational force  $\mathbf{F}_D$  of average magnitude

$$F_D = 2.343141 \times 10^9 \text{ N}. \quad (44)$$

*CGA-secular perigee precession of the Moon:* In paper [3], we have already investigated the CGA-secular perigee precession of the Moon using formula (47) in [3]. We obtained a value in good agreement with 0.060 arcsec/cy found by De Sitter [9,10,11] who used GRT and some inaccurate

data available at that time (1916). However, if we employ the GR-formula, which is identical to (31), and modern accurate data, we find  $\Delta\varphi_{\text{GR}} = 0.060889 \text{ arcsec/cy}$ . The correct computation of the secular perigee precession of the Moon should be seen as a fact of extreme significance, particularly for alternative gravity theories. In what follows we perform the calculation with the help of the CGA-formula (39). Hence, by inserting the orbital and physical parameters into (39), we obtain

$$\begin{aligned}\Delta\varphi_{\text{CGA}} &= 2.215076 \times 10^{-10} \text{ rad/rev} \\ &= 2.215076 \times 10^{-10} \times \frac{180}{\pi} \times 3600 \times \left(\frac{36525}{27.322}\right) = 0.061109 \text{ arcsec/cy} .\end{aligned}\quad (45)$$

As we can see, the CGA-formula (39) gives the numerical value (45), which is good in agreement with the above prediction.

*CGA-effects on the orbital motion of asteroid Ceres:* Ceres is the largest and first discovered asteroid, found by Italian astronomer Piazzi on January 1, 1801. The following orbital and physical parameters are from NASA ([nssdc.gsfc.nasa.gov/planetary/factsheet/asteriodfact.html](http://nssdc.gsfc.nasa.gov/planetary/factsheet/asteriodfact.html)):  $e = 0.075$ ,  $a = 2.767 \text{ AU}$ ,  $1 \text{ AU} = 149.597870 \times 10^9 \text{ m}$ ,  $p = 4.60 \text{ yr}$ ,  $1 \text{ yr} = 365.25 \text{ d}$ ,  $m_{\text{C}} = 9.393 \times 10^{20} \text{ kg}$ . For the Sun's mass, we have  $m_{\odot} = 1.9891 \times 10^{30} \text{ kg}$  and for the mass-ratio, we have  $q = \frac{m_{\text{C}}}{m_{\odot}} = 4.722 \times 10^{-10}$ . Since  $q \cong 0$ , thus according to (42), we find, after inserting the orbital and physical parameters into (41):

$$\Lambda = 2.764521 \times 10^{-12} \text{ ms}^{-2} .\quad (46)$$

This means, among other things, that the Sun, as a principal gravitational source, exerts on Ceres, during its orbital motion, an extra-gravitational field  $\Lambda$  of average magnitude (46). Or according to (22), the Sun exerts on Ceres an extra-gravitational force  $\mathbf{F}_{\text{D}}$  of average magnitude

$$F_{\text{D}} = 2.60 \times 10^9 \text{ N} .\quad (47)$$

*CGA-secular perihelion precession of Ceres:* In addition to the importance of the secular perigee precession of the Moon, the investigation of Ceres' secular perihelion precession is itself a significant test-case for alternative theories of gravity. For example, the GR-secular perihelion precession is  $\Delta\varphi_{\text{GR}} = 0.303484 \text{ arcsec/cy}$ . This GR-prediction should be, of course, compared with the CGA-prediction in order to be sure of the CGA-formalism. So, by substituting the orbital and physical parameters into (39), we obtain

$$\Delta\varphi_{\text{CGA}} = 0.303324 \text{ arcsec/cy} .\quad (48)$$

This is in excellent accordance with the GR-prediction.

### 5.3. CGA-effects on the orbital motion of four major asteroids

In order to rigorously assess the theoretical predictions of the CGA-formalism, we have selected four major asteroids from the JPL small body database (<http://ssd.jpl.nasa.gov/sbdb.cgi>) and from ([nssdc.gsfc.nasa.gov/planetary/factsheet/asteroidfact.html](http://nssdc.gsfc.nasa.gov/planetary/factsheet/asteroidfact.html)). Their masses are extremely small compared to that of the Sun. This characteristic allows us to apply the same computation process

used for Ceres. The orbital and physical parameters are listed in Table 1, and the observed perihelion precession of each asteroid and CGA-effects are listed in Table 2.

