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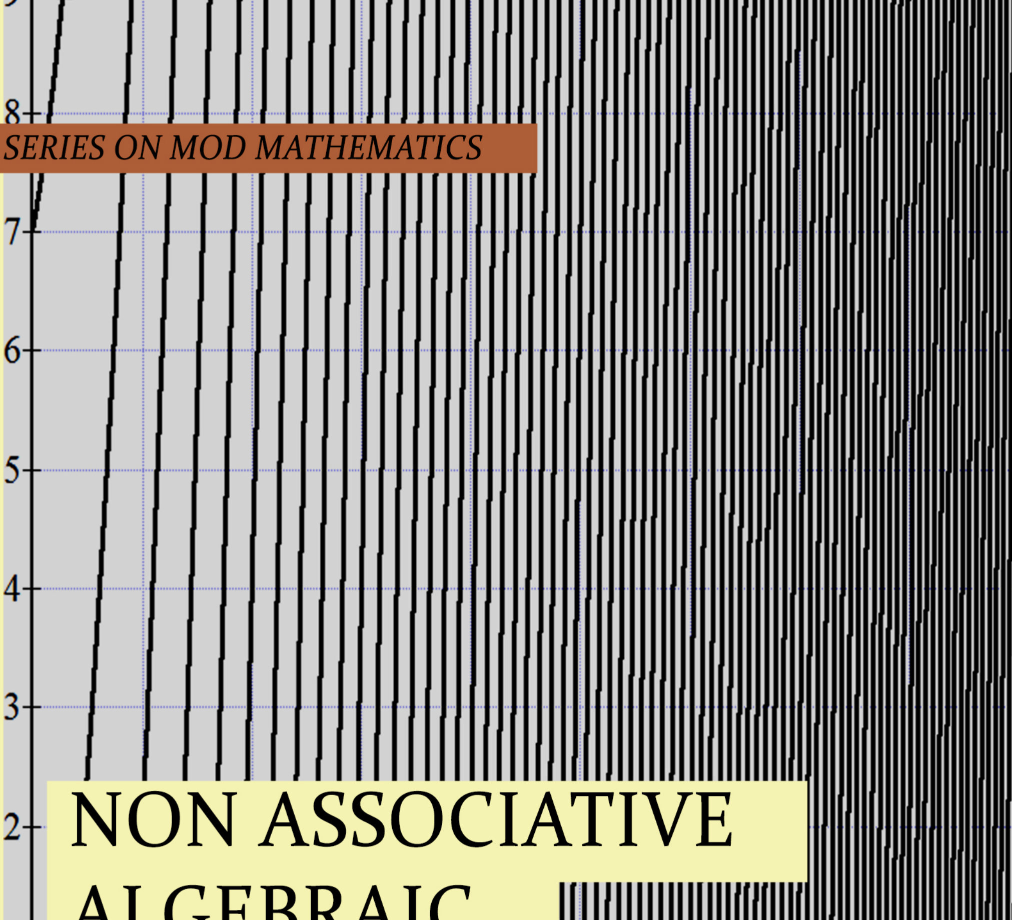

NON ASSOCIATIVE ALGEBRAIC STRUCTURES ON MOD PLANES

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Non-Associative Algebraic Structures on MOD Planes

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PREFACE

In this book authors for the first time construct non-associative algebraic structures on the MOD planes. We are not in position to build non-associative rings using the MOD planes $R_n(m)$, $R_n^I(m)$ ($I^2 = I$); $R_n^g(m)$ ($g^2 = 0$), $R_n^h(m)$ ($h^2 = h$), $R_n^k(m)$ ($k^2 = (m - 1)k$)) and $C_n(m)$.

It is interesting to note that using MOD planes we can construct infinite number of groupoids for a fixed m and all these MOD groupoids are of infinite cardinality.

Special identities satisfied by these MOD groupoids build using the six types of MOD planes are studied. Further the new concept of special pseudo zero of these groupoids are defined, described and developed. Also conditions for these MOD groupoids to have special elements like idempotent, special pseudo zero divisors and special pseudo nilpotent are obtained.

Further non-associative MOD rings are constructed using MOD groupoids and commutative rings with unit.

That is the MOD groupoid rings gives infinitely many non-associative ring. These rings are analysed for substructures and special elements. This study is new and innovative and several open problems are suggested.

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Chapter One

BASIC CONCEPTS

In this book authors for the first time build non associative MOD structures like MOD groupoids and MOD non associative rings.

Further this is developed to MOD subset groupoids and MOD subset non-associative rings.

To build MOD groupoids we use the real MOD planes $R_n(m)$; $2 \leq m < \infty$ which are infinite in number.

MOD groupoids are built in a similar fashion as that groupoids built using Z_m; $2 \leq m < \infty$.

For instance $G = \{Z_{15}, *, (10,2)\}$ is a groupoid of order fifteen. For more about these notions refer [30].

But in case of MOD groupoids

$$G = \{R_n(15), *, (t, s)\}$$

we get infinite order MOD groupoids and in fact using the MOD plane $R_n(15)$ alone one can build infinite number of MOD real groupoids for the pair (t, s) can take infinite number of values from $R_n(15)$ or t, s $\in$ [0, 15).

This is one of the easy ways of getting infinite order groupoids. All MOD real groupoids are of infinite order.

Of course depending on the pair (t, s) we can obtain conditions for the MOD real groupoid to contain zero divisors, idempotents and nilpotents.

For more about MOD real planes please refer [45].

Next the concept of MOD finite complex modulo integers $C_n(m)$; $2 \leq m < \infty$ are defined in [54]

Using these complex modulo integer MOD planes one can build infinite number of MOD complex groupoids.

For instance $C_n(m) = \{a + bi_F \mid a, b \in [0, m), i_F^2 = m - 1\}$.

$G = \{a + bi_F \mid a + bi_F \in C_n(m), (x, y), *\}$ is a MOD complex groupoid and $x, y \in C_n(m)$.

Thus for any fixed m one can have infinite number of MOD complex groupoids.

By imposing conditions on x and y we get idempotents, zero divisors and nilpotents in MOD complex groupoids.

Next we have defined MOD neutrosophic plane

$R_n^I(m) = \{a + bI \mid I^2 = I, a, b \in [0, m)\}$.

In fact we have infinite number of MOD neutrosophic planes as $2 \leq m < \infty$.

For more about MOD neutrosophic planes please refer [45].

However for a given m say m = 18 we can have infinite number of MOD neutrosophic groupoids as

$$G = \{a + bI \mid a, b \in R_n^I(18); (t, s), *\} \; t, s \in R_n^I(18);$$

and as cardinality of the MOD neutrosophic plane R_n^I (18) is infinite we have infinite number of distinct MOD neutrosophic groupoids of infinite order.

Next using the MOD dual number plane $R_n(m)g$; $g^2 = 0$ that is $R_n(m)g = \{a + bg \mid a, b \in [0, m), g^2 = 0\}$ is defined as the MOD dual number plane and order of $R_n(m)g$ is infinite.

In fact we have infinite number of MOD dual number planes for more refer [55]. This using these MOD dual number plane $R_n(m)g$ for any fixed m we get infinite number of MOD dual number groupoids each of infinite order.

Likewise we can built using $R_n(m)h$, $h^2 = h$ and $R_n(m)k$; $k^2 = (m - 1)k$ are MOD special dual like number plane and special quasi dual like number plane respectively.

Finally we indicate the method of building MOD subset real groupoid the subsets of MOD real plane $R_n(m)$.

Let M = {Collection of all subsets from G = $\{R_n(m), (t, u), *\}$ where $R_n(m) = \{(a, b) \mid a, b \in [0, m)\}$ and $t, u \in [0, m)$. {M, *} is the MOD real subset MOD groupoid under the induced operation *.

Subsets MOD gropoids are built using subsets of each of the six MOD planes $R_n(m)$, $C_n(m)$, $R_n^I(m)$, $R_n(m)g$; $g^2 = 0$, $R_n(m)$ h; $h^2 = h$ and $R_n(m)$ k ; $k^2 = (m - 1)k$.

Thus subset MOD groupoids are built. These also are of infinite order and for a given m we can have infinite number of subset MOD groupoids.

One illustration to this effect is given.

Let G = {Collection of all subsets from $C_n(m) = \{a + bi_F \mid a, b \in [0,m); (i_F)^2 = m - 1, 2 \leq m < \infty\}$, (t, u), *} where $t, u \in C_n(m)$ is a MOD complex modulo integer subset groupoid.

Clearly G is of infinite order and by varying (t, u) one can get infinite number of such MOD subset complex groupoids.

Likewise one can built using the rest of the five MOD planes.

Next using these MOD groupoids we can built MOD non associative groupoid rings.

This is carried out in last chapter of this book.

Also using subset MOD groupoids using these six MOD planes we get MOD subset non associative rings and the properties of these rings are also studied.

Chapter Two

GROUPOIDS USING MOD PLANES

In this chapter we for the first time introduce the new notion of MOD groupoids built using MOD planes. For a given MOD plane we can in general have infinite number of groupoids this is the first marked difference between MOD semigroups and MOD groupoids. For a given MOD plane we can have utmost 4 types of distinct semigroups but however in case of groupoids for a given MOD plane we can have infinite number of MOD groupoids built using the MOD plane.

We proceed on to define MOD groupoids and illustrate them by examples.

DEFINITION 2.1: *Let* $G = R_n(m) = \{(a, b) \mid a, b \in [0, m);$ *m, a positive integer*$\}$ *be the real MOD plane. Define* $(a, b) * (c, d) = s\,(a, b) + t\,(c, d)$ *where* $(a, b), (c, d) \in R_n(m)$ *and* $s, t \in [0, m)$.

i. *G is a closed binary operation on* $R_n(m)$.
ii. *G is non associative operation on* $R_n(m)$ *in general.*
iii. *G in general is non commutative.*

We define $G = \{R_n(m), *(s, t)\}$ *to be the MOD real groupoid built on the MOD plane* $R_n(m)$.

We will give examples of MOD real groupoids.

Example 2.1: Let G = {$R_n(5)$, $*$ (3, 0.2)} be the MOD real groupoid.

Clearly G is of infinite order; G is non commutative and non associative.

For let x = (2, 1) and y = (0.3, 0.2) $\in$ $R_n(5)$.

$$\begin{aligned} x * y &= (2, 1) * (0.3, 0.2) \\ &= 3\,(2, 1) + 0.2\,(0.3, 0.2) \\ &= (1, 3) + (0.06, 0.04) \\ &= (1.06, 3.04) \qquad \ldots \qquad \text{I} \end{aligned}$$

$$\begin{aligned} \text{Now } y * x &= (0.3, 0.2) * (2, 1) \\ &= 3\,(0.3, 0.2) + 0.2\,(2, 1) \\ &= (0.9, 0.6) + (0.4, 0.2) \\ &= (1.3, 0.8) \qquad \ldots \qquad \text{II} \end{aligned}$$

Clearly I and II are different; hence G is a non commutative MOD real groupoid.

