Functions of multivectors in 3D Euclidean geometric algebra via spectral decomposition (for physicists and engineers)

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Geometric algebra is a powerful mathematical tool for description of physical phenomena. The article [3] gives a thorough analyze of functions of multivectors in $Cl_3$ relaying on involutions, especially Clifford conjugation and complex structure of algebra. Here is discussed another elegant way to do that, relaying on complex structure and idempotents of algebra. Implementation of $Cl_3$ using ordinary complex algebra is briefly discussed.

Keywords: function of multivector, idempotent, nilpotent, spectral decomposition, unipodal numbers, geometric algebra

1. Numbers

Geometric algebra is a promising platform for mathematical analysis of physical phenomena. The simplicity and naturalness of the initial assumptions and the possibility of formulation of (all?) main physical theories with the same mathematical language imposes the need for a serious study of this beautiful mathematical structure. Many authors have made significant contributions and there is some surprising conclusions. Important one is certainly the possibility of natural defining Minkowski metrics within Euclidean 3D space without the need for the introduction of negative signature, that is, without the need to define time as the fourth dimension ([1, 6]).

In Euclidean 3D space we define orthogonal unit vectors $e_1, e_2, e_3$ with the property

$$e_i^2 = 1, e_ie_j + e_je_i = 0,$$

so one could recognize the rule for multiplication of Pauli matrices. Each element of algebra ($Cl_3$) can be expressed as linear combination of elements of $2^3$ – dimensional basis (Clifford basis)

$$\{1, e_1, e_2, e_3, e_1e_2, e_3e_1, e_2e_3, e_1e_2e_3\},$$

where we have a scalar, three vectors, three bivectors and pseudoscalar. According to the number of unit vectors in the product we are talking about odd or even elements. If we define $j = e_ie_j$, it is easy to show that pseudoscalar $j$ has two interesting properties in $Cl_3$: 1) $j^2 = -1, \ 2) \ jX = Xj$, for any element $X$ of algebra, and behaves like an ordinary imaginary unit, which enables as to study a rich complex structure of $Cl_3$. This property we have for $n = 3, 7, \ldots$ [3]. Bivectors can be expressed as product of pseudoscalars and vectors, $j\bar{v}$.

We define a general element of algebra (multivector)

$$M = t + \bar{x} + j\bar{u} + jb = z + F, \quad z = t + jb, \quad F = \bar{x} + j\bar{u}$$
where $z$ is complex scalar and an element of center of algebra, while $F$, by analogy, is a complex vector. Complex conjugation is $z^* = z^t - jb$, $F^* = F^t = \bar{x} - j\bar{n}$. The complex structure allows different ways of expressing multivectors, one is

$$M = t + \bar{x} + j\bar{n} + jb = t + j\bar{n} + j(b - j\bar{x}),$$

where multivector of the form $a + vj$ belongs to even part of algebra and can be associated with rotations, spinors or quaternions. Also we could treat multivector as $(\mathbf{1})$ and implement it relying on ordinary complex numbers.

Main involutions in Clifford algebra are:

1) grade involution: $M^t = t - \bar{x} + j\bar{n} - jb$
2) reverse (adjoint): $M^* = t + \bar{x} - j\bar{n} - jb = z^* + F^*$
3) Clifford conjugation: $\bar{M} = t - \bar{x} - j\bar{n} + jb = \bar{z} + \bar{F} = z - F$,

where asterisk stands for a complex conjugate. Grade involution is transformation $\hat{x} = -\bar{x}$ (space inversion), while reverse in $\text{Cl}_3$ is like complex conjugation, $\bar{x}^t = \bar{x}$, $j^t = -j$. Clifford conjugation is combination of two involutions $\bar{M} = M^t$, $\bar{x} = -\bar{x}$, $j = j$. Bivectors given as a wedge product could be expressed as $\bar{x} \wedge \bar{y} = j\bar{x} \times \bar{y}$, where $\bar{x} \times \bar{y}$ is a cross product. Application of involutions is easy now.

