An exact radial smooth type A solution to the Navier-Stokes equation.

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Abstract. In this paper it is demonstrated that the Navier Stokes equation has a smooth type A nontrivial exact solution combining two radial solutions inside and outside the unit sphere.

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1 Introduction

One of the Clay institute millenium problems is the yes or no existence of an exact solution of the Navier-Stokes equation for the velocity vector, with elements \( \{u_i\}_{i=1}^{3} \), matched with the pressure \( p \). We have \( u_i = u_i(x_1,x_2,x_3,t), (i = 1,2,3) \) and \( p(x_1,x_2,x_3,t) \) in the Navier stokes equation

\[
\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i - \nu \nabla^2 u_i + \frac{\partial}{\partial x_i} p = f_i
\] (1.1)

The function \( f_i \) is considered externally given. Furthermore, the solution, \( u_i \) in (1.1) must have finite energy. We have \( \nu > 0 \) and

\[
\int_{\mathbb{R}^3} \sum_{i=1}^{3} u_i^2(x_1,x_2,x_3,t) d^3x \leq C(t)
\] (1.2)

and a vanishing divergence \( \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i = 0 \). The idea is to demonstrate that an exact solution is possible or not given the requirements and the zero time initial conditions \( u_0,i(x_1,x_2,x_3) = u_i(x_1,x_2,x_3,0) \)

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2 Solution

Let us start to define $x_i = r_β_i$ for fixed $β_i, (i = 1, 2, 3)$ and $\sum_{i=1}^{3} β_i^2 = 1$. Here, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Subsequently, let us define a heuristic solution for $u_i = u_i(x_1, x_2, x_3, t)$, with,

$$u_i = \begin{cases} 
 c_i \exp[-at - b/r], & 0 < r \leq 1 \\
 (c_i/r) \exp[-at - br], & r \geq 1 
\end{cases} \quad (2.1)$$

with, $a > 0, b > 0$ real and $c_i \in \mathbb{R}$. The initial value function equals $u_{0,i}(x_1, x_2, x_3) = u_i(x_1, x_2, x_3, 0)$. The function in equation (2.1) is "sufficiently smooth" for $r > 0$ and $t > 0$.

2.1 Finite energy

In the inspection of the requirements, given in the introductory section, let us check (2.1) for finite energy. We note that generally the solution must show,

$$\int_{\mathbb{R}^3} \sum_{i=1}^{3} u_i^2(x_1, x_2, x_3, t) d^3x \leq C(t) \quad (2.2)$$

The $C(t)$ is finite. The angular terms of (2.1) give a finite contribution to the energy. Below it will be demonstrated that the velocity in radial terms, including the $r^2$ from the Jacobian $J = r^2 \sin \theta$, gives finite energy too. Firstly,

$$\int_{0}^{\infty} r^2 u_i^2(r, t) dr \leq C_i(t) \quad (2.3)$$

From the definition in (2.1) the requirement is

$$\int_{0}^{1} r^2 u_i^2(r, t) dr + \int_{1}^{\infty} r^2 u_i^2(r, t) dr \leq C_i(t) \quad (2.4)$$

Inside the unit sphere we see, for $b > 0$, $r^2 \leq 1$ together with $-\frac{b}{r} \leq -b$

$$\int_{0}^{1} r^2 u_i^2(r, t) dr \leq c_i^2 \exp[-2(at + b)] \quad (2.5)$$

Secondly, for $r \geq 1$, including the $r^2$ from the Jacobian

$$\int_{1}^{\infty} r^2 u_i^2(r, t) dr = c_i^2 e^{-2at} \int_{1}^{\infty} e^{-2br} dr \leq c_i^2 e^{-2at} \frac{e^{-2b}}{2b} \quad (2.6)$$