Now let z = (0.1, 0.5) $\in$ R_n (5). We find (x $*$ y) $*$ z and x $*$ (y $*$ z), using equation I

$$\begin{aligned} (x * y) * z &= (1.06, 3.04) * (0.1, 0.5) \\ &= 3\,(1.06, 3.04) + (0.2)\,(0.1, 0.5) \\ &= (3.18, 4.12) + (0.02, 0.1) \\ &= (3.2, 4.22) \qquad \ldots \qquad a \end{aligned}$$

Now

$$\begin{aligned} x * (y * z) &= x * [(0.3, 0.2) * (0.1, 0.5)] \\ &= x * [3\,(0.3, 0.2) + 0.2\,(0.1, 0.5)] \end{aligned}$$

$$= \quad (2, 1) * [(0.9, 0.6) + (0.02, 0.1)]$$

$$= \quad (2, 1) * (0.92, 0.7)$$

$$= \quad 3(2, 1) + 0.2\ (0.92, 0.7)$$

$$= \quad (1, 3) + (0.184, 0.14)$$

$$= \quad (1.184, 3.14) \qquad \ldots \qquad b$$

Clearly a and b are different hence $(x * y) * z \neq x * (y * z)$ in general. That is the MOD real groupoid in general is non associative.

Example 2.2: Let $G = \{R_n(10), *, (0, 0.3)\}$ be the MOD real groupoid. G is both non commutative and non associative.

If $x = (2, 0.115)$ and $y = (7.2, 4.5) \in G$,

$$x * y = (2, 0.115) * (7.2, 4.5)$$

$$= 0\ (2, 0.115) + 0.3\ (7.2, 4.5)$$

$$= (0, 0) + (2.16, 1.35)$$

$$= (2.16, 1.35) \in G.$$

This is the way operations are performed on G.

Example 2.3: Let $G = \{R_n(12), * (5, 7)\}$ be the real MOD groupoid.

For $x = (0.312, 0.87)$ and $y = (0.501, 0.999) \in G$.

We find $x * y$ and $y * x$.

$$
\begin{aligned}
x * y &= (0.312, 0.87) * (0.501, 0.999) \\
&= 5\,(0.312, 0.87) + 7\,(0.501, 0.999) \\
&= (1.560, 4.35) + (3.507, 6.993) \\
&= (5.067, 11.343) \in G.
\end{aligned}
$$

$$
\begin{aligned}
\text{Now } y * x &= (0.501, 0.999) * (0.312, 0.87) \\
&= 5\,(0.501, 0.999) + 7\,(0.312, 0.87) \\
&= (2.505, 4.995) + (2.184, 6.09) \\
&= (4.689, 11.085) \in G.
\end{aligned}
$$

Clearly $x * y \neq y * x$.

Thus G is a non commutative MOD real groupoid of infinite order.

Example 2.4: Let $G = \{R_n(12), * (0.001, 0.5)\}$ be the MOD real groupoid. G is of infinite order and non commutative.

Example 2.5: Let $G = \{R_n(19), *, (0.3399, 0)\}$ be the MOD real groupoid.

Example 2.6: Let $G = \{R_n(242), * (100.33, 0.30001)\}$ be the MOD real groupoid.

Now we can have subgroupoids and ideals of these MOD real groupoids.

In fact these MOD real groupoid give an infinite groupoid.

Example 2.7: Let $S_1 = \{R_n(10), *, (1, 0.52)\}$ be the MOD real groupoid.

Let $x = (8, 2)$ and $y = (0.1, 0.7) \in R_n(10)$

$$\begin{aligned} x * y &= (8, 2) * (0.1, 0.7) \\ &= 1\,(8, 2) + 0.52\,(1, 0.7) \\ &= (8, 2) + (0.52, 0.364) \\ &= (8.52, 2.52) \qquad \ldots \qquad \text{I} \end{aligned}$$

Now $S_2 = \{R_n(10), *, (3, 7)\}$.

For the same x, y we find

$$\begin{aligned} x * y &= (8, 2) * (0.1, 0.7) \\ &= 3\,(8, 2) + 7\,(0.1, 0.7) \\ &= (4, 6) + (0.7, 4.9) \\ &= (4.7, 0.9) \qquad \ldots \qquad \text{II} \end{aligned}$$

From I and II it is clear S_1 and S_2 are two different groupoids.

Let $S_3 = \{R_n(10), *, (0.25, 6)\}$ be the MOD real groupoid.

$$\begin{aligned} x * y &= (8, 2) * (0.1, 0.7) \\ &= 0.25 \times (8, 2) + 6\,(0.1, 0.7) \end{aligned}$$

$= (2, 0.5) + (0.6, 4.2)$

$= (2.6, 4.7)$ … III

Clearly I, II and III are different. Hence the 3 MOD groupoids S_1, S_2 and S_3 are different but all of them is built using the real MOD plane $R_n(10)$.

Let $S_4 = \{R_n (10), *, (0, 5)\}$ be the MOD real groupoid.

For the same $x, y \in S_4$

$x * y = (8, 2) * (0.1, 0.7)$

$= 0 (8, 2) + 5 (0.1, 0.7)$

$= (0, 0) + (0.5, 3.5)$

$= (0.5, 3.5)$ … IV

I, II, III and IV are different. Hence all the four MOD real groupoids are distinct.

Let $S_5 = \{R_n (10), *, (0, 0.2)\}$ be the MOD real groupoid.

For the same x and y we find $x * y$ in S_5;

$x * y = (8, 2) * (0.1, 0.7)$

$= 0 (8, 2) + 0.2 (0.1, 0.7)$

$= (0, 0) + (0.02, 0.14)$

$= (0.02, 0.14)$ … V

Clearly V is different from I, II, III and IV.

Thus S_5 is a different real MOD groupoid from S_1, S_2, S_3 and S_4.

Let $S_6 = \{R_n(10), *, (2, 2)\}$ be the real MOD groupoid.

For the same x, y we find

$$
\begin{aligned}
x * y &= (8, 2) * (0.1, 0.7) \\
&= 2(8, 2) + 2(0.1, 0.7) \\
&= (6, 4) + (0.2, 1.4) \\
&= (6.2, 5.4) \qquad \ldots \qquad \text{VI}
\end{aligned}
$$

Clearly VI is different from I, II, III, IV and V.

Further this implies S_6 is a different real MOD groupoid from S_1, S_2, S_3, S_4 and S_5.

Let $S_7 = \{R_n(10), *, (0.5, 0.5)\}$ be the real MOD groupoid. For the same x and y we find

$$
\begin{aligned}
x * y &= (8, 2) * (0.1, 0.7) \\
&= 0.5(8, 2) + 0.5(0.1, 0.7) \\
&= (4, 1) + (0.05, 0.35) \\
&= (4.05, 1.35) \qquad \ldots \qquad \text{VII}
\end{aligned}
$$

We have give 7 distinct MOD real groupoids all of them are distinct. Groupoids S_1, S_2, S_3, S_4 and S_5 are non commutative. However S_6 and S_7 are commutative MOD real groupoids. Both S_6 and S_7 are non associative real groupoids.

For take $x = (3, 0.2)$, $y = (0.5, 1)$ and $z = (0.01, 0.301) \in S_6$

$$(x * y) * z = [(3, 0.2) * (0.5, 1)] * (0.01, 0.301)$$

$= [2\,(3, 0.2) + 2\,(0.5, 1)] * (0.01, 0.301)$

$= [(6, 0.4) + (1, 2)] * (0.01, 0.301)$

$= (7, 2.4) * (0.01, 0.301)$

$= 2\,(7, 2.4) + 2\,(0.01, 0.301)$

$= (4, 4.8) + (0.02, 0.602)$

$= (4.02, 5.402) \qquad \ldots \qquad A$

Consider

$x * (y * z) = (3, 0.2) * [(0.5, 1) * (0.01, 0.301)]$

$= (3, 0.2) * [2\,(0.5, 1) + 2\,(0.01, 0.301)]$

$= (3, 0.2) * [(1, 2) + (0.02, 0.602)]$

$= (3, 0.2) * (1.02, 2.602)$

$= 2\,(3, 0.2) + 2\,(1.02, 2.602)$

$= (6, 0.4) + (2.04, 5.204)$

$= (8.04, 5.604) \qquad \ldots \qquad B$

Clearly A and B are distinct. That is $x * (y * z) \neq (x * y) * z$. Thus the MOD real groupoid S_6 is non associative but commutative.

Next we give some examples of subgroupoids.

Example 2.8: Let $S = \{R_n(7), *, (3, 5)\}$ be the real MOD groupoid.

S has subgroupoids of both infinite and finite order.

Let B = $\{Z_7, *, (3, 5)\} \subseteq S$; clearly B is a groupoid of finite order. Thus B is a finite subgroupoid of S.

Take $R_1 = \{(x, 0) \mid x \in [0, 7)\} \subseteq S$. R_1 is a subgroupoid of infinite order.

For take x $= (0.31, 0)$ and $y = (4.33, 0) \in R_1$

$$
\begin{aligned}
x * y &= (0.31, 0) * (4.33, 0) \\
&= 3\,(0.31, 0) + 5\,(4.33, 0) \\
&= (0.93, 0) + (0.65, 0) \\
&= (1.58, 0) \in R_1.
\end{aligned}
$$

Hence R_1 is a MOD real subgroupoid of S of infinite order. Further R_1 is non associative and non commutative.

Let $x = (0.31, 0)$ and $y = (4.33, 0) \in R_1$

$$
\begin{aligned}
y * x &= (4.33, 0) * (0.31, 0) \\
&= 3\,(4.33, 0) + 5\,(0.31, 0) \\
&= (5.99, 0) + (1.55, 0) \\
&= (0.54, 0) \in R_1.
\end{aligned}
$$

Clearly $x * y \neq y * x$ so our claim R_1 is a non commutative MOD real subgroupoid is justified.

Now we take $z = (0.4, 0)$ we find

$$
\begin{aligned}
(x * y) * z &= [(0.31, 0) * (4.33, 0)] * (0.4, 0) \\
&= (1.58, 0) * (0.4, 0)
\end{aligned}
$$

(from the earlier working)

$$= 3\,(1.58, 0) + 5\,(0.4, 0)$$

$$= (5.74, 0) + (2, 0)$$

$$= (0.74, 0) \qquad \ldots \qquad \text{I}$$

Consider $x * (y * z)$

$$= (0.31, 0) * [(4.33, 0) * (0.4, 0)]$$

$$= (0.31, 0) * [3\,(4.33, 0) + 5\,(0.4, 0)]$$

$$= (0.31, 0) * [(5.99, 0) + (2, 0)]$$

$$= (0.31, 0) * (0.99, 0)$$

$$= 3\,(0.31, 0) + 5\,(0.99, 0)$$

$$= (0.93, 0) + (4.95, 0)$$

$$= (5.88, 0) \qquad \ldots \qquad \text{II}$$

Clearly I and II are distinct so our claim the subgroupoid is non associative is justified.

Similarly $R_2 = \{(0, y) \mid y \in [0, 7)\} \subseteq S$ is a MOD real subgroupoid of S of infinite order.

Clearly $R_1 \cap R_2 = \{(0, 0)\}$, so both are distinct MOD real subgroupoid of S.

Now we proceed onto define MOD neutrosophic groupoids using MOD neutrosophic planes $R_n^I(m) = \{a + bI \mid a_1\ b \in [0, m), I^2 = I\}$.

DEFINITION 2.2: *Let $R_n^I(m)$ be the MOD neutrosophic plane, G = { $R_n^I(m)$, *, (s, t), s, t ∈ [0, m)} where * is a closed binary operation on $R_n^I(m)$ defined by, for x, y ∈ R_n^I ; x * y = (s(x) + t(y)) (mod m) * is non associative.*

Thus G is defined as the MOD neutrosophic real groupoid.