Defining paravector $p = t + \bar{x}$ we have $p\bar{p} = |t + \bar{x}|^2 = (t + \bar{x})(t - \bar{x}) = t^2 - \bar{x}^2$ and we have usual metric of special relativity.

From $M = \hat{M} \Rightarrow M = t + j\bar{n}$, even part of algebra (spinors).

From $M = M^t \Rightarrow M = t + \bar{x}$, paravector; Reverse is anti-automorphism $(MM^t)^t = MM^t$, so $MM^t$ (square of multivector magnitude, (2)) is paravector.

From $M = \bar{M} \Rightarrow M = t + jb = z$, complex scalar. Clifford conjugation is anti-automorphism, $\bar{M}\bar{M} = M\bar{M}$, so $M\bar{M}$ (square of multivector amplitude, (2)) is a complex scalar and there is no other “amplitude” with such a property (1).

We define a multivector amplitude $|M|$ (hereinafter MA)

$$M\bar{M} = |M|^2 = t^2 - \bar{x}^2 + n^2 - b^2 + 2j\langle tb - \bar{x} \cdot \bar{n} \rangle, \quad |M| \in \mathbb{C} \quad (3)$$

which one could express as

$$\sqrt{M\bar{M}} = |M| = \sqrt{(z + F)(\bar{z} - F)} = \sqrt{z^2 - F^2}, \quad F^2 = x^2 - n^2 + 2j\bar{x} \cdot \bar{n} \in \mathbb{C}.$$
algebra ([1, 9]). Every idempotent in $Cl_3$ can be expressed as $u_\pm = p_\pm + N_\pm, N_\pm^2 = 0$, where $p_\pm$ are simple idempotents, like $p_\pm = (1 \pm e_1)/2$. For example, $u_+ = (1 + e_1 + e_2 + je_3)/2$, $N = e_2 + j e_3$

2. Implementation

From $M = \alpha_0 + \sum_{i=1}^3 \alpha_i e_i = \alpha_0 + A$, $\alpha_i \in \mathbb{C}$, it is easy to implement algebra on computer using ordinary complex numbers only. In [15] are defined products:

$$A \circ B = \sum_{i=1}^3 \alpha_\beta_i \text{ (generalized inner product)},$$
$$A \otimes B = \text{det} [e_i, \alpha_i, \beta_i] \text{, (generalized outer product)},$$ and
$$AB = A \circ B + A \otimes B \text{ (generalized geometric product)}.$$

Now we have $(\alpha_A + A)(\alpha_B + B) = \alpha_A \alpha_B + \alpha_B A + \alpha_A B + AB$. We can find $\alpha_i$ for multivector $M = t + \bar{x} + j \bar{u} + j b$ using linear independency.

3. Spectral decomposition

Starting from multivector

$$M = z + f = z + \sqrt{F^2} f = x + y f, \quad F^2 \neq 0$$

we see the form of unipodal numbers. Defining $M_x = x \pm y$ and recalling easy proofing relation $fu_\pm = \pm u_\pm$ follows

$$M u_\pm = (x \pm y f) u_\pm = (x \pm y) u_\pm = M_x u_\pm,$$

so we have projection. Spectral basis $u_\pm$ is very useful because binomial expansion of multivector is very simple

$$M^2 = (M_x u_+ + M_- u_-)^2 = M_x^2 u_+ + M_-^2 u_- \Rightarrow M^n = M_x^n u_+ + M_-^n u_-,$$ $n \in \mathbb{Z}$,

where $n < 0$ is possible for $|M| \neq 0$ ($M^{-1} = \bar{M}/(M \bar{M})$).

Defining conjugation $(a + bf)^\dagger = a - bf$ (obviously Clifford conjugation) we have
\[(a+bf)(a-bf) = a^2 - b^2, \text{ where } \sqrt{a^2 - b^2} = \sqrt{MM} = \sqrt{MM} \text{ is a multivector amplitude.}\]

In spectral basis using \(u_\pm = u_\pm\) we have \(MM = (M_+u_+ + M_+u_-)(M_-u_- + M_-u_+) = M_+M_-\).