ASTEROID	$a$ (AU)	$e$	$P$ (yr)	$m$ (kg)
<b>1862 Apollo</b>	1.470110	0.55993	1.785	$2.00 \times 10^{12}$
<b>2101 Adonis</b>	1.874470	0.76381	2.570	$1.80 \times 10^{12}$
<b>433 Eros</b>	1.457970	0.22263	1.760	$6.69 \times 10^{15}$
<b>1566 Icarus</b>	1.077903	0.82683	1.119	$1.00 \times 10^{12}$

**Table 1:** Orbital and physical parameters of four major asteroids.

Using the data listed in Table 1 and the CGA-formalism, we obtain the CGA-predictions, which are listed in Table 2 along with the observed secular perihelion precession of each asteroid.

Asteroid	$\Delta\varphi_{\text{obs}}$ (arcsec/cy)	Predicted CGA-effects		
		$\Lambda$ ( $\text{ms}^{-2}$ )	$F_D$ (N)	$\Delta\varphi_{\text{CGA}}$ (arcsec/cy)
<b>1862 Apollo</b>	2.1239	$1.842055 \times 10^{-11}$	36.8411	2.144459
<b>2101 Adonis</b>	1.9079	$8.892244 \times 10^{-12}$	16.0000	1.918676
<b>433 Eros</b>	1.6000	$1.889735 \times 10^{-11}$	$1.26423 \times 10^5$	1.573589
<b>1566 Icarus</b>	10.007	$4.676354 \times 10^{-11}$	46.76354	10.05435

**Table 2:** Observed secular perihelion precession of each asteroid and the predicted CGA-effects.

Table 2 provides a more conclusive illustration that, in addition to the average magnitude of couple the  $(\Lambda, F_D)$ , the predicted CGA-secular perihelion precession  $\Delta\varphi_{\text{CGA}}$  of each asteroid is in excellent agreement with the observed value  $\Delta\varphi_{\text{obs}}$ .

The reader, who is already familiar with CGA-formalism, has the natural right to ask about the status of CGA vis-à-vis the scientific community. Certainly, CGA as an alternative gravity theory could be regarded as a generalization and refinement of Newton's theory of gravitation, as we have previously seen in [2,3]. CGA is, in fact, a newborn gravitational model formulated in 2009 [1] compared to the old ones. However, if CGA is capable of predicting and explaining some old and new gravitational effects [1,2,3,4], this is due to the originality, simplicity, and coherence of its formalism, which has no adjustable parameters. The well-informed reader can judge CGA in its own context. Historically, GRT succeeded in providing a numerical value for the perihelion precession of Mercury [12] and attributed this additional rotation of the line of apsides to the space-time that curves around the Sun.

However, as demonstrated in [3], CGA has predicted and explained the same effect in the framework of Euclidean geometry and the Galilean relativity principle. We have already shown in the present paper that the CGA-effects are mainly due to the couple  $(\mathbf{\Lambda}, \mathbf{F}_D)$ . Again, in order to convince the reader more conclusively of the excellent agreement between the CGA-predictions and observations, let us investigate the perihelion precession of the inner planets using the newly derived formula (39). The average magnitude of the couple  $(\mathbf{\Lambda}, \mathbf{F}_D)$  has been previously calculated (see, *e.g.*, Table 1 in Ref. [3]).

#### 5.4. CGA-secular perihelion precession of inner planets

In Table 3, the orbital and physical parameters of each inner planet are listed. Some data are adopted from [3].

PLANET	$a$ (m)	$e$	$P$ (yr)
<b>Mercury</b>	$57.92 \times 10^9$	0.2056	0.241
<b>Venus</b>	$108.25 \times 10^9$	0.0068	0.615
<b>Earth</b>	$149.60 \times 10^9$	0.0167	1.000
<b>Mars</b>	$227.95 \times 10^9$	0.0934	1.881

**Table 3:** Orbital and physical parameters of inner planets.

With the help of the data listed in Table 3 and the newly derived formula (39), we obtain the CGA-predictions  $\Delta\varphi_{CGA}$  displayed in Table 4 along with the observed secular perihelion precession  $\Delta\varphi_{obs}$  of each inner planet.