If instead of s, t ∈ [0, m); s, t ∈ [0, mI) we call that G as a MOD neutrosophic - neutrosophic groupoid.

If s, t ∈ $R_n^I(m)$; we call G as the MOD neutrosophic real groupoid.

All these situations will be described by appropriate examples in the following.

Example 2.9: Let B = { $R_n^I(6)$, * (2,2)} be the MOD neutrosophic groupoid, for x = 0.3 + 5.2I and y = 2.1 + 0.7I we see

x * y = (0.3 + 5.2I) * (2.1 + 0.7I)

= 2(0.3 + 5.2I) + 2(2.1 + 0.7I)

= (0.6 + 4.4I) + (4.2 + 1.4I)

= 4.8 + 5.8I ∈ B.

This is the way * operation is performed on B.

Example 2.10: Let X = $R_n^I(11)$m, *, (0.3, 0.05)} be the MOD neutrosophic groupoid.

For x = 8 + 0.5I and y = 0.7 + 9I in X we find x * y.

x * y = (8 + 0.5I) * (0.7 + 9I)

$$= \quad 0.3\ (8 + 0.5I) + 0.05\ (0.7 + 9I)$$

$$= \quad (2.4 + 0.15I) + (0.035 + 0.45I)$$

$$= \quad (2.435 + 0.60I) \in X.$$

Example 2.11: Let M = $\{R_n^I(15), *, (0.3, 8)\}$ be the MOD neutrosophic groupoid.

Let x = 0.9 + 0.5I and y = 7 + 8I $\in$ M;

$$x * y \quad = \quad (0.9 + 0.5I) * (7 + 8I)$$

$$= \quad 0.3\ (0.9 + 0.5I) + 8\ (7 + 8I)$$

$$= \quad 0.27 + 0.15I + 11 + 4I$$

$$= \quad 11.27 + 4.15\ I \in M.$$

This is the way operation * is performed on M.

Example 2.12: Let P = $\{R_n^I(10), *, (0, 0.002)\}$ be the MOD neutrosophic groupoid.

Let x = 7.2 + 6.81I and y = 0.31 + 5.3 I $\in$ P

$$x * y \quad = \quad (7.2 + 6.81I) * (0.31 + 5.3I)$$

$$= \quad 0\ (7.2 + 6.81I) + 0.002\ (0.31 + 5.3I)$$

$$= \quad 0 + 0.00062 + 0.0106I$$

$$= \quad (0.00062 + 0.0106I) \in P.$$

P is easily verified to be the non commutative MOD neutrosophic groupoid of infinite order.

However it remains an interesting problem to find subgroupoids of finite order if (s, t) is such that both s and t are proper decimals.

Example 2.13: Let M = $\{R_n^I(12), *, (4,0)\}$ be the MOD neutrosophic groupoid.

Let x = 0.3 + 4.5I and y = 7.2 + 5.3I $\in$ M.

$$\begin{aligned} x * y &= (0.3 + 4.5I) * (7.2 + 5.3I) \\ &= 4(0.3 + 4.5I) + 0(7.2 + 5.3I) \\ &= (1.2 + 6I) \in M. \end{aligned}$$

Now we show by varying the pair (s, t) we can get for a given MOD neutrosophic plane infinite number of MOD neutrosophic groupoids.

Example 2.14: Let B_1 = $\{R_n^I(12), *(4,6)\}$ be the MOD neutrosophic groupoid.

Let x = 2 + 0.3I and y = 0.4 + 3I $\in$ B_1.

$$\begin{aligned} x * y &= (2 + 0.3I) * (0.4 + 3I) \\ &= 4(2 + 0.3I) + 6\,(0.4 + 3I) \\ &= 8 + 1.2\,I + 2.4 + 18I \\ &= 10.4 + 7.2\,I \qquad \ldots \qquad I \end{aligned}$$

Let B_2 = $\{R_n^I(12), *, (0.4, 7)\}$ be the MOD neutrosophic decimal groupoid

$$\begin{aligned} x * y &= (2 + 0.3I) * (0.4 + 3I) \\ &= 0.4\,(2 + 0.3I) + 7(0.4 + 3I) \end{aligned}$$

$= (0.8 + 0.12I) + (2.8 + 21I)$

$= 3.6 + 9.12I \quad \ldots \quad II$

I and II are different so B_1 and B_2 are two different groupoids.

Let $B_3 = \{R_n^I(12), *, (0.2, 0.01)\}$ be the MOD neutrosophic decimal groupoid.

For the same $x, y \in B_3$, we find

$$
\begin{aligned}
x * y &= (2 + 0.3I) * (0.4 + 3I) \\
&= 0.2\,(2 + 0.3I) + 0.01\,(0.4 + 3I) \\
&= 0.4 + 0.06I + 0.004 + 0.03I \\
&= 0.404 + 0.09\,I \quad \ldots \quad III
\end{aligned}
$$

Clearly III is different from I and II so B_3 is a different groupoid from B_1 and B_2.

Let $B_4 = \{R_n^I(12), *, (0,6)\}$ be the MOD neutrosophic groupoid.

For the same $x, y \in B_4$,

$$
\begin{aligned}
x * y &= (2 + 0.3I) * (0.4 + 3I) \\
&= 0(2 + 0.3I) + 6(0.4 + 3I) = 2.4 + 6I \quad \ldots \quad IV
\end{aligned}
$$

IV is different from I, II and III. Thus B_4 is a MOD neutrosophic groupoid different from B_1, B_2 and B_3.

Take $B_5 = \{ R_n^I (12), *, (0.002,0)\}$ be the MOD neutrosophic decimal groupoid.

For $x, y \in B_5$ we have

$$x * y = (2 + 0.3I) * (0.4 + 3I)$$

$$= 0.002 (2 + 0.3I) + 0 (0.4 + 3I)$$

$$= 0.004 + 0.0006I \qquad \ldots \qquad V$$

We see V is different from I, II, III and IV. So B_5 is a different groupoid from B_1, B_2, B_3 and B_4.

Example 2.15: Let $B = \{ R_n^I (10), *, (0.3I, 5I)\}$ be the MOD neutrosophic-neutrosophic decimal groupoid.

For $x = 3 + 0.7 I$ and $y = 2.1 + 4.2I$ in B we have

$$x * y = (3 + 0.7 I) * (2.1 + 4.2I)$$

$$= 0.3I (3 + 0.7I) + 5I(2.1 + 4.2I)$$

$$= 0.9I + 0.21I + 10.5I + 21I$$

$$= 2.61I \in B.$$

We see for every $x, y \in B$ we have $x * y$ to be only a pure neutrosophic value in $[0, mI)$.

Hence we see in case of MOD a neutrosophic-neutrosophic groupoid for every $x, y \in B$; $x * y \in [0, mI)$.

Example 2.16: Let $S = \{ R_n^I (13), * (6I, 0.6I)\}$ be the MOD neutrosophic-neutrosophic decimal groupoid.

Let $x = 7$ and $y = 6.32 + 9.2I \in S$.

$$\begin{aligned} x * y &= 7 * 6.32 + 9.2I \\ &= 6I \times 7 + 0.6I \times 6.32 + 9.2I \\ &= 42I + 3.792I + 5.52 \\ &= 12.312I \in S. \end{aligned}$$

Example 2.17: Let M = $\{R_n^I(10), *, (0.7I, 0)\}$ be the MOD neutrosophic-neutrosophic decimal groupoid.

For $x = 3.2 + 6.4I$ and $y = 8.2 + 0.9I \in M$;

$$\begin{aligned} \text{we find } x * y &= (3.2 + 6.4I) * (8.2 + 0.9I) \\ &= 0.7I\,(3.2 + 6.4I) + 0 \\ &= 2.24\,I + 4.48I \\ &= 6.72I \in M. \end{aligned}$$

Example 2.18: Let P = $\{R_n^I(8), * (0, 8I)\}$ be the MOD neutrosophic-neutrosophic groupoid.

For $x = 0.8 + 0.9I$ and
$y = 0.19 + 0.11\,I \in P$, we find

$$\begin{aligned} x * y &= (0.8 + 0.9I) * (0.19 + 0.11I) \\ &= (0\,(0.8 + 0.9I) + 8I\,(0.19 + 0.11I) \\ &= 1.52I + 0.88I = 2.40\,I \in P. \end{aligned}$$

This is the way product or * operation is performed on P.

Now we know by varying (s, t) the MOD neutrosophic-neutrosophic groupoids also are distinct.

THEOREM 2.1: *Let*
$G = \{R_n(m), * (s, t)\}$ and $G' = \{ R_n^I (m), * (s, t)\}$ be two MOD real groupoid and MOD neutrosophic groupoid respectively. G and G′ contain finite subgroupoids if s and t are integers. If s and t are decimals G and G′ do not contain finite subgroupoids.

Proof is direct and hence left as an exercise to the reader.

Example 2.19: Let $G = \{ R_n^I (20), *, (3 + 0.2I, 0.4 + 0.6I)\}$ be the MOD neutrosophic real decimal groupoid.

For $x = 2 + 0.5I$ and $y = 0.7 + 1.2I \in G$ we see

$$\begin{aligned} x * y &= (2 + 0.5I) * (0.7 + 1.2I) \\ &= (3 + 0.2I) \times (2 + 0.5I) + (0.4 + 0.6I) \times (0.7 + 1.2I) \\ &= (6 + 0.4I + 1.5I + 0.1I) + (0.28 + 0.42I \\ &\qquad\qquad + 48I + 0.72I) \\ &= (6.28 + 6.94I) \in G. \end{aligned}$$

We can build using $R_n^I (20)$ infinite number of MOD neutrosophic real decimal groupoids.

Example 2.20: Let $B = \{ R_n^I (16), * (2I + 0.8, 4)\}$ be the MOD neutrosophic real decimal groupoid.

For $x = 10 + 3I$ and $y = 0.7 + 0.8I$ in B

$$\begin{aligned} \text{we have } x * y &= (10 + 3I) * (0.7 + 0.8I) \\ &= (2I + 0.8) \times (10 + 3I) + 4 (0.7 + 0.8I) \\ &= (20I + 8 + 6I + 2.4I) + 2.8 + 3.2I \end{aligned}$$

$= 10.8 + 15.6I \in B$ … I

This is the way * is defined on B.

$y * x = (0.7 + 0.8I) * (10 + 3I)$

$= (2I + 0.8) \times (7 + 0.8I) + 4 (10 + 3I)$

$= 14I + 5.6 I + 70 + 21I + 2.4 I + 8I$

$= 6 + 3I$ … II

Clearly I and II are distinct. Hence B is a non commutative MOD real groupoid.

Example 2.21: Let M = $\{ R_n^I (12), *, (10 + 0.3I, 10 + 0.3I)\}$ be the MOD neutrosophic real decimal groupoid.

For $x = 0.7 + 5I$ and $y = 2 + 0.6I \in M$ we find

$x * y = (0.7 + 5I) * (2 + 06I)$

$= (10 + 0.3I) (0.7 + 5I) + (10 + 0.3I) (2 + 0.6I)$

$= 7 + 0.21I + 50I + 1.5I + 20 + 0.6I + 6I + 0.18I$

$= 3 + 9.49I$ … I

It is easily verified $x * y = y * x$ for all $x, y \in M$.