Starting from \(M = z + F\) and \(\rho = \sqrt{x^2 - y^2}\) we have

\[
M = z + \sqrt{F^2} f = x + yf = \rho \left( \frac{x}{\rho} + \frac{y}{\rho} f \right) = \rho (\cosh \varphi + f \sinh \varphi),
\]

obtaining polar form of multivector. If multivector amplitude is zero we have light-like multivector and there is no polar form. Now defining \(\text{tanh } \varphi = \vartheta \) ("velocity") we have

\[
M = \rho \gamma (1 + \vartheta f), \quad \gamma^{-1} = \sqrt{1 - \vartheta^2}
\]

and in spectral basis

\[
M = \rho \gamma (1 + \vartheta f) = k_ux_+ + k_-u_- \Rightarrow k_+ = \rho \gamma (1 \pm \vartheta) = \rho K^{\pm 1},
\]

where \(K = \sqrt{(1 + \vartheta)/ (1 - \vartheta)}\) is generalized Bondi factor (\(\vartheta = \log K\)). Now we have

\[
(K_+u_+ + K_-u_-)(K_-u_- + K_-u_+) = K_1K_2u_+ + K_-K_-u_- \Rightarrow K = K_1K_2,
\]

or

\[
\gamma_1 \gamma_2 (1 + \vartheta_1 f)(1 + \vartheta_2 f) = \gamma_1 \gamma_2 (1 + (\vartheta_1 + \vartheta_2)f + \vartheta_1 \vartheta_2) = \\
\gamma_1 \gamma_2 (1 + \vartheta_1 \vartheta_2)(1 + (\vartheta_1 + \vartheta_2)f) \Rightarrow \\
\gamma = \gamma_1 \gamma_2 (1 + \vartheta_1 \vartheta_2), \quad \vartheta = (\vartheta_1 + \vartheta_2)/(1 + \vartheta_1 \vartheta_2),
\]

and we have "velocity addition rule". So, every multivector could be mathematically treated as an ordinary boost in special relativity. For \(\vartheta = 1\) we have a "boost" \(\gamma (1 + \vartheta f) = Ku_+ + K^{-1}u_-\) as transformation that preserves multivector amplitude and should be considered as part of Lorentz group \((1, 2)\). As a simple example of unit complex vector we already mentioned \(f = e_i + e_2 + je_3 = e_i + e_2 + e_i e_2\), completely in \(Cl_2\), suggesting that one could analyze problem in basis \((1, F_i)\) or related spectral basis and rotate all elements to obtain relations for arbitrary orientation of plane, using powerful apparatus of geometric algebra for rotations.

Mapping basis \((1, f)\) to \((e^{\vartheta f}, f e^{\vartheta f})\) we obtain new orthogonal basis and new components of multivector

\[
a + bf \rightarrow a'e^{\vartheta f} + b'fe^{\vartheta f} = (a' + b'f)e^{\vartheta f}, \quad a' = ae^{-\vartheta f}, \quad b' = be^{-\vartheta f}, \quad |a + bf| = |a' + b'f|.
\]

4. Functions of multivectors

Using series expansion it is straight forward to find closed formulae for (analytic at least) functions. If \(F^2 = 0 \Rightarrow f (M) = f (z)\) we have \(f (M) = f (z)\) and it is easy to find closed form using theory of functions on complex field. Otherwise, from the series expansion

\[
f (x) = f (0) + \sum_n \frac{f^{(n)} (0)}{n!} x^n
\]
using $M^2 = M^n u_+ + M^n u_-$ we have

$$f(M) = f(M_+) u_+ + f(M_-) u_-$$

and, again, it is easy to find closed form because of $M_\pm \in \mathbb{C}$. For $M = F = \sqrt{F^2}$ we have $M_\pm = \pm \sqrt{F^2} \Rightarrow f(M) = f\left(\sqrt{F^2}\right) u_+ + f\left(-\sqrt{F^2}\right) u_-$. If function is even we have $f(M) = f\left(\sqrt{F^2}\right)(u_+ + u_-) = f\left(\sqrt{F^2}\right) u_+$ and $f(F) = f\left(\sqrt{F^2}\right)(u_+ - u_-) = f\left(\sqrt{F^2}\right) f$ for odd functions. For $M = z + F$, $F^2 = N^2 = 0$ there is no spectral decomposition ($f$ is not defined), but we have $M^n = (z + N)^n = z^n + n z^{n-1} N$, giving $f(z + N) = f(z) + f'(z) N$. We also have special cases