PLANET	$\Delta\varphi_{obs}$ (arcsec/cy)	$\Delta\varphi_{CGA}$ (arcsec/cy)
<b>Mercury</b>	43.1000	43.042580
<b>Venus</b>	8.0000	8.622262
<b>Earth</b>	5.0000	3.844435
<b>Mars</b>	1.3624	1.352923

**Table 4:** Observed secular perihelion precession and CGA-predictions for each inner planet are listed for comparison.

As before, from Table 4, the reader can easily observe the good agreement between CGA-theoretical predictions and observations. Since GRT has already arrived at the same results, it is necessary to say a few words about CGA and GRT. In the CGA-context, the secular perigee precession of the Moon and secular perihelion precession of planets and asteroids, as extra-gravitational effects, are originally caused by the action of the couple  $(\mathbf{\Lambda}, \mathbf{F}_D)$ . However, in the GR-context, the above-mentioned secular effects are causally attributed to the curvature of space-time around the Sun. Regarding CGA, the reader can refer back to Eqs.(30) and (39) to see the dependence of orbital eccentricity  $e$  on the average magnitude of  $\mathbf{F}_D$  and the dependence of perigee and/or perihelion precession  $\Delta\varphi$  on the average magnitude of  $\mathbf{\Lambda}$ . Remembering the discussion in connection with a certain similarity between CGA and GRT, particularly Eqs.(18) and (19) containing CGA-effects and GR-effects, respectively.

### 5.5. CGA-effects on the orbital motion of binary pulsars

Before the advent of the CGA, it was usually claimed that the study of compact stellar objects like, neutron stars and pulsars exclusively belonged to the GR-domain because their strong compactness is enough to bend local space-time in such a way that observable GR-effects should occur. However, this claim has already been refuted in [3]. Once again, we consider the reader as an intellectual witness to focus his attention on the fact that the CGA is very capable of investigating its proper effects, even, in compact stellar objects. More precisely, we will see that in addition to the couple  $(\mathbf{\Lambda}, \mathbf{F}_D)$ , there is another important effect, namely, the CGA-apsidal motion in binary pulsar systems which, has been studied earlier in [3]. This effect is very similar to perihelion precession and should be defined by the same newly derived formula (39). Indeed, if we take the usual notation for the apsidal motion rate  $\dot{\omega}$  instead of  $\Delta\varphi$ , we obtain

$$\dot{\omega} = \left[ \frac{3p^2}{2\pi a(1-e^2)} \right] \Lambda. \quad (49)$$

The high compactness of pulsars implies that the resulting gravitational fields near the pulsar's surface are large, thus enabling strong-field tests of alternative gravity theories. Furthermore, pulsars and their orbiting companions are generally compact enough that their motion can be treated as that of two point-masses. Therefore, in the CGA-context, we can logically consider each pulsar as the main gravitational source and the companion as the test-body. Consequently, the causal source of CGA-effects in the binary pulsar systems is exactly of the same nature as for ordinary (non-compact) eclipsing binary star systems [2,3,4].

Hence, the combined gravitational field (21) becomes stronger as the pulsar and its companion are so close together that an ordinary star like the Sun could not fit in their orbits. As a direct result, the couple  $(\mathbf{\Lambda}, \mathbf{F}_D)$  should have its intensity amplified drastically. That is why, for example, the value of CGA-apsidal motion rate of binary pulsar systems should be more significant than that of ordinary eclipsing binary star systems. Just like in previous investigations [3], the determination of GCA-effects in binary pulsars should show us, among other things, that the usual GRT-interpretation of gravity as a deformation of space-time is not a physical reality but a pure topological property of Riemann geometry, which is conceptually non-Euclidean. We have displayed in Table 5 the orbital and physical parameters of each binary pulsar. The Sun's mean radius ( $R_\odot = 695700$  km) and mass ( $m_\odot = 1.9891 \times 10^{30}$  kg) are used as units for each binary pulsar's semi-major axis and mass.

PULSAR	$P$ (d)	$e$	$a/R_\odot$	$m_A/m_\odot$	$m_B/m_\odot$	REF.
<b>PSR B 1913+16</b>	0.322997	0.6171	2.803849	1.4414	1.3867	a
<b>PSR B 1534+12</b>	0.420	0.2740	3.280619	1.3400	1.3400	b
<b>PSR J 0737-3039</b>	0.102251	0.0877	1.265262	1.3380	1.2490	c

**Table 5:** Orbital and physical parameters of each pulsar and its companion  
**Ref.:** a) Weisberg and Taylor [12]; b) Nice *et al.*, [13]; c) Kramer *et al.*, [14]

Since the mass-ratio,  $q = m_B/m_A$ , of each binary pulsar is of the order of unity, according to (42), we can take  $M = m_A + m_B$  as the total mass of the system for the average magnitude of  $\Lambda$  and also for the CGA-apsidal motion rate  $\dot{\omega}_{\text{CGA}}$ . So, with the aid of Table 5, we have calculated and displayed in Table 6 the CGA-effects with the observed apsidal motion rate  $\dot{\omega}_{\text{obs}}$  of each binary pulsar.