Here, $b > 0$ and finite real. Hence, from the previous equations (2.3)-(2.6) it follows that $C_i(t) \geq \max\{1, \frac{1}{2b}\} c_i^2 \exp[-2(at + b)]$ can be finite. The finite energy requirement is correctly observed for the solution in (2.1).
2.2 Vanishing divergence of the solution

If we suppose \(0 < r \leq 1\) then

\[
\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{3} e^{-at} c_i \frac{\partial}{\partial x_i} e^{-b/r} = \frac{be^{-at}}{r^2} \sum_{i=1}^{3} \beta_i c_i
\]

(2.7)

Hence, from the assumption

\[
\sum_{i=1}^{3} \beta_i c_i = 0
\]

(2.8)

it follows that \(\nabla \cdot u = 0\). Suppose then that, \(r \geq 1\). The requirement for \(r \geq 1\), is to have,

\[
\sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i = 0
\]

In this equation the product \(c_i \beta_i\) is identified and note, \(\sum_{i=1}^{3} c_i \beta_i = 0\). Hence, the required vanishing divergence also applies to the \(r \geq 1\) case.

2.3 Navier-Stokes for \(0 < r \leq 1\)

In the first part of the solution we have \(\frac{\partial}{\partial t} u_i = -au_i\). Subsequently, from \(\frac{\partial}{\partial x_i} u_i = \frac{bx_i}{r^3} u_i\)

\[
\sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i = \sum_{j=1}^{3} u_j \frac{bx_j}{r^3} u_i
\]

(2.10)

In (2.10) we may note the co-occurrence of \(c_j\) and \(x_j = \beta_j r\), so from (2.8) it follows that for \(0 < r \leq 1\) we have \(\sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i = 0\). In addition, the algebraic consequence of (2.1) for the Navier - Stokes is

\[
\frac{\partial^2}{\partial x_j^2} u_i = b \left\{ \frac{1}{r^3} - \frac{3r^2 x_j^2}{r^5} \right\} u_i + \frac{b^2 x_j^2}{r^6} u_i
\]

(2.11)

The previous algebraic excercise gives the following

\[
\nabla^2 u_i = \frac{b^2}{r^4} u_i
\]

(2.12)

Looking back at equation (1.1) gives for \(\frac{\partial}{\partial x_i} p\)

\[
- au_i - \nu r^4 u_i + \frac{\partial}{\partial x_i} p = f_i
\]

(2.13)

When \(p = p(r,t)\) it is \(\frac{\partial}{\partial x_i} p = \beta_i p'(r,t)\) with the prime indicating the \(r\) derivation. Hence,

\[
\sum_{i=1}^{3} \left( -a \beta_i u_i - \nu \frac{b^2}{r^4} \beta_i u_i + \beta_i^2 p'(r,t) \right) = \sum_{i=1}^{3} \beta_i f_i
\]

(2.14)
From this equation the \( \beta_i u_i \) in the sum warrants the vanishing of the first two terms in (2.14) based on the vanishing divergence (2.8). Hence, because \( \sum_{i=1}^{3} \beta_i^2 = 1 \), we see

\[
p(r, t) = p(0, t) + \sum_{i=1}^{3} \beta_i \int_{0}^{r_1} f_i(r_1, t) dr_1
\]

(2.15)

Given \( 0 < r \leq 1 \) it then follows that (2.1) contains the \((u_1, u_2, u_3)^T\) solution associated with \( p = p(r, t) \) in (2.15). The choice of \( f_i \) in (2.15) is still "free".