$$f(u_\pm) = f(\pm1) u_\pm,
\quad f(f) = f(u_+ - u_-) = f(1) u_+ + f(-1) u_-,
\quad f(\hat{F}) = f(-ju_+ + ju_-) = f(-j) u_+ + f(j) u_-.$$

Obviously for odd functions is $f(f) = f(1) f$, $f(\hat{F}) = -f(j) f$, while for even functions is $f(f) = f(1)$, $f(\hat{F}) = f(j) f$.

For inverse functions we have

$$f^{-1}(y) = x \Rightarrow f(x) = y \Rightarrow f(x_\pm) = y_\pm \Rightarrow x_\pm = f^{-1}(y_\pm).$$

For light-like multivectors $(M M = 0)$ we have

$$M = z + \sqrt{F^2} f = z + z_F f, \quad z^2 - F^2 = 0 = (z - z_F)(z + z_F),$$

and two possibilities:

1) $z = z_F \Rightarrow M_+ = 2z_F$, $M_- = 0 \Rightarrow f(M) = f(2z_F) u_+$
2) $z = -z_F \Rightarrow M_+ = 0$, $M_- = -2z_F \Rightarrow f(M) = f(-2z_F) u_-$

Once a spectral decomposition of function is analyzed there remains just to use well known properties of functions of complex variables.

5. Examples

Here we define $M = z + F = z + \sqrt{F^2} f = z + z_F f$ and $M_+ = z + z_F$, $M_- = z - z_F$.

For inverse of $M$ we have $(|M| \neq 0)$

$$M^{-1} = \frac{1}{M_+ u_+ + M_- u_-} = \frac{M_+ u_+ + M_- u_-}{(M_+ u_+ + M_- u_-)(M_+ u_+ + M_- u_-)} = \frac{M_+ u_+ + M_- u_-}{M_+ M_-} = \frac{u_+}{M_+} + \frac{u_-}{M_-},$$

as expected. Now it is obvious that
\[ M^{-n} = \frac{1}{(M_+ u_+ + M_- u_-)^n} = \frac{u_+}{(M_+)^n} + \frac{u_-}{(M_-)^n} \].

We can find square root using
\[ \sqrt{M} = S = S_+ u_+ + S_- u_- \Rightarrow M = M_+ u_+ + M_- u_- = (S_+)^2 u_+ + (S_-)^2 u_- \Rightarrow S_\pm = \pm \sqrt{M_\pm} . \]

So, generally we have \( M^{\pm 1/n} = S \Rightarrow S_\pm = (M_\pm)^{\pm 1/n} \), \( n \in \mathbb{N} \). As a simple example (reader could compare with [3]) \( e^{1/n} \).

Exponential function is easy one, \( e^M = e^{M_+ u_+} + e^{M_- u_-} \) and now we have exponentials of complex numbers (just use ordinary \( i = \sqrt{-1} \) replacing \( i \to j \) at the end). Logarithm is inverse function to exponential, so we have
\[ \log M = X \Rightarrow e^X = M = M_+ u_+ + M_- u_- \Rightarrow X_\pm = \log M_\pm . \]

In [3] is derived formula \( \log M = \log |M| + \varphi \hat{F} \), \( \varphi = \arctan(|F|/z) \), but those are equivalent:
\[ z_F = \sqrt{F^2} = -j|F|, \quad \hat{F} = -jf \Rightarrow \]
\[ \log (M_+) u_+ + \log (M_-) u_- = \frac{1}{2} \log (M_+) \log (M_-) + \frac{1}{2} \log (M_+) - \log (M_-) = \frac{1}{2} \log |M| - jF \log \left( \frac{1 - j|F|/z}{1 + j|F|/z} \right) = \log |M| + \hat{F} \arctan(|F|/z) = \log |M| + \varphi \hat{F} . \]