Consider $z = 1 + I$ we find

$(x * y) * z = (3 + 9.49 I) * (1 + I)$ using I

$= 10 + 0.3I \times 3 + 9.49 I + 10 + 0.3I \times 1 + I$

$= 30 + 0.9I + 94.9I + 2.847I + 10 +$
$0.3I + 10I + 0.3I$

$= 4 + 1.247\,I \qquad \ldots \qquad II$

Consider

$x * (y * z)$

$= (0.7 + 5I) * [(2 + 0.6I) * (1 + I)]$

$= (0.7 + 5I) * [10 + 0.3I \times (2 + 0.6I) + 10 + 0.3I \times I + 1]$

$= (0.7 + 5I) * [20 + 0.6I + 6I + 0.18I$
$10 + 0.3I + 10I + 0.3I]$

$= (0.7 + 5I) * (4 + 4.3I)$

$= (10 + 0.3I)\,(0.7 + 5I) + (10 + 0.3I) \times$
$(4 + 4 – 3I)$

$= 7 + 0.21I + 50I + 1.5I + 40 + 1.2I$
$+ 43I + 1.29\,I$

$= 11 + 1.2I \qquad \ldots \qquad III$

Clearly II and III are distinct hence M is a non associative groupoid but M is commutative.

THEOREM 2.2: *Let S = $\{R_n^I(m), *, (s,t)\}$ be the MOD neutrosophic groupoid.*

S is commutative if and only if s = t.

Proof is direct and hence left as an exercise to the reader.

Example 2.22: Let M = $\{C_n(20), *, (5,0.3)\}$ be the MOD complex modulo integer decimal groupoid.

M is of infinite order.

We just show for $x = 2 + 3i_F$ and $y = 10 + 16i_F \in M$.

$$
\begin{aligned}
x * y &= (2 + 3i_F) * (10 + 16i_F) \\
&= 5\,(2 + 3i_F) + 0.3\,(10 + 16i_F) \\
&= 10 + 15i_F + 3 + 4.81_F \\
&= 13 + 198\, i_F \in M.
\end{aligned}
$$

In view of this we make the formal definition.

DEFINITION 2.3: *Let* $S = \{C_n(m), * (s, t), i_F^2 = (m - 1), s, t \in [0, m)\}$. *S is defined as the MOD complex modulo integer real decimal groupoid.*

If $s, t \in C_n(m)$ *then we define S to be a MOD complex modulo integer complex groupoid.*

We will give a few examples of this definition.

Example 2.23: Let M = $\{C_n(10); *, (8, 0.3)\}$ be the MOD complex modulo integer real decimal groupoid.

For $x = 3 + 0.2i_F$ and $y = 0.7 + 8\, i_F \in M$ we see

$$
\begin{aligned}
x * y &= (3 + 0.2i_F) * (0.7 + 8i_F) \\
&= 8(3 + 0.2i_F) + 0.3\,(0.7 + 8i_F) \\
&= 24 + 1.6\, i_F + 0.21 + 2.4\, i_F \\
&= 4.21 + 4i_F \in M.
\end{aligned}
$$

Example 2.24: Let S = $\{C_n(17), * (0.7, 1), i_F^2 = 16\}$ be the MOD complex modulo integer real decimal groupoid.

For x $= 0.9 + 8i_F$ and y $= 16 + 2i_F \in$ S we find

$$x * y = (0.9 + 8i_F) * (16 + 2i_F) = 0.7\,(0.9 + 8i_F) + 1(16 + 2i_F)$$

$$= 0.72 + 5.6\, i_F + 16 + 2i_F$$

$$= 16.72 + 7.6\, i_F \in S.$$

Example 2.25: Let B = $\{C_n\,(9), *\,(2i_F, 4 + 0.3i_F)\,,\, i_F^2 = 8\}$ be the MOD complex modulo integer complex decimal groupoid.

For x $= 6.8 + 2.1\, i_F$ and y $= 3 + 2i_F \in$ B

we find $x * y = (6.8 + 2.1i_F) * (3 + 2i_F)$

$$= 2i_F\,(6.8 + 2.1i_F) + (4 + 0.3i_F)\,(3 + 2i_F)$$

$$= 13.6\, i_F + 4.2 \times 8 + 12 + 0.9i_F + 8i_F + 0.6 \times 8$$

$$= 3.6i_F + 33.6 + 3 + 4.8 + 0.9i_F$$

$$= 4.5i_F + 5.4 \in B.$$

Thus B is a MOD complex modulo integer complex decimal groupoid.

Example 2.26: Let S = $\{C_n(6), *, (3 + 2i_F, 5.1 + 2.3i_F), i_F^2 = 5\}$ be the complex modulo integer complex decimal groupoid.

Let x $= 3 + 0.5\, i_F$ and y $= 0.7 + 2i_F \in$ S

$$x * y = (3 + 0.5i_F) * (0.7 + 2i_F)$$

$$= (3 + 2i_F)\ (3 + 0.5i_F) + (5.1 + 23i_F) \times (0.7 + 2i_F)$$

$$= 9 + 1.5i_F + 6i_F + 5 + 3.57 + 1.61i_F + 10.2\ i_F + 4.6 \times 5$$

$$= 3.77 + 23.0 + (4.2 + 1.61 + 1.5)i_F$$

$$= 2.77 + 1.31i_F \in S.$$

Now having seen examples of MOD complex modulo integer complex groupoids we now proceed into define using MOD dual number planes the notion of MOD dual number real groupoids and MOD dual number groupoids.

DEFINITION 2.4: *Let* $M = \{R_n(m)(g) \mid g^2 = 0, *, (s,t);\ s,t \in [0, m)\}$ *be the MOD dual number real decimal groupoid.*

If s and t $\in R_n(m)(g)$ *then we define M as the MOD dual number groupoids.*

We will first illustrate these situations by some examples.

Example 2.27: Let $P = \{R_n(10)(g) \mid g^2 = 0; *, (3,7)\}$ be the MOD dual number real decimal groupoid.

For $x = 0.3 + 0.71g$ and $y = 6.5 + 21g \in P$

we find

$$x * y = (0.3 + 0.71g) * (6.5 + 2.1g)$$

$$= 3\ (0.3) + 0.71g) + 7\ (6.5 + 2.1g)$$

$$= 0.9 + 2.13\ g + 45.5 + 14.7\ g$$

$$= 6.4 + 6.83g \in P.$$

This is the way operation * is performed on P.

Example 2.28: Let M = $\{R_n(13)g \mid g^2 = 0, *, (2.1, 0.12)\}$ be the MOD dual number real decimal groupoid.

Let x = 6 + 2.1g and y = 0.2 + 7g $\in$ M;

$$\begin{aligned} x * y &= (6 + 2.1g) * (0.2 + 7g) \\ &= (6 + 2.1g)\ 2.1 + (0.2 + 7g)\ 0.12 \\ &= 12.6 + 4.41g + 0.024 + 0.84\ g \\ &= 12.624 + 5.25\ g \in M. \end{aligned}$$

This is the way * operation is performed on M.

Example 2.29: Let M = $\{R_n(12)(g) \mid g^2 = 0, *, (0, 2.5)\}$ be the MOD dual number real decimal groupoid.

Example 2.30: Let B = $\{R_n(15)g;\ g^2 = 0, *, (10 + g, 0.2 + 0.4g)\}$ be the MOD dual number decimal groupoid.

For x = 7 + 0.8g and 0.4 + 0.5g $\in$ B;

$$\begin{aligned} x * y &= (7 + 0.8g) * (0.4 + 5g) \\ &= (10 + g)\ (7 + 0.8g) + (0.2 + 4g)\ (0.4 + 5g) \\ &= 70 + 7g + 8g + 0.08 + 0.8g + g \\ &= 10.08 + 1.88\ g \in B \qquad \ldots \qquad \mathrm{I} \end{aligned}$$

B is clearly non associative.

Take z = 3.1 = 5.2 g $\in$ B;

$(x * y) * z = (10.08 + 1.88g) * (3.1 + 5.2g)$
(using equation I for $x * y$)

$= (10 + g) \times (10.08 + 1.88\,g) + (0.2 + 0.4g) \times (3.1 + 5.2g)$

$= 100.8 + 18.8g + 10.08g + 0.62 + 1.24g + 1.04g$

$= 11.42 + 0.16g$ … II

Now $x * (y * z) = x * [(0.04 + 0.5g) * (3.1 + 5.2g)]$

$= x * [(10 + g)\,(0.4 + 0.5g) + (0.2 + 0.4g)\,(3.1 + 5.2g)]$

$= [7 + 0.8g] * [0.08 + 0.1g + 0.16g + 0.62 + 1.24g + 1.04g]$

$= (7 + 0.8g) * (0.7 + 2.54g)$

$= (10 + g)\,(7 + 0.8g) + (0.7 + 2.54g) \times (0.2 + 0.4g)$

$= 70 + 7g + 8g + 0.14 + 0.508g + 0.28g$

$= 10.14 + 0.785g$ … II

Clearly I and II are different hence * operation on B is non associative.

Example 2.31: Let $B = \{R_n(11)g,\ g^2 = 0;\ *,\ (3.1 + 2g,\ 6 + 0.8g)\}$ be the MOD dual number decimal groupoid.

Let $x = 6 + 5g$ and $y = 5.5 + 5.5g \in B$,

$x * y \quad = (3.1 + 2g)\,(6 + 5g) + (6 + 0.8g)\,(5.5 + 5.5g)$

$= (18.6 + 12g + 15.5g) + 33.0 + 33g + 4.4g)$

$= 7.6 + 9.9\,g \in B.$

This is the way * operation is performed on B.

Example 2.32: Let $M = \{R_n(10)g; g^2 = 0, * (6, 7g)\}$ be the MOD dual number decimal groupoid.

Let $x = 6 + 4g$, $y = 4.5 + 5.5g \in M$.

$$x * y = (6 + 4g) * (4.5 + 5.5g)$$

$$= 6(6 + 4g) + 7g(4.5 + 5.5g)$$

$$= (36 + 24g) + 31.5g + 0$$

$$= 6 + 5.5g \in M.$$

Now we proceed onto define MOD special dual like number real decimal groupoids and MOD special dual like number decimal groupoids.

DEFINITION 2.5: *Let $M = \{R_n(m)(g) \mid g^2 = g, *, (s, t) ; s, t \in [0, m)\}$ be the MOD special dual like number real decimal groupoid.*

If $s, t \in R_n(m)$ then we define M to be the MOD special dual like number groupoid.

We will illustrate this situation by some examples.

Example 2.33: Let $M = \{R_n(12)(g); g^2 = g, *, (4,3)\}$ be the MOD special dual like number real decimal groupoid.

Let $x = 0.3g + 4.2$ and $y = 5g + 0.5 \in M$

$$x * y = (0.3g + 4.2) * (5g + 0.5)$$

$$= 4(4.2 + 0.3g) + 3 (5g + 0.5)$$

$= 16.8 + 1.2g + 15g + 1.5g$

$= 6.3 + 4.2g.$

Example 2.34: Let P = $\{R_n (18) g; g^2 = g, *, (10.1, 0.3)\}$ be the MOD special dual like number real decimal groupoids.

Let $x = 6 + 2g$ and $y = 0.7 + 7.5 g \in P$.