PULSAR	$\dot{\omega}_{\text{OBS}}$ (deg/yr)	Predicted CGA-effects		
		$\dot{\omega}_{\text{CGA}}$ (deg/yr)	$\Lambda$ ( $\text{ms}^{-2}$ )	$F_{\text{D}}$ (N)
<b>PSR B 1913+16</b>	4.226595	4.223614	$2.114670 \times 10^{-4}$	$5.832862 \times 10^{26}$
<b>PRS B 1534+12</b>	1.756	1.760	$1.185548 \times 10^{-4}$	$3.159952 \times 10^{26}$
<b>PSR J 0737-3039</b>	16.900	16.83559	$1.925609 \times 10^{-3}$	$4.783955 \times 10^{27}$

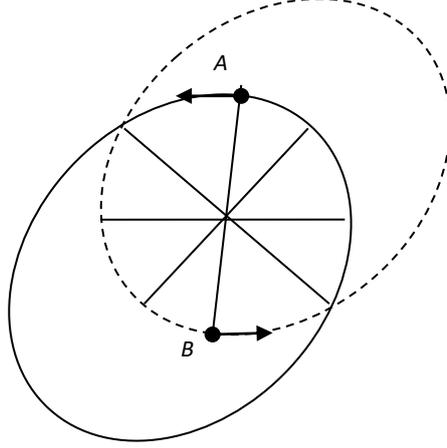
**Table 6:** Observed apsidal motion rate and predicted values of CGA-effects.

As the reader can easily see, in addition to the other CGA-effects, Table 6 reveals the excellent agreement between the predicted CGA-apsidal motion rate and the observed rate for each binary pulsar.

## 6. Conversion Rate of Orbital Energy into Gravitational Radiation

In the GRT-context, the orbital energy loss is attributed to gravitational radiation, specifically gravitational waves emitted by bound binary systems. In the CGA-context, we prefer to use the term "conversion" instead of "loss" because the CGA attributes the mechanism of converting orbital energy into gravitational radiation to the variation of work done by the dynamic gravitational force  $\mathbf{F}_{\text{D}}$  during the orbital motion of the binary system  $\{A, B\}$ . This is because  $\mathbf{F}_{\text{D}}$  is considered to play a dual role: as an extra-gravitational force and as a perturbation force.

*Explanation:* Since the relative orbit of the main gravitational source  $A$  of mass  $m_A$  and the test-body  $B$  of mass  $m_B$  is elliptical, more precisely, the binary system  $\{A, B\}$  may contain either a single elliptical orbit or each body would travel in its own separate elliptical orbit as illustrated in Figure 1 below.



**Figure 1:** Orbit of the hypothetical binary star system  $\{A, B\}$ .

Thus the two stars are closer together at some times than at others, so the couple  $(\mathbf{A}, \mathbf{F}_D)$  alternately strengthens at periastron and weakens at apastron. Therefore, dynamically, a certain amount of work  $\delta W$  is performed by  $\mathbf{F}_D$  and is defined as follows

$$\delta W = \mathbf{F}_D \cdot \delta \mathbf{s} , \quad (50)$$

where  $\delta \mathbf{s}$  is a displacement vector.

Or equivalently, we can rewrite Eq.(50) as follows

$$\delta W = F_D \cdot \delta s \cos \theta , \quad (51)$$

where  $\theta$  is the angle between the force vector and the displacement vector and

$$F_D = \frac{\mu}{a} \left[ \frac{GM}{ca} \right]^2 , \quad \mu = m_A m_B M^{-1}, \quad M = m_A + m_B . \quad (52)$$

The presence of the reduced mass  $\mu$  is quite natural since we are dealing with a binary system consisting of two bodies rotating about their common center of mass (Fig.1). Now, supposing that Kepler's third law ( $r^3/p^2 = GM/4\pi^2$ ) is accurate enough to be applied here<sup>4</sup>. Thus, according to the above considerations,  $\delta W$  should have its maximum value at periastron  $r \rightarrow r_{\min}$  and its minimum value at apastron  $r \rightarrow r_{\max}$ . Conversely, at periastron, the orbital period should be  $p \rightarrow p_{\min}$  and at apastron  $p \rightarrow p_{\max}$ . Taking into account the fact that  $\mathbf{F}_D$  as a perturbation force will cause a change in the work  $\delta W$  during the time interval  $\delta p = p_{\max} - p_{\min}$ . Therefore, the rate of change in  $\delta W$  is called, in the CGA-context, *average gravitational radiation* and is given by

$$\dot{\epsilon} = \frac{\delta W}{\delta p} . \quad (53)$$

<sup>4</sup> Actually, Kepler's third law  $r^3/p^2 = GM/4\pi^2$  has been generalized in the context of CGA and takes the form:  $r^3/p^2 = GM/4\pi^2 [1 + GM/c^2 r]$ .