### 2.4 Navier-Stokes for \( r \geq 1 \)

Similarly to the previous algebraic construction we may observe that

\[
\frac{\partial u_i}{\partial x_j} = c_i e^{-at} \left\{ -\frac{x_j}{r^3} e^{-br} - \frac{b x_j}{r^2} e^{-br} \right\}
\]

(2.16)

In the previous equation we see that \( \beta_j = x_j/r \) occurs. Together with \( c_j \) from the pre-multiplication with \( u_j \) the product \( c_j \beta_j \) occurs. We have \( \sum_{j=1}^{3} c_j \beta_j = 0 \). Hence the term \( \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i = 0 \). Subsequently we note that in the radial terms of \( u_i \),

\[
\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right)
\]

This leads us to \( \nabla^2 u_i = b^2 u_i \). Hence,

\[
-(a + \nu b^2) u_i + \frac{\partial}{\partial x_i} p = f_i
\]

(2.17)

If \( f_i \) conveniently can be selected for \( r \geq 1 \) such that

\[
f_i = g_i - (a + \nu b^2) u_i
\]

(2.18)

then \( p(x_1, x_2, x_3, t) = (x \cdot g) \) for \( g \) a real constant vector in \( r \geq 1 \).

### 3 Requirements for \( f_i \)

In the previous two sections two reduced forms for \( p(x_1, x_2, x_3, t) \) were obtained. In (2.15) the selected \( f_i \) is "free". So, regarding the requirement that \( f_i \) must be multiply differentiable, let us take

\[
f_i = g_i - (a + \nu b^2) u_i
\]

(3.1)

for \( r > 0 \) and the \( u_i \) come from (2.1). Suppose, for \( r \geq 1 \) we have \( \varphi(r) = \frac{1}{r} e^{-br} \). Then

\[
\frac{\partial \varphi(r)}{\partial r} = - \left( \frac{1}{r} + b \right) \varphi(r)
\]

(3.2)
for $b > 0$ finite. Then noting radial dependence only in $r \geq 1$, we may repeatedly apply $\frac{\partial}{\partial r}$ to (3.2) and be convinced that $\left| \frac{\partial^n}{\partial x^n} f_i \right|$, with $n = 0, 1, 2, \ldots$ and $i, j = 1, 2, 3$, will remain finite for $\mathbb{R}^3$ where $r \geq 1$. For $0 < r \leq 1$, we have for $\psi(r) = e^{-b/r}$ the limit behavior $\lim_{r \to 0} \psi(r) = 0$. The multiple application of $\frac{\partial}{\partial r}$ to $\psi(r)$ provides powers of $1/r$. Note that, $\frac{\partial}{\partial r} \psi(r) = \frac{b}{r^2} \psi(r)$.

Hence, for $\frac{\partial^n}{\partial x^n} \psi(r)$, with $n$ finite but perhaps large, we will have $(1/r)^n \psi(r)$ forms and for $r \to 0$ see a vanishing of differentials. Hence, for $n = 1, 2, \ldots N$ with $N$ finite integer possibly large, $\left| \frac{\partial^n}{\partial x^n} f_i \right|$ will be finite for $\mathbb{R}^3$. If $\mathbb{R}^3 \setminus (0,0,0)$ may be taken for physical space then $\left| \frac{\partial^n}{\partial x^n} f_i \right|$ will be finite for $n = 1, 2, 3, \ldots$. It appears that the $\left| \frac{\partial^n}{\partial x^n} f_i \right|$ requirement is also fulfilled by the heuristic in (2.1). Because, $\sum_{j=1}^{3} c_j \beta_j = 0$, from (3.1) and (2.15) it follows for $0 < r \leq 1$ that $p(r, t) = p(0, t) + r \sum_{j=1}^{3} \beta_j g_j$. Note $x_j = r \beta_j$, while, we already established, for $r \geq 1$, $p(x_1, x_2, x_3, t) = \sum_{j=1}^{3} x_j g_j = (x \cdot g)$.

4 Conclusion

The claim is that in the previous sections an exact smooth nontrivial type A solution to the Navier-Stokes equation is presented. Perhaps that the exclusively radial dependence will prove to be an unphysical form for solution. However, as far as the author can see this is not a reason to reject the mathematics. The author would also like to refer to another approach of getting exact nontrivial solutions of the Navier-Stokes equation in [2].

References