$x * y = (6 + 2g) * (0.7 + 7.5g)$

$= 10.1 \times (6 + 2g) + 0.3 (0.7 + 7.5g)$

$= 60.6 + 20.2g + 021 + 2.25 g$

$= 6.81 + 4.45g \in P.$

Example 2.35: Let S = $\{R_n(10)g; g^2 = g, * (6.3, 0.1)\}$ be the MOD special dual like number real decimal groupoid.

Let $x = 0.9 + 2.4g$ and $y = 5.1 + 5g \in S$.

$x * y = (0.9 + 2.4g) * (5.1 + 5g)$

$= 6.3 (0.9 + 2.4g) + 0.1 \times (5.1 + 5g)$

$= 2.07 + 5.12g + 0.5g + 0.51$

$= 2.58 + 5.62g \in S.$

Example 2.36: Let S = $\{R_n(12)g, g^2 = g, *, (10 + 2g, 6g + 6)\}$ be the MOD special dual like number groupoid.

Let $x = 5.4 + 3.2g$ and $y = 6 + 0.3g \in S$

$x * y = (5.4 + 3.2g) * (6 + 0.3g)$

$= (10 + 2g) (5.4 + 3.2g) + (6 + 6g) (6 + 0.3g)$

$= 54 + 10.8g + 32g + 6.4g + 36 + 1.8g + 36g + 1.8g$

$= 6 + 4.8g \in S.$

This is the way * operation is performed on S.

Example 2.37: Let B = $\{R_n(8)g;\ g^2 = g,\ * (0, 4.2)\}$ be the MODp special dual like number real decimal groupoid.

Let $x = 3 + 5g$, $y = 0.8 + 0.6g$
and $z = 5.2 + 6.1\,g \in B$.

We find (x * y) * z

$= [(3 + 5g) * (0.8 + 0.6g)] * (5.2 + 6.1g)$

$= [0\,(3 +5g) + 4.2\,(0.8 + 0.6g)] * (5.2 + 6.1g)$

$= 0 + (3.36 + 2.52\,g) * (5.2 + 6.1g)$

$= 0\,(3.36 + 2.52g) + (5.2 + 6.1g) \times 4.2$

$= 21.84 + 25.62\,g$

$= 5.84 + 1.62\,g \qquad \ldots \qquad \text{I}$

Consider x * (y * z)

$= (3 + 5g) * [0.8 + 0.6g * 5.2 + 6.1g]$

$= (3 + 5g) * [0 + 4.2\,(5.2 + 6.1g)]$

$= 3 + 5g * (21.84 + 25.62)$

$= (3 + 5g) * (5.84 + 1.62g)$

$= 0(3 + 5g) + 4.2\,(5.84 + 1.62g)$

$= 24.528 + 6.804g$

$= 0.528 + 6.804g$ … II

Clearly I and II are distinct.

Hence B is a non associative groupoid.

Example 2.38: Let W = $\{R_n(15)g, g^2 = g, *, (5, 5)\}$ be the MOD special dual like number real groupoid. W is commutative but non associative.

Let x = 3.1 + 8.3 g, y = 8.5 + 2.6 g and z = 2.4 + 3.4 g $\in$ W

(x * y) * z

$= [(3.1 + 8.3g) * (8.5 + 2.6g)] * (2.4 + 3.4g)$

$= [5 (3.1 + 8.3g) + 5(8.5 + 2.6g)] * (2.4 + 3.4g)$

$= [15.5 + 41.5 g + 42.5 g + 42.5 + 13g] * (2.4 + 3.4g)$

$= (13 + 9.5g) * (2.4 + 3.4g)$

$= 5 (13 + 9.5g) + 5 \times (2.4 + 3.4g)$

$= 65 + 47.5g + 12 + 17.0g$

$= 5 + 2.5 g + 12 + 2g$

$= 3 + 4.5 g$ … I

Consider x * (y * z)

$= (3.1 + 8.3g) * [(8.5 + 2.6g) * (2.4 + 3.4g)]$

$= (3.1 + 8.3g) * [5 (8.5 + 2.6g) + 5(2.4 + 3.4g)]$

$= (3.1 + 8.3g) * (42.5 + 13.0g + 12 + 17g)$

$= (3.1 + 8.3\ g) * (9.5 + 0)$

$= 5\ (3.1 + 8.3g) + 5\ (9.5)$

$= (15.5 + 41.5g) + 47.5$
$= 3 + 11.5\ g$ … II

Clearly I and II are distinct.

Hence w is commutative but is non associative.

THEOREM 2.3: *Let $G = \{R_n(m)(g),\ g^2=g,\ *\ (s, t)\}$ be a MOD special dual like number real groupoid (MOD special dual like number groupoid).*

G is a commutative groupoid if and only if $s = t$.

Proof is direct and hence left as an exercise to the reader.

Next we define MOD special quasi dual number real groupoid and MOD special quasi dual number groupoid.

DEFINITION 2.6: *Let $B = \{R_n(m)\ g;\ g^2 = (m - 1)\ g,\ *,\ (s, t),\ s, t \in [0, m)\}$. B is defined as the MOD special quasi dual number real decimal groupoid.*

If $s, t \in R_n(m)g$ we define B to be the MOD special quasi dual number groupoid.

We give examples of them.

Example 2.39: Let $S = \{R_n(20)\ g;\ g^2 = 19g, *, (10, 0.2)\}$ be the MOD special quasi dual number real decimal groupoid.

Let $x = 0.8 + 4g$ and $y = 9.2 + 5.6g \in S$

$$\begin{aligned} x * y &= (0.8 + 4g) * (9.2 + 5.6g) \\ &= 10\,(0.8 + 4g) + 0.2\,(9.2 + 5.6g) \\ &= 8 + 40g + 18.4 + 1.12g \\ &= 9.84 + 1.12g \in S. \end{aligned}$$

Example 2.40: Let M = $\{R_n(10)\ g;\ g^2 = 9g, (0.4, 0.8), *\}$ be the MOD special quasi dual number decimal groupoid.

$$\begin{aligned} \text{Let } x &= (5.2 + 8.6g) \text{ and} \\ y &= 0.3 + 0.9\,g \in M. \end{aligned}$$

$$\begin{aligned} x * y &= 0.4\,(5.2 + 8.6g) + 0.8\,(0.3 + 0.9g) \\ &= 2.08 + 3.44g + 0.24 + 0.72g \\ &= 2.32 + 4.16\,g \in M. \end{aligned}$$

Example 2.41: Let S = $\{R_n(13)g;\ g^2 = 12g, *, (10, 10)\}$ be the MOD special quasi dual number groupoid. S is commutative but is non associative.

In view of all these we have the following theorem.

THEOREM 2.4: *Let S = S = $\{R_n(m)g\ ;\ g^2 = (m - 1)g, (s, t), *\}$ be the MOD special quasi dual number groupoid. S is commutative if and only if s = t.*

The proof is direct and hence left as an exercise to the reader.

Example 2.42: Let S = $\{R_n(10)\ g,\ g^2 = 9g, *, (0.3, 0.3)\}$ be the MOD special quasi dual number groupoid.

Let $x = 8.3 + 4.6\,g$ and $y = 2.6 + 1.5\,g \in S$

$$(x * y) = (8.3 + 4.6g) * (2.6 + 1.5g)$$
$$= (8.3 + 4.6g)\, 0.3 + (2.6 + 1.5g)\, 0.3$$
$$= (2.49 + 1.38\, g) + (0.78 + 0.45g)$$
$$= 3.27 + 1.83\, g \in S.$$

We see $x * y = y * x$ for all $x, y \in S$.

Next we proceed onto describe some special properties associated with these MOD groupoids built using MOD planes like satisfying special identity and so on.

In the first place for properties about groupoid refer [30]. Before we put them into theory we just show these properties by some examples.

Example 2.43: Let $S = \{R_n(12), *, (3, 9)\}$ be the MOD real groupoid. This has subgroupoids of finite order.

Now we study the special identities related with them.

We see if S contains a subgroupoid which satisfies any of the identities [3-4, 9, 30] then we define S to be Smarandache Moufang weak MOD groupoid and so on.

We will define this first.

DEFINITION 2.7: *Let $S = \{R_n(m), *, (s, t)\}$ be the MOD real groupoid. We say S is a MOD Smarandache groupoid if S contains a proper subset H such that (H, *) is a semigroup.*

All MOD real groupoids are not in general Smarandache groupoids.

We will give one or two examples of Smarandache MOD real groupoids for more ref.

Example 2.44: Let G = {$R_n(6)$, *, (4,5) be the MOD real Smarandache groupoid.

Example 2.45: Let G = {$R_n^I(8)$, *, (2,4)} be the MOD neutrosophic Smarandache groupoid.

Example 2.46: Let S = {$C_n(12)$, *, $i_F^2 = 11$, (1,3)} be the MOD complex modulo integer Smarandache groupoid.

Example 2.47: Let S = {$R_n(14)g$, $g^2 = 0$, * (7,8)} be the MOD dual number Smarandache groupoid.

Example 2.48: Let W = {$R_n(12)g$, $g^2 = g$, * (1, 6)} be the MOD special dual like number Smarandache groupoid.

Example 2.49: Let P = {$R_n(6)g$ | $g^2 = 5g$, *, (4,5)} be the MOD special quasi dual number Smarandache groupoid.

We have seen examples of Smarandache MOD groupoids. Next we proceed to give the following theorems.

THEOREM 2.5: *Let S = {$R_n(m)$ (($C_n(m)$ or $R_n^I(m)$, $R_n(m)g$ with $g^2 = 0$, $R_n(m)$ g with $g^2 = g$ or $R_n(m)g$ with $g^2 = (m – 1)g$, * (s, t); s, t $\in Z_m$} be the MOD real (complex, neutrosophic, dual number, special dual like, number special quasi dual number) groupoid.*

S is a Smarandache MOD groupoid if Z_m is a Smarandache groupoid.

Proof is direct hence left as an exercise to the reader.

It is left as an open conjecture.

Conjecture 2.1: Can {$R_n(m)$, *, (s, t), s, t $\in$ [0, m) \ Z_m} be a Smarandache MOD real groupoid?

ii) Can $\{R_n(m)\ *, (s, t) / s, t \in [0, m) \setminus Z_m\}$ have finite order subgroupoids?

iii) Can $G = \{R_n(m), *, (s, t)\}$ be a S-groupoid such that G is a Moufang groupoid?

Several such conjectures are also proposed in the problems portion that follows this chapter.

Next we proceed onto define describe and develop the notion of S-MOD Moufang groupoid, S-MOD Bol groupoid, S-MOD P-groupoid, S-MOD left alternative groupoid, S-MOD right alternative groupoid and so on.

DEFINITION 2.8: *Let $S = \{R_n(m), *, (s,t), s, t \in [0, m)\}$ be a MOD real decimal groupoid. S is a MOD weak P-groupoid if and only if S is a S-groupoid and S has a subgroupoid H which satisfies the identity $(x * y) * x = x * (y * x)$ for all $x, y \in H$.*

We will give examples of them.

Example 2.50: Let $S = \{R_n(6), *, (3,5)\}$ be the MOD Smarandache groupoid. Clearly S is a S-MOD P-groupoid.

Example 2.51: Let $S = \{R_n(12), *, (5, 10)\}$ be the MOD real groupoid. S is a MOD Smarandache real weak P-groupoid.