Because  $\delta s$  should be infinitesimal compared to the relative separation between  $A$  and  $B$ , it is best to take  $\delta s = \frac{1}{\pi} r_G$  where  $r_G = GM/c^2$  is the gravitational radius of the system  $\{A, B\}$ . Thus, for the case  $\theta \cong 0^\circ$ , Eq. (51) becomes  $\delta W = F_D \cdot \delta s$ , and consequently, Eq. (53) takes the form:

$$\dot{\epsilon} = F_D \frac{\delta s}{\delta p} = \frac{1}{\pi} \frac{\mu}{a} \left[ \frac{GM}{ca} \right]^2 \frac{r_G}{\delta p}. \quad (54)$$

Let us focus our attention on the orbital period. We have from Kepler's third law:

$$p_{\min}(r \rightarrow r_{\min}) = \frac{2\pi r_{\min}^{3/2}}{\sqrt{GM}} \quad \text{and} \quad p_{\max}(r \rightarrow r_{\max}) = \frac{2\pi r_{\max}^{3/2}}{\sqrt{GM}}.$$

Since we are dealing with an elliptical orbit, we have  $r_{\max} = a(1+e)$  and  $r_{\min} = a(1-e)$ , and  $\delta p$  takes the form

$$\delta p = p_{\max} - p_{\min} = \frac{2\pi a^{3/2}}{\sqrt{GM}} \left[ (1+e)^{3/2} - (1-e)^{3/2} \right]. \quad (55)$$

By substituting (55) into (54), and performing some algebraic calculation, we get

$$\dot{\epsilon} = \frac{1}{2\pi^2} \frac{\mu}{\sqrt{GMa}} \left[ \frac{GM}{ca} \right]^4 \mathcal{G}(e), \quad (56)$$

with

$$\mathcal{G}(e) = \left[ (1+e)^{3/2} - (1-e)^{3/2} \right]^{-1}. \quad (57)$$

Because of the principle of conservation of energy, the average gravitational radiation  $\dot{\epsilon}$  should be equal to the change in the total orbital energy  $\dot{E}_{\text{tot}}$  of the system under consideration, *i.e.*, we should have

$$\dot{\epsilon} = \dot{E}_{\text{tot}}. \quad (58)$$

In view of the fact that the average orbital velocity is  $v_{\text{orb}} = \sqrt{GM/a}$ , according to Eq.(57), the change in orbital energy is of the order  $(v_{\text{orb}}/c)^4$ . Furthermore, it is clear from the expression in Eq.(56) that the average amount of gravitational radiation explicitly depends on the orbital eccentricity via the factor  $f(e)$  defined by (58). Finally, as we can see from Eq.(56), it defines the average rate of gravitational emission energy from the system  $\{A, B\}$ . This average rate  $\dot{\epsilon}$  has the physical dimensions of power, and its units are  $\text{J s}^{-1}$  or Watt. In the CGA-context, we can refer to  $\dot{\epsilon}$  as gravitational power. Also,  $\dot{\epsilon}$  is strictly positive, which is not new because the natural sign of power in electricity and mechanics is always positive. Moreover, Peters and Mathews, in their very interesting work [15], derived a formula for the average rate at which the system radiates energy. This rate is also strictly positive. In [15], their formula is numbered (16) and located on page 437, first column. Here, we rewrite it exactly in its original form as follows:

$$\langle P \rangle = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5 (1-e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right). \quad (i)$$

Thus, originally, Peters and Mathews derived a formula with a strictly positive sign and used  $\langle P \rangle$  as an average power. Unfortunately, many authors wrote formula (i) with a negative sign (-) and wrongly attributed it to Peters and Mathews [15]. It seems they had not even seen the original article

in which the formula in question was expressed without a negative sign. In passing, with the help of formula (i), the total orbital energy, and Kepler's third law, one can easily derive, after differentiation with respect to time, the following expressions for the semi-major axis decay rate and orbital period decay rate of the binary system  $\{ m_1, m_2 \}$ :

$$\dot{a} = \frac{64}{5} \frac{G^3 m_1 m_2 (m_1 + m_2)}{c^5 a^3 (1 - e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right), \quad (\text{ii})$$

$$\dot{p}_{\text{orb}} = \frac{192\pi}{5} \frac{G^{5/2} m_1 m_2 (m_1 + m_2)^{1/2}}{c^5 a^{5/2} (1 - e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right). \quad (\text{iii})$$

As we can see, both expressions (ii) and (iii) have preserved the positive sign of expression (i). This is contrary to certain authors who wrote these formulae with a negative sign.