Now we can find for \( a \in \mathbb{R} \) \( M^a = X \Rightarrow \log X = a \log M \Rightarrow X = e^{a \log M} \),
but the same appears to be correct for \( a = z + F \) and one can find, for example,
\[ M^v = X \Rightarrow \log X = \bar{v} \log M \Rightarrow X = e^{\bar{v} \log M} , \]
although here some caution is needed because of possibility \( \log X = (\log M) \bar{v} \Rightarrow X = e^{(\log M) \bar{v}} \). Also, relation \( M = e^{\log M} \) is generally not valid and needs some care due to the multivalued nature of the logarithm operation. Nevertheless, expressions like \( j^v = \exp(e_j \log j) = \exp(j \pi e_j/2) = je_j \), or \( (e_j e_2)^v = j \) are quite possible in \( Cl_3 \).

Trigonometric and hyperbolic trigonometric functions are straightforward and ctg one could obtain as inverse of tan. For example
\[ \sin(F) = \sin \left( \sqrt{F^2} \right) u_+ + \sin \left( -\sqrt{F^2} \right) u_- = \sin \left( \sqrt{F^2} \right) (u_+ - u_-) = F_+ \sin \left( \sqrt{F^2} \right) , \]
\[ \cos(F) = \cos \left( \sqrt{F^2} \right) u_+ + \cos \left( -\sqrt{F^2} \right) u_- = \cos \left( \sqrt{F^2} \right) (u_+ + u_-) = \cos \left( \sqrt{F^2} \right) . \]

In fact, series in powers of argument are crucial for all analytic functions (in geometric algebra too) and we can use presented spectral decomposition to obtain components of such functions in spectral basis.

**Conclusion**

Geometric algebra of Euclidean 3D space \( (Cl_3) \) is really rich in structure and gives possibility to analyze functions defined on multivectors, extending thus theory of functions of
real and complex variables, providing intuitive geometrical interpretation also. From simple fact that for complex vector \((F^2 \neq 0)\) we can write \(F / \sqrt{F^2} = f,\) \(f^2 = 1,\) \(F^2 \in \mathbb{C}\) follows nice possibility to explore idempotent structure \(u_\pm = (1 \pm f)/2\) and spectral decomposition of multivectors. Using orthogonality of spectral basis vectors (idempotents) \(u_\pm\) it is shown that all multivectors (even rotation operators, \([1]\)) can be treated as unipodal numbers (i.e. hypercomplex numbers over complex field). Definition of functions is then quite simple and natural and strongly counts on the theory of functions of complex variable. Complex numbers and vectors (bivectors, trivectors) are thus united in one promising system.

Appendix

A1. Bilinear transformations

Regarding that bilinear transformations of multivectors do not change some property of multivectors one could ask yourself: what property? In \([2]\) was shown that multivector amplitude, defined using Clifford conjugation which is unique involution that is commutative, belongs to center of algebra.

Now, let \(T\) be an operator on multivectors that does not change some property of multivectors. We have \(T(M') = T(XMY)\) so

1) \(X, M, Y\) generally have different orientations, so we could demand \(T(X) \in \mathbb{C}\) and
\[T(M') = T(X)T(M)T(Y)\]
2) \(T\) is linear and \(T(X) = 1\)
3) \(T(X)\) is well defined quantity for all (interesting) \(X\) in an unique way

From spectral decomposition we see that there is such quantity, namely, multivector amplitude, defined as \(M_+M_- \in \mathbb{C}\) using just natural conjugation \(a + bf \rightarrow a - bf\), where \(f\) is our hypercomplex unit. But this is just Clifford conjugation and we see now new meaning for it: just “hypercomplex conjugation”. This is strong argument for regarding Lorentz transformations to be group of bilinear transformations that preserve multivector amplitude. It is verified on paravectors giving known special relativity, but now we should extend it on whole multivector including in this way all multivector symmetries of Euclidean 3D space.