Example 2.52: Let $S = \{R_n(6), *\ (4,3)\}$ be the MOD real groupoid. S is a MOD Smarandache weak P-groupoid.

Example 2.53: Let $M = \{C_n(6), *, (4,3)\}$ be the MOD complex modulo integer groupoid which is a MOD Smarandache weak P-groupoid.

Example 2.54: Let $M = \{R_n^I(12), *, (5,10)\}$ be the MOD neutrosophic modulo integer groupoid which is a MOD Smarandache weak P-groupoid.

Example 2.55: Let T = {$R_n(12)$ (g), $g^2 = 0$, *, (5,10) be the MOD dual number groupoid which is a MOD Smarandache weak P-groupoid.

Example 2.56: Let S = {$R_n(12)(g)$, $g^2 = g$, *, (5, 10)} be the MOD special dual like number groupoid. S is a S-MOD special dual like number weak groupoid.

Example 2.57: Let B = {$R_n(6)$ g; $g^2 = 5g$, *, (3,5)} be the MOD special quasi dual number groupoid.

B is a S-MOD special quasi dual number weak groupoid.

Now we proceed onto define S weak alternative groupoids in the following.

DEFINITION 2.9: *Let S = {$R_n(m)$, *, (s, t); s, t $\in Z_m$} be the MOD real groupoid. Suppose S has a subgroupoid H such that H satisfies (x * y) * y = x *(y * y) for all x, y $\in$ H. Then we call S to be a weak MOD right alternative groupoid.*

*If (x * x) * y = x* (x * y) for all x, y $\in$ H then we define S to be a weak Smarandache left alternative MOD real groupoid.*

If S is both left alternative as well as right alternative MOD weak groupoid then we define S to be a Smarandache weak MOD alternative groupoid.

We will give example of all these situations.

Example 2.58: Let S = ($R_n(14)$, *, (7,8)} be the MOD real groupoid. Clearly S is a MOD Smarandache real weak alternative groupoid.

Example 2.59: Let S = {($R_n(12)$, *, (4,9)} be the MOD real groupoid.

Clearly S is a S-MOD real alternative groupoid of infinite order.

Example 2.60: Let S = {$R_n(6) \cup \{e\}$ where $e * e = x$ and $x * x = e$ for all $x \in R_n(6)$, (3, 5)} be the MOD groupoid with identity. S is a S-MOD weak right alternative groupoid.

Since we are not in a position naturally to find an identity element in the MOD groupoids for that matter in groupoids built using Z_n we adjoin an element e to Z_n, that is $Z_n \cup \{e\}$ where $e * e = e$, $x * x = e$ for all $x \in Z_n$ and $x * e = e * x = x$ for all $x \in Z_n$.

In the same way for the plane $R_n(m)$ adjoin a special element e = (e, e) such that $e * x = x \times e = x$ $e * e = e$ and $x * x = e$, for all $x \in R_n(m)$.

That is if $x = (0.779, 8.8031) \in R_n(9)$

$$\begin{aligned} x * x &= (e, e) = e \\ &= (0.7, 0.8001) * (e, e) \\ &= (0.7, 0.8001) \\ &= (e, e) * (0.7, 0.8001). \end{aligned}$$

Now this situation will be illustrated by some examples.

Example 2.61: Let S = {$R_n(12) \cup (e, e)$, *, (7, 5)} is a special MOD groupoid with identity.

For

$$(0.311, 8.011) * (e, e) = (0.311, 8.011).$$

Now can these special MOD groupoids with identity, e = (e, e) satisfy any special identities partially for the choices of (s, t). The answer is yes.

This situation is illustrated by some examples.

Example 2.62: Let S = {R_n (6) ∪ {(e, e)}, *, (5,5)} be the special MOD groupoid with identity.

Clearly $5 + 5.5 \equiv 0 \pmod 6$ and $5^2 = 1 \pmod 6$. Thus S is S-weak left (right) alternative groupoid of infinite order only weak as the identity is true only for a proper finite subset of S.

Example 2.63: Let S = {R_n(12) ∪ {(e, e)}, *, (5,11)} be the special interval MOD groupoid with identity.

S is a S-weak MOD right alternative groupoid.

Example 2.64: Let S = {R_n(12) ∪ {(e, e)}, *, (5,3)} be the S-special interval MOD groupoid with identity $5 + 5.3 = 0$ (mod 6) and $5^2 = 1$ (mod 6).

So S is a S-special interval MOD weak right alternative groupoid.

THEOREM 2.6: *Let S = {$R_n(m)$ ∪ {(e, e)}, *, (t, u), t, u ∈ Z_m} be the S-special MOD interval groupoid with identity. S is a S-weak right alternative MOD interval groupoid if and only if $t^2 \equiv 1 \pmod m$ and $tu + u \equiv 0 \pmod m$.*

Proof: Let x = (x, x_2) and y = (y_1, y_2) ∈ S.

$$(x * y) * y$$

$$= [(x_1, x_2) * (y_1, y_2)] * (y_1, y_2)$$

$$= (tx_1 + uy_1, tx_2 + uy_2) * (y_1, y_2)$$
$$(t^2x_2 + tuy_1 + uy_1, t^2x_2 + tuy_2 + uy_2)$$

$$= (x_1, x_2) \qquad \ldots \qquad \text{I}$$

(as $t^2 \equiv 1 \pmod m$ and $tu + u \equiv 0 \pmod m$).

Consider $x * (y * y)$

$$= x * y = x \qquad \ldots \qquad \text{II}$$

I and II are identical hence the claim so S is a MOD weak right alternative groupoid.

Likewise one can prove the following theorem.

THEOREM 2.7: *$G = \{R_n(m) \cup \{(e, e)\}, *, (t, u)\}$ is the MOD S-special left alternative groupoid if and only if $u^2 \equiv 1 \pmod n$ and $(t + tu) = 0 \pmod n$.*

Proof: To prove G is a S-MOD left alternative groupoid one has to prove $(x * x) * y = x * (x * y)$.

Now $(x * x) * y = y$ as $x * x = (e, e)$.

Consider $x * (x * y)$

$$= \quad x * (t(x_1,x_2) + u(y_1, y_2))$$

$$= \quad x * (tx_1 + xy_1, tx_2 + uy_2)$$

$$= \quad (tx_1 + tux_1 + u^2y_1, tx_1 + tux_2 + u^2y_2)$$

$$= \quad ((t + tu)\, x_1 + y_1, (t + tu)\, x_2 + y_2)$$

$$= \quad (y_1 \;\; y_2) = y,$$

as $\quad t + tu \equiv 0 \bmod n$

and $\quad u^2 \equiv u \bmod n.$

Hence the claim.

We will illustrate these two situations by some examples.

Example 2.65: Let G= $\{R_n(12) \cup \{e\} * (11, 2)\}$ be a S-right alternative MOD interval groupoid as $11^2 \equiv 1$ and $11.2 + 2 = 0$ (mod 12).

Further $G' = \{R_n\ (12) \cup \{(e, e)\}, *, (2, 5)$ is a S-left alternative MOD groupoid as $5^2 = 1$ (mod 12) and

$$t + tu = 2 + 10 \equiv 0 \bmod 12.$$

Thus the theorems are proved for both G and G′.

Example 2.66: Let $G = \{R_n(12) \cup \{(e, e)\}, *, (4,9), 4, 9 \in Z_{12}\}$ be a S-MOD special interval groupoid with identity G is a S-MOD special interval Bol groupoid.

Example 2.67: Let $G' = \{R_n(15) \cup \{(e, e)\}, *, (10,6)\}$ be the S MOD special interval groupoid with identity G′ is a S-MOD P-interval groupoid.

THEOREM 2.8: *Let $G = \{R_n(m) \cup \{(e,e)\}, *, (t, u), t, u \in Z_m,$ m a non prime$\}$, $t^2 = t$ (mod m) and $u^2 = u$(mod m)$\}$ be the S-MOD special interval groupoid with identity.*

i) G is a weak MOD interval P-groupoid.
ii) G is a weak S-MOD interval Bol groupoid
iii) G is a weak S-MOD interval Moufang groupoid.

Proof of (i)

To show G is a S MOD special interval P-groupoid we have to show $(x * y) * x = x * (y * x)$ for all $x, y \in G$.

Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in G$.

Consider

$$(x * y) * x = [(x_1, x_2) * (y_1, y_2)] * [(x_1, x_2)]$$

$$= (tx_1 + uy_1, tx_2 + uy_2) * (x_1, x_2)$$

$$= t^2x_1 + tuy_1 + ux_1, t^2x_2 + tuy_2 + ux_2).$$

Clearly $t^2 = t \pmod m$ so

$$(tx_1 + tuy_1 + ux_1, tx_2 + tuy_2 + ux_2) \quad \ldots \text{ I}$$

Consider

$$x * (y * x) = x * ((y_1, y_2) * (x_1, x_2))$$

$$= (x_1, x_2) [ty_1 + ux_1, ty_2, + ux_2]$$

$$= (tx_1 + uty_1 + u^2x_1, tx_2 + tuy_2 + u^2x_2)$$

$$(\because u^2 \equiv u \pmod m)$$

$$= (tx_1 + uty_1 + ux_1, tx_2 + tuy_2 + ux_2) \quad \ldots \text{ II}$$

I and II are identical so G is a S-MOD interval strong P-groupoid.

Proof of (ii)

To prove G is a S-MOD interval Bol groupoid.

We have to show $((x * y) * z * y = x * ((y * z) * y)$ for all $x, y, z \in G$.

Consider $((x * y) * z) * y$

$= (((x_1, x_2) * (y_1, y_2) * (z_1, z_2)) * (y_1, y_2)$

$= ((tx_1 + uy_1, tx_2 + uy_2) * (z_1, z_2)) * (y, y_2))$

$= (t^2 x_1 + tuy_1 + uz_1, t^2x_2 + tuy_2 + uz_2) * (y_1, y_2)$

$= (t^3x_1 + t^2uy_1 + tuz_1 + uy_1, t^3x_2 + t^2uy_2 + tuz_2 + uy_2)$

$= (tx_1 + tuy_1 + tuz_1 + uy_1, tx_2 + uy_2 + tuz_2 + uy_2)$ … I

Consider

$x*((y * z) *y)$

$= x*((y_1, y_2) * (z_1, z_2)) * (y_1, y_2)$

$= (x_1, x_2) ((ty_1 + uz_1, ty_2 + uz_2) * (y_1, y_2))$

$= (x_1, x_2) (t^2y + tuz_1 + uy_1, t^2y_2 + tuz_2 + uy_2)$

$= (tx_1 + ut^2y_1 + u^2tz_1 + u^2y_1 , tx_2 + t^2uy_2 + tu^2z_2 + u^2y_2)$

(using $t^2 \equiv t \pmod m$ and $u^2 \equiv u \pmod m$)

$= (tx_1 + uty_1 + utz_1 + uy_1, tx_2 + tuy_2 + utz_2 + uy_2)$ … II

I and II are identical hence the claim.

Proof of (iii)

To prove G is a S-MOD special interval Moufang groupoid we have to prove

$$(x * y) * (z * x) = (x * (y * z)) * x$$

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2) \in G$.