## 6.1. Consequences of Gravitational Radiation Damping

Five consequences may occur from gravitational radiation damping, namely, (1) a decrease in the orbital period: The progressive conversion of orbital energy of the binary system  $\{A, B\}$  into gravitational radiation causes a gradual decay of the orbital period; (2) a decrease in the orbital separation: Similar to case (1), the conversion of orbital energy leads to a decrease in the orbital separation between  $A$  and  $B$ ; (3) change in the orbital eccentricity; (4) change in the orbital angular momentum; (5) gravitational coalescence: Due to the conversion of orbital energy, the orbits are gradually shrinking, and coalescence should be imminent, at least in the long term.

It is worth noting that in the framework of Newton's gravity theory, these orbital parameters are constants of motion. However, in the CGA-context, they will be functions of time that will gradually vary.

### 6.1.1. Orbital period decay rate

The parameters  $a$ ,  $p$ , and  $e$  are related to the total orbital energy  $E_{\text{tot}}$ , Kepler's third law, and orbital angular momentum, respectively, through the following equations:

$$E_{\text{tot}} = -\frac{Gm_A m_B}{2a}, \quad (59)$$

$$\frac{p^2}{a^3} = \frac{4\pi^2}{GM}, \quad (60)$$

$$\ell = \mu \sqrt{GMa(1 - e^2)}. \quad (61)$$

By combining equations (59) and (60) via differentiation with respect to time, and taking into account equation (58), we obtain – after substitution and some algebraic calculation – the following expression for the orbital period decay rate caused by gravitational radiation damping:

$$\dot{p} = \frac{3}{\pi} \left[ \frac{GM}{c^2 a} \right]^2 \mathcal{G}(e). \quad (62)$$

For the case of very low-orbital eccentricity ( $e \ll 1$ ), we can estimate the orbital period decay rate by:

$$\dot{p} \approx \frac{7}{2\pi} \left[ \frac{G\mu}{c^2 a} \right]^2. \quad (63)$$

### 6.1.2. Semi-major axis decay rate

By following the above process exactly, we obtain an expression for the semi-major axis decay rate:

$$\dot{a} = \frac{2}{\pi} \left[ \frac{GM}{c^2 a} \right]^2 \frac{a}{p} \mathcal{G}(e). \quad (64)$$

For the case of very low-orbital eccentricity ( $e \ll 1$ ), we can estimate the semi-major axis decay rate by:

$$\dot{a} \approx \frac{2}{\pi} \left[ \frac{G\mu}{c^2 a} \right]^2 \frac{a}{p}. \quad (65)$$

### 6.1.3. Secular change in the orbital eccentricity

To calculate the rate of secular change in the orbital eccentricity, we must differentiate Eq.(55) with respect to time. With the help of Eq.(64), we find

$$\dot{e} = -\frac{\mathcal{h}(e) (GM)^{5/2}}{\pi^2 c^4 a^{7/2}}, \quad (66)$$

where

$$\mathcal{h}(e) = [(1+e)^{1/2} + (1-e)^{1/2}]^{-1}. \quad (67)$$

### 6.1.4. Secular change in the orbital angular momentum

By differentiating Eq.(61) with respect to time, we obtain

$$\dot{\ell} = \ell \left[ \frac{\dot{a}}{2a} - \frac{e}{(1-e^2)} \dot{e} \right], \quad (68)$$

substituting Eqs.(64) and (66) into (68) yields the following expected expression

$$\dot{\ell} = \frac{\ell(GM)^{5/2}}{\pi^2 c^4 a^{7/2}} \left[ \frac{1}{2} \mathcal{G}(e) + \frac{e}{(1-e^2)} \mathcal{h}(e) \right]. \quad (69)$$