For multivector \(X\) (transformation) now we have \(T(X) = X_+X_- = 1\) and
\[X = e^M \Rightarrow M = \log X = \log |X| + \phi \hat{F} = \log 1 + \phi \hat{F} = \phi \hat{F} = F,\]
giving thus general bilinear (twelve parameters) transformation \(M' = e^{\psi_+\psi_-} M e^{\psi_+\psi_-}\).

A2. Hyperbolic inner and outer products

Given two multivectors \(M_1 = z_1 + z_1Ff\) and \(M_2 = z_2 + z_2Ff\) we define a square of multivector distance (conjugate products, \([13]\)) as
\[M_1M_2 = (z_1 - z_1Ff)(z_2 + z_2Ff) = z_1z_2 - z_1z_2Ff + (z_1z_2Ff - z_1z_2Ff) = hi + ho f,\]
where \(hi\) and \(ho\) stands for hyperbolic inner and hyperbolic outer products. If \(M_1 = M_2 = M\) we have \(M^\dagger M = z^2 - z^2_F + (zz_F - zz_F) f = z^2 - z^2_F\) is just square of multivector amplitude.
This suggests that \( hi \) and \( ho \) have to do something about being “parallel” or “orthogonal” besides being “near” and “close”. For complex and hypercomplex plane (with real coordinates) meaning is obvious (fig A2).

With \( ho = 0 \) multivectors are said to be “h-parallel”, while for \( hi = 0 \) they are “h-orthogonal”. For \( hi = ho = 0 \) multivector distance is null and we said it to be “h-light-like”, where h- stands for hyperbolic.

In “boost” formalism we have

\[
M_1 = M_2 + \rho_1 \rho_2 \gamma_1 \gamma_2 (1 - \delta_1 \delta_2),
\]

\[
ho = \rho_1 \rho_2 \gamma_1 \gamma_2 (\delta_2 - \delta_1) / (1 - \delta_1 \delta_2).
\]

So, “h-parallel” multivectors have equal “velocities” and \( hi = \rho_1 \rho_2 \), while for “h-orthogonal” multivectors “velocities” are reciprocal and \( ho \) becomes infinite (orthogonal multivectors belong to different hyperquadrants delimited by light-like hyper-planes).

**Lema:** Let \( M_1 M_2 = 0 \) for \( M_1 \neq 0 \) and \( M_2 \neq 0 \). Then \( M_1 M_2 \neq 0 \), and vice versa.

\[
M_1 M_2 = (M_1, u_+ + M_1, u_-) (M_2, u_+ + M_2, u_-) = M_1, M_2, u_+ + M_1, M_2, u_-,
\]

\[
M_1 M_2 = (M_1, u_- + M_1, u_+) (M_2, u_+ + M_2, u_-) = M_1, M_2, u_+ + M_1, M_2, u_-,
\]

and we have \( M_1^2 M_2 = 0 \) or \( M_1 M_2^2 = 0 \), which means \( M_1 = 0 \) and \( M_2 = 0 \) or \( M_1 = 0 \) and \( M_2 = 0 \), but either case gives \( M_1 M_2 \neq 0 \). Converse proof is similar.

**A3. Polynomials**

Suppose we have simple equation \( M^2 + 1 = 0 \), then objects that squares to -1 are solutions. Using spectral decomposition we could explore it further, so,

\[
M^2 + 1 = (M, u_+ + M, u_-)^2 + u_+ + u_- = (M^2 + 1) u_+ + (M^2 + 1) u_- = 0 \Rightarrow
\]

\[
M^2 + 1 = 0, \ M^2 + 1 = 0, \text{ so we have two equations over complex numbers. Obvious solutions are } \sqrt{-1}, j, je, \hat{F}, \text{ but it is possible to investigate further. There is an infinite number of solutions, obviously, due to algebraically expanded paradigm of number.}
\]

Another simple equation is \( M^2 = M^2 u_+ + M^2 u_- = 0 \). One obvious solution is \( M = N \) (nilpotent) which we cannot obtain using spectral decomposition (because there is no one for a multivector \( N \)), so, some caution is necessary.

In physics we are using a lot of special polynomials and their roots and we probably should reconsider those in geometric algebra.
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