Consider $(x * y) * (z * x)$

$$= [(x_1, x_2) * (y_1, y_2)] * [(z_1, z_2) * (x_1, x_2)]$$

$$= (tx_1 + uy_1, tx_2 + uy_2) * (tz_1 + ux_1, tz_2 + ux_2)$$

$$= (t^2x_1 + tuy_1 + utz_1 + u^2x_1, t^2x_2 + tuy_2 + utz_2 + u^2x_2)$$

(using $t^2 \equiv t \pmod m$ and $u^2 = u \pmod m$)

$$= (tx_1 + tuy_1 + utz_1 + ux_1, tx_2 + uty_2 + utz_2 + ux_2) \quad \ldots \quad I$$

Consider $(x * (y * z)) * x$

$$= ((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) * (x_1, x_2)$$

$$= ((x_1, x_2) * [ty_1 + uz_1, ty_2 + uz_2)] * (x_1, x_2)$$

$$= (tx_1 + uty_1 + u^2z_1, tx_2 + uty_2 + u^2z_2) + (x_1, x_2)$$

$$= (t^2 x_1 + ut^2y_1 + u^2tz_1 + ux_1, t^2x_2 + ut^2y_2 + utz_2 + ux_2)$$

$$=(tx_1 + uty_1 + utz_1 + ux_1, tx_2 + uty_2 + utz_2 + ux_2) \quad \ldots \quad \text{II}$$

I and II are identical hence G is a S-strong special MOD interval Moufang groupoid.

However it remains as an open conjecture to find S-special interval MOD subset groupoids in which (t, u) $\in$ $[0, m) \setminus Z_m$ to satisfy any of the special identities; $3 < m < \infty$.

Now the study using in these MOD subset groupoids t, u $\in$ $[0, m) \setminus Z_m$ is carried out. Such MOD groupoids we call as fraction / decimal MOD subset groupoids.

This is first illustrated by some examples.

Example 2.68: Let S = {Collection of all subsets from the set $R_n(10)$, *, (3.24, 0.009)} be the MOD subsets fractional / decimal groupoids.

Let x = {(0, 2), (5, 1), (6, 0.5)} and
y = ((7, 0) (0.2, 0.1), (0.7, 1)} $\in$ S.

x * y = {(0,2), (5,1), (6,0.5)} * {(7,0), (0.2,0.1), (0.7, 1)}

= (0, 2) * (7, 0), (5, 1) * (7, 0), (6, 0.5) * (7, 0), (0, 2) * (0.2, 0.1), (5, 1) * (0.2, 0.1), (6, 0.5) * (0.2, 0.1), (6, 0.5) * (0.7, 1), (0, 2) * (0.7, 1), (5, 1) * (0.7, 1)}

$$= \quad (0 * 7, 2 * 0), (5 * 7, 1 * 0), (6 * 7, 0.5 * 0), (0 * 0.2, 2 * 0.1), (5 * 0.2, 1 * 0.1), (6 * 0.2, 0.5 * 0.1), (6 *0.7, 0.5 * 1), (0 * 0.7, 2 * 1), (5 * 0.7, 1 * 1)\}$$

$$= \quad (0.063, 6.48), (6.263, 3.24), (9.503, 1.62), (0.018, 6.4809), (6.2018, 3.2409), (9.4418, 1.6209), (9.503, 1.6209), (0.0063, 6.489), (6.2063, 3.249)\} \in S.$$

This is the way product is performed on S.

It can be easily verified S is a non associative structure.

Example 2.69: Let S = {Collection of all subsets from the MOD plane $R_n(7)$, (0, 0.03), *} be the MOD subset decimal groupoid.

S is non commutative and is of infinite order.

Example 2.70: Let S = {Collection of all subsets from the MOD plane $R_n(13)$, * (0, 0.03)} be the MOD subset decimal groupoid of infinite order.

S is non commutative.

Let x = {(0, 0.2)} and y = {(5,7)} $\in$ S;

$$x * y \quad = \{(0, 0.2)\} * \{(5, 7)\}$$

$$= \{(0 * 5, 0.2 * 7)\}$$

$$= \{(0.15, 0.21)\} \in S.$$

Clearly

$$y * x \neq x * y \text{ for } y * x = \{(5,7)\} * \{(0, 0.2)\}$$

$$= \{(5 * 0, 7 * 0.2)\}$$

$$= \{(0, 0.06)\} \in S$$

$$\neq x * y.$$

Clearly $x * (y * z) \neq (x * y) * z$

For let $x = \{(0.3, 0.7)\}$, $y = \{(0,0.5)\}$ and $z = \{(0.6, 1)\} \in S$;

$x * (y * z)$

$$= \{(0.3, 0.7)\} * (\{0, 0.5)\} * \{(0.6, 1)\})$$

$$= \{(0.3, 0.7)\} * \{(0 * 0.6, 0.5 * 1)\}$$

$$= \{(0.3, 0.7)\} * \{(0.018, 0.03)\}$$

$$= \{(0.3 * 0.018, 0.7 * 0.03)\}$$

$$= \{(0.00054, 0.0009)\} \quad \ldots \quad \mathrm{I}$$

Consider $(x * y) * z$

$$= [\{(0.3, 0.7)\} * \{(0, 0.5)\}] * \{(0.6, 1)\}$$

$$= \{ (0.3, 0.7) * (0, 0.5)\} * \{(0.6, 1)\}$$

$$= \{(0.3 * 0, 0.7 * 0.5)\} * \{(0.6, 1)\}$$

$= \{(0, 0.015)\} * \{(0.6, 1)\}$

$= \{(0.018, 0.03)\}$ …. II

Clearly I and II are distinct.

Thus the operation is non associative decimal interval MOD groupoid.

Example 2.71: Let S = {(Collection of all subsets from the MOD real plane $R_n(25)$, *, (0.7772, 0.543)} be the MOD subset decimal groupoid.

S is of infinite order.

Clearly * is both non associative and non commutative on S.

Example 2.72: Let M = {(Collection of all subsets from the real MOD plane $R_n(16)$, *, (0.311, 6.8)} be the MOD subset groupoid.

Clearly M is non commutative.

We see finding idempotents, units and zero divisors in decimal subset MOD real groupoids happens to be a challenging problem.

Now we proceed onto suggest a few problems for this chapter.

Problem

1. Find some special features enjoyed by MOD real plane groupoids.

2. Let M = {$R_n(20)$, *, (10, 11)} be the MOD real plane groupoid.

   i) Find subgroupoids of finite order.
   ii) Prove M has idempotents.
   iii) Can M have S-idempotents?
   iv) Can M have units?
   v) Find S-units if any in M.
   vi) Can M have zero divisors?
   vii) Can M have zero divisors, which are not S-zero divisors?
   viii) Obtain any other special feature associative with M.
   ix) Can M have ideals of finite order?
   x) Can M have S-ideals?
   xi) Is every right ideal a left ideal?
   xii) Find S-right ideals if any in M.

3. Let N = {$R_n(20)$, *, (18, 2)} be the real MOD plane interval groupoid.

   Study questions (1) to (xii) of problem (2) for this N.

4. Let X = {$R_n(12)$, *, (4, 9)} be the real MOD plane groupoid.

   Study questions (1) to (xii) of problem (2) for this X.

5. Characterize all those MOD groupoids built on $R_n(m)$ to satisfy the special identities like Moufang, Bol, Bruck, idempotents, alternative and so on.

6. Characterize those real MOD plane groupoids which are P-groupoids.

7. Characterize those MOD real plane groupoids which are idempotent groupoids.

8. Obtain conditions on the pair (t, s) of the MOD groupoids G so that they satisfy Bol identity.

9. Similarly find those real MOD plane groupoids which satisfy Moufang identity.

10. Characterize those MOD real plane groupoids which are Smarandache groupoids.

11. Let T = $\{R_n(23), *, (11,0)\}$ be the MOD real plane groupoid.

    Study questions (1) to (xii) of problem (2) for this T.

12. Let S = $\{R_n(15), *, (10.003, 0.792)\}$ be the MOD decimal real plane groupoid.

    Study questions (1) to (xii) of problem (2) for this S

13. Let V = $\{R_n(10), *, (0.77703, 5)\}$ be the MOD decimal real plane groupoid.

    Study questions (1) to (xii) of problem (2) for this V.

14. Let M = $\{R_n(127), *, (100.32, 10.032)\}$ be the MOD real plane interval decimal groupoid.

    Study questions (1) to (xii) of problem (2) for this M.

15. Let B = $\{R_n(45), *, (0, 0.000757)\}$ be the MOD real plane decimal interval groupoid.

    Study questions (1) to (xii) of problem (2) for this B.

16. Obtain some special features associated with S-MOD special interval groupoid with identity.

17. Let S = $\{C_n(9), *, (8, 1.02)\}$ be the finite complex modulo integer MOD decimal groupoid.

    Study questions (1) to (xii) of problem (2) for this S.

18. Let M = $\{C_n(27), *, (8 + 4i_F, 20i_F + 2)\}$ be the complex MOD modulo integer groupoid.

    Study questions (1) to (xii) of problem (2) for this M.

19. Let W = $\{R_n^I(27), *, (3,10)\}$ be the MOD neutrosophic interval groupoid.

    Study questions (1) to (xii) of problem (2) for this W.

20. Let S = $\{R_n^I(43), *, (20 + 3I, 19I + 23)\}$ be the MOD neutrosophic interval groupoid.

    Study questions (1) to (xii) of problem (2) for this S.

21. Let T = $\{R_n(15)g,\ g^2 = 0, *, (10, 7.2)\}$ be the MOD dual number interval decimal groupoid.

    Study questions (1) to (xii) of problem (2) for this T.

22. Let W = $\{R_n(16)\ g, *, (10g + 7, 9g + 0.32)\}$ be the MOD dual number interval decimal groupoid.

    Study questions (1) to (xii) of problem (2) for this W.

23. Let Z = $\{R_n(27)\ g;\ g^2 = g, *, (10, 15)\}$ be the MOD special dual like number interval groupoid.

    Study questions (1) to (xii) of problem (2) for this Z.

24. Let T = $\{R_n(151)g,\ g^2 = g, *\ (0.331, 4.12\ g)\}$ be the MOD special dual like interval number decimal groupoid.

    Study questions (1) to (xii) of problem (2) for this T.

25. Let U = $\{R_n(9)\ g \mid g^2 = 8g, *, (2.3, 8)\}$ be the MOD special quasi dual number interval decimal groupoid.

    Study questions (1) to (xii) of problem (2) for this U.

26. Let M = $\{R_n(23)\ g;\ g^2 = 22g, *, (10 + 2.3g, 2.3 + 10g)\}$ be the MOD special dual number like interval decimal groupoid.

    Study questions (1) to (xii) of problem (2) for this M.

27. Let P = $\{R_n(14)\ g;\ g^2 = 13g, *, (g, 10)\}$ be the MOD special dual like number interval groupoid.

    Study questions (1) to (xii) of problem (2) for this P.

28. Obtain some special features enjoyed by MOD dual number (MOD special dual like number, MOD quasi dual number) interval groupoid.

Chapter Three

ALGEBRAIC STRUCTURES ON MOD SUBSETS OF MOD PLANES

The MOD planes was introduced in [45]. Here we proceed on to introduce the notion of algebraic structures on MOD subsets of MOD planes.

The non associative algebraic structure, groupoids and non associative pseudo rings of two types are introduced.

We know the real MOD plane is a small or mini form of the real plane.