For the case of very low-orbital eccentricity ( $e \ll 1$ ), we can estimate the secular change in the orbital angular momentum by:

$$\dot{\ell} \approx \frac{\ell(G\mu)^{5/2}}{\pi^2 c^4 a^{7/2}}. \quad (70)$$

### 6.1.5. Gravitational coalescence (total) time

*Gravitational coalescence mean-time:* In the CGA-context, the gravitational coalescence mean-time is the mean-lifetime of a binary system's orbit. We can derive an expression for the gravitational coalescence mean-time by integrating Eq.(64). Thus, by assuming that the initial binary separation decreases and that Eq.(64) is always valid, we obtain the following equation after integration:

$$t = \frac{2\pi^2 c^4 a^{7/2}}{7(GM)^{5/2}} \mathcal{G}^{-1}(e), \quad (71)$$

where

$$\mathcal{G}^{-1}(e) = [(1+e)^{3/2} - (1-e)^{3/2}].$$

Eq.(71) is called “gravitational coalescence mean-time” because it only predicts the mean-time for the orbital radius to shrink by 50% of the initial radius. Finally, the gravitational coalescence (total) time should be equal to two times the gravitational coalescence mean-time (71):

$$\tau = \frac{4\pi^2 c^4 a^{7/2}}{7(GM)^{5/2}} \mathcal{G}^{-1}(e) . \quad (72)$$

For the case of very low-orbital eccentricity ( $e \ll 1$ ), we can estimate the gravitational coalescence (total) time by:

$$\tau \approx \frac{4\pi^2 c^4 a^{7/2}}{7(G\mu)^{5/2}} . \quad (73)$$

## 7. Fourth Numerical Test of CGA

It is worth recalling that the CGA, as an alternative gravity theory, has already passed three numerical tests inside and outside the Solar System. These tests are summarized in Tables 2, 4, and 6. Now, we are concerned with the fourth numerical test of CGA by investigating the effects of gravitational radiation damping on the variation of orbital elements of the binary pulsar PSR B1913+16 and the double pulsar PSR J0737-3039, respectively. Thus, this fourth numerical test should be split into two separate parts.

*First Part:* Since their discovery, binary pulsars have been predominantly considered an exceptional astrophysical laboratory, particularly for testing alternative and competing gravitational theories. Let us focus our attention on the most famous binary pulsar, PSR B1913+16. For instance, as shown in Table 6, there is excellent agreement between the predicted CGA-apsidal motion rate and the observed one for each binary pulsar. We have selected this binary pulsar because, over the past 40 years, among the best-known results are the values of its orbital parameters and coalescence time. With the aid of Table 5 and by using Eqs.(62), (64), and (72), the predicted CGA-values of  $\dot{p}_{CGA}$ ,  $\dot{\omega}_{CGA}$ , and  $\tau_{CGA}$  can be calculated and compared with the observed/measured values of orbital elements and coalescence time from [16]. The outcomes are displayed in Table 7.

PULSAR	Observed/Measured values			Predicted CGA-values		
	$\dot{p}_{OBS}$ (s/yr)	$\dot{a}_{OBS}$ (m/yr)	$\tau_{measured}$ (yr)	$\dot{p}_{CGA}$ (s/yr)	$\dot{a}_{CGA}$ (m/yr)	$\tau_{CGA}$ (yr)
<b>PSR B 1913+16</b>	$7.6 \times 10^{-5}$	3.5	$3 \times 10^8$	$7.603 \times 10^{-5}$	3.540	$3.146 \times 10^8$

**Table 7:** Observed values of orbital parameters and measured coalescence time are compared with the predicted CGA-values for the binary pulsar PSR B1913+16.

Once again, as the reader can notice, Table 7 shows the good agreement between the observations/measurements and CGA-predictions. To distinguish CGA from other alternative and competing gravitational theories, it is useful to summarize the results of Table 7 by the ratio of observed/measured values to expected CGA-values:

$$\frac{\dot{p}_{obs}}{\dot{p}_{CGA}} = 0.999 \quad ; \quad \frac{\dot{\omega}_{obs}}{\dot{\omega}_{CGA}} = 0.988 \quad ; \quad \frac{\tau_{measured}}{\tau_{CGA}} = 0.953 .$$

*Second Part:* PSR J0737-3039 is the first double pulsar discovered in April 2003[17]. It is particularly considered an ideal stellar laboratory for testing alternative and competing gravity theories in an extremely strong gravitational field ( $g \approx 2 \times 10^{12} \text{ ms}^{-2}$ ). As we can see from Table 5, the orbital eccentricity of the double pulsar is ( $e \ll 1$ ), which is why we are required to use Eqs.(63) and (73) to calculate the orbital period decay rate and gravitational coalescence (total) time for comparison with the observed/measured values from [17,18]. The results are displayed in Table 8.

PULSAR	Observed/Measured values		Predicted CGA-values	
	$\dot{p}_{\text{OBS}}$ (s/yr)	$\tau_{\text{measured}}$ (yr)	$\dot{p}_{\text{CGA}}$ (s/yr)	$\tau_{\text{CGA}}$ (yr)
PSR J0739 - 3039	$3.951 \times 10^{-5}$	$85 \times 10^6$	$4.134 \times 10^{-5}$	$85.62 \times 10^6$

**Table 8:** Observed value of orbital period decay rate and measured coalescence time compared with the predicted CGA-values for the double pulsar PSR J0739-3039.

Like before, to distinguish between CGA and other alternative and competing gravity theories, we recapitulate the results of Table 8 by calculating the ratio of observed/measured values to expected CGA-values:

$$\frac{\dot{p}_{\text{obs}}}{\dot{p}_{\text{CGA}}} = 0.955 \quad ; \quad \frac{\tau_{\text{measured}}}{\tau_{\text{CGA}}} = 0.993 .$$

It is important to remember that before the emergence of the CGA as an alternative gravity theory, it was always emphasized that the study of compact stellar objects exclusively belonged to the domain of GR because their strong compactness was sufficient to bend local space-time in a way that observable GR-effects should occur. However, as we have seen, the CGA is also capable of investigating, predicting, and explaining the same type of effects in compact stellar objects, all within the framework of Euclidean geometry and the Galilean relativity principle. This demonstrates a concrete fact that the propagation of the gravitational field and the action of gravitational force are both independent of the topology of space-time, and the *geometrization* of gravity is unnecessary.

## 8. Conclusion

Based solely on the CGA-formalism, we have demonstrated the causal dependence of orbital eccentricity on the average magnitude of dynamic gravitational force. Additionally, a new formula for the secular perigee and perihelion precession has been derived. This formula is proven to be exclusively dependent on the average magnitude of the dynamic gravitational field and is successfully applied inside and outside the Solar System. Furthermore, it is found that the CGA-predictions are, at the same time, in excellent agreement with observations and GR-predictions. This fact is mainly due to the existence of a certain similarity between the approximate GR-equations of motion and CGA-ones. Finally, the conversion rate of orbital energy into gravitational radiation is calculated, and its effects are studied.

## References

- [1] M.E. Hassani, *Galilean Electrodynamics* **20**, Special Issues 3, 54 (2009)
- [2] M.E. Hassani, *Galilean Electrodynamics* **22**, Special Issues 3, 43 (2011)
- [3] M.E. Hassani, *Galilean Electrodynamics* **26**, Special Issues 6, 42 (2015)  
Available at: <https://doi.org/10.6084/m9.figshare.1336063.v1>
- [4] M.E. Hassani, *Galilean Electrodynamics* **27**, Special Issues 1, 3 (2016)  
Available at: <https://doi.org/10.6084/m9.figshare.1336076.v1>
- [5] B. Shahib-Saless and D.K. Yeomans, *AJ*, **107**, 1885 (1994)
- [6] J.D. Norton, *Archive for History of Exact Sciences*, 15. XII, volume **45**, Issue 1, pp 17-94 (1992)
- [7] H.A. Lorentz, *Proceedings of the Royal Netherlands Academy of Arts and Sciences*, **2**, 559 (1900)
- [8] V.B. Braginsky, M.C. Caves and K.S. Thorn, *Phys. Rev D*. **15**, 2047 (1977)
- [9] W. De Sitter, *MNRAS* **76**, 699 (1916)
- [10] W. De Sitter, *MNRAS* **77**, 155 (1916)
- [11] W. De Sitter, *MNRAS* **78**, 3 (1917)
- [12] J.M. Weisberg and J.H. Taylor, arXiv: 0407149v1 (2004)
- [13] D.J. Nice, R.W. Sayer and J. H. Taylor, *ApJ*, **466**, L87 (1996)
- [14] M. Kramer et al., in: P. Chen, E. Bloom, G.Madjeski, V.Patrosian. (Eds.), *Proceedings of the 22<sup>nd</sup> Texas Symposium on Relativistic Astrophysics*. (2005)
- [15] P.C. Peters and J. Mathews, *Phys.Rev.* **131**, 445 (1963)
- [16] J.M. Weisberg, J.H. Taylor and L.A. Fowler, *Scientific American* **245**, 74 (1981)
- [17] M. Burgay, *et al.*, *Nature* **426**, 531 (2003)
- [18] M. Kramer, *Séminaire Poincaré IX*, 63-89 (2006)