Further on the MOD real plane we can get the subsets of them which we choose to call as MOD real subsets.

Using MOD subsets of the plane for the first time we build MOD subset groupoids.

Example 3.1: Let

S = {Collection of all subsets from the MOD real plane R_n (5)}.

= {subsets from the pair (a, b), (c, d), (e, f), … | a, b, c, d, e, f ∈ [0, 5)}.

Let

S_1= {(4, 0.004), (0.221, 0.0001), (0.001, 0.2), (0.6, 0.21)}

and S_2 = {(0, 0), (1, 4.2), (0.1, 0.4)} ∈ S.

We can perform non associative operations on S.

Example 3.2: Let M = {Collection of all subsets from the MOD plane R_n(12)}.

S_1 = {(3.221, 0), (0.21, 0.0051), (10.0001, 6.0009), (1.2107, 0.11075)} and

S_2 = {(0, 0.772), (0.21, 11.2045), (0.0011107, 6.2), (7, 4), (9, 2), (0.17, 2.17801)} are two MOD subsets from the MOD subsets of MOD plane M.

Example 3.3: Let S = {Collection of all subsets from the MOD plane R_n (28)}.

We see S has infinite number of subsets some are of finite order and some subsets are of infinite order.

Example 3.4: Let S = {Collection of all subsets from the MOD plane R_n(14)}. Collection of all subsets from the subsets of the MOD plane [0, 14).

Let A = {(0, 5), (7, 3), (0.331, 0.225), (6.3, 13.451)} and
B = {(0.7, 6.53), (9.23, 10.07), (10.007, 9.083)} be elements of S.

Now having seen examples of subsets of the MOD planes we proceed on to define non associative binary operations on them.

DEFINITION 3.1: *Let S = {Collection of all subsets from the MOD real plane $R_n(m)$}, define a operation $*$ on S such that $s_1 * s_2 = ts_1 + us_2$; $t, u \in Z_m$ and $s_1, s_2 \in S$. $G = \{S, *, (t, u)\}$; G is defined as the subset MOD groupoid on $R_n(m)$.*

Clearly $(0, 0) \in R_n(m)$ does not act like zero for $(x, y) * (0, 0) \neq (0, 0)$ in general and $(0, 0) * (x, y) \neq (0, 0)$.

So $(0, 0)$ is not the zero of these groupoids we call them as MOD pseudo zero of $R_n(m)$.

So we say if

$(x, y) * (a, b) = (0, 0)$ I

and $(a, b) * (x, y) = (0, 0)$ II then

(x, y) is a MOD pseudo zero divisor of $R_n(m)$. If only one of I or II is true we call (x, y) is a one sided MOD pseudo zero divisor.

If $(x, y) * (x, y) = (0, 0)$ we call (x, y) the MOD pseudo nilpotent element of order two.

First we will illustrate this situation by an example or two.

Example 3.5: Let S = {Collection of subsets from $R_n(20)$} be the subsets of the MOD plane $R_n(20)$. $G = \{S, *, (10, 5)\}$ is the subset real MOD plane groupoid.

Let $s_1 = \{(3, 2.1), (0.31, 0), (0, 1.5)\}$ and
$s_2 = \{(1, 0.4), (4, 0.2), (5, 0.7), (0.3, 0), (0, 0.1)\} \in G$.

$$s_1 * s_2 = 10s_1 + (5s_2)$$

= {(10, 1), (3.1, 0), (0, 15)} + {(5,2), (0, 1), (5, 3.5), (1.5, 0), (0, 0.5)}

= {(15, 3), (8.1, 2), (5, 17), (10, 2), (3.1, 1), (0, 16), (15, 4.5), (8.1, 3.5), (5, 18.5), (11.5,0), (4.6, 0), (1.5, 15), (10, 1.5), (3.1, 0.5), (0, 15.5)} $\in$ S.

This is the way '$*$' operation is performed on the MOD subset groupoids on $R_n(20)$.

Clearly the $*$ operation is both non commutative and non associative.

Consider $s_2 * s_1 = 10\, s_2 + 5\, s_1$

= {(10, 4), (0, 2), (10, 7), (3, 0), (0, 1)} + {(15, 10.5), (1.55, 0), (0, 7.5)}

= {(5, 14.5), (15, 12.5), (5, 17.5), (18, 10.5), (15, 11.5), (11.55, 4), (1.55, 2), (11.55, 7), (4.55, 0), (1.55, 1), (10, 11.5), (0, 9.5), (10, 14.5), (3, 7.5), (0, 8.5)} $\in$ S.

Clearly $s_1 * s_2 \neq s_2 * s_1$ for this $s_1, s_2 \in$ G.

Now let x = {(0.3, 0), (5, 1), (0, 2)},

y = {(0.7 1), (1, 0), (2, 2)} and

z = {(0.5, 0.2), (3, 0)} $\in$ G.

Consider x $*$ (y $*$ z) = x $*$ (10 y + 5 z)

= x $*$ [{(7, 10), (10, 0), (10, 10)} + {(2.5, 1), (15, 0)}]

= {(0.3, 0), (5, 1), (0, 2)}$*$ {(9.5, 11), (12.5, 1), (12.5, 11), (22, 10), (5, 0), (5, 10)}

= {(3, 0), (10, 10), (0, 0)} + {(7.5, 15), (2.5, 5), (2.5, 15), (10, 10), (5, 0), (5, 10)}

= {(10, 15), (17.5, 5), (7.5, 15), (5.5, 5), (12.5, 15), (2.5, 5), (2.5, 15), (5.5, 15), (12.5, 5), (13, 10), (10, 10), (0, 0), (15, 10), (8, 0), (8, 10), (18, 0), (15, 0), (5, 0)} … I

Consider (x ∗ y) ∗ z

= (10 x + 5 y) ∗ z

= ({(3, 10), (10, 10), (0, 0)} + {(3.5, 5), (5, 0), (10, 10)})∗ z

= {(6.5, 15), (13.5, 15), (3.5, 5), (8, 10), (15, 0), (5, 10), (10, 10), (0, 0), (13, 0)} ∗ {(0.5, 0.2), (3, 0)}

= {(5, 10), (15, 10), (0, 0), (10, 0)} + {(2.5, 1), (15, 0)}

= {(7.5, 11), (17.5, 11), (2.5, 1), (12.5, 1), (0, 10), (10, 10), (15, 0), (5, 0)} … II

Clearly I and II are distinct so $(x * y) * z \neq x * (y * z)$ in general for x, y, z ∈ G.

Thus this real MOD subset plane groupoid is of infinite order and is both non commutative and non associative.

Example 3.6: Let G = {Collection of all subsets from the MOD plane $R_n(7)$, ∗, (4, 3)} be the real MOD plane interval groupoid. G is of infinite order and G is both non commutative and non associative.

Clearly if s_1 = {(1, 1)} and s_2 = {(3, 3)} ∈ G,

$s_1 * s_2$ = {(1, 1)} ∗ {(3, 3)}

$= \{4\ (1, 1)\} + \{3\ (3, 3)\}$

$= \{(4, 4)\} + \{(9, 9)\}$

$= \{6, 6\}.$

$s_1 * s_1 \quad = \{(1, 1)\} * \{(1, 1)\}$

$= \{4\ (1, 1)\} \times \{3\ (1, 1)\}$

$= \{(4, 4)\} + \{(3, 3)\}$

$= \{(0, 0)\}.$

Thus this groupoid has MOD pseudo nilpotent elements of order two.

$s_2 * s_2 \quad = \{(3, 3)\} * \{(3, 3)\}$

$= \{4\ (3, 3)\} + \{3\ (3, 3)\}$

$= \{(5, 5)\} + \{(2, 2)\}$

$= \{(0, 0)\}.$

Thus $s_2 \in G$ is also MOD pseudo nilpotent of order two.

Let $p_1 = \{(5, 2)\} \in G$.

Clearly

$p_1 * p_1 \quad = \{(5, 2)\} * \{(5, 2)\}$

$= \{(20, 8)\} + \{(15, 6)\}$

$= \{(0, 0)\}.$

Consider $q_1 = \{(3.11, 6.02)\} \in G$

$$q_1 * q_1 = \{(3.11, 6.02)\} * \{(3.11, 6.02)\}$$

$$= \{(12.44, 24.08)\} + \{(9.33, 18.06)\}$$

$$= \{(21.77, 42.14)\}$$

$$= \{(0.77, 0.14)\} \neq \{(0, 0)\}.$$

Thus G has MOD pseudo nilpotent elements of order two which are only finite in number. If the pair in the subset of G is a decimal pair or not an integer than that element in general is not a MOD pseudo nilpotent element of order two.

It is important and interesting to note that $t + u = 4 + 3 = 0 \pmod 7$; if this condition is not satisfied it will not be a MOD pseudo nilpotent element of order two.

Let $s_1 = \{(1, 1), (3, 3)\}$ and $s_2 = \{(4, 4), (6, 6)\} \in G$.

$$s_1 * s_2 = \{(1, 1), (3, 3)\} * \{(4, 4), (6, 6)\}$$

$$= \{(4, 4), (5, 5)\} + \{(5, 5), (4, 4)\}$$

$$= \{(2, 2), (3, 3), (1, 1)\}.$$

Clearly this is not a MOD pseudo zero divisor of G.

Example 3.7: Let M = {Collection of all subsets from the real MOD plane $R_n(12)$; $*$, (8, 4)} be the real MOD plane groupoid of infinite order. M has infinite number of MOD pseudo nilpotent subsets of order two. M has MOD pseudo zero divisors.

For take $s_1 = \{(3, 6), (6, 0), (3, 3), (6, 6)\}$ and

$$s_2 = \{(6, 0), (0, 6), (0, 3)\} \in M.$$

Clearly $s_1 * s_2 = 0 = s_2 * s_1 = 0 = s_1 * s_1 = s_2 * s_2$.

Thus apart from MOD pseudo nipotents of order two M has MOD pseudo zero divisors.

Example 3.8: Let M = {Collection of all subsets from the MOD real plane $R_n(9)$, $*$, (3, 6)} be the MOD subset interval groupoid. M has MOD pseudo zero divisors as well as MOD pseudo nilpotents.

In view of all these we have the following theorem.

THEOREM 3.1: *Let G = {Collection of all subsets from the MOD real plane $R_n(m)$, $*$ (t, u); t, u $\in Z_m$} be the MOD real interval groupoid on [0, m).*

i. G has MOD pseudo nilpotents of order two if $t + u \equiv 0 \ (\text{mod } m)$.

ii. G has MOD pseudo zero divisors if m is a composite number and t, u in Z_m are such that they are $t^2 = 1$ and $u^2 = 1$, $t + u = 0 \ (\text{mod } m)$.

Proof of all theorem is direct and hence left as an exercise to the reader.

Example 3.9: Let G = {Collection of all subsets from the MOD real plane $R_n(15)$, $*$, (11, 4)} be the MOD real plane groupoid of infinite order.

G has nilpotents of order two. However $R_n(15)$ has no MOD pseudo zero divisors other than MOD pseudo nilpotents of order two in this case. For it is to be noted both 11 and 4 are units in [0, 15).

Example 3.10: Let G = {Collection of all subsets from the MOD real plane $R_n(24)$, $*$, (5, 7)} be the groupoid.

Let s_1 = {(x, y)} and s_2 = {(s, r)} be elements of G such that

$s_1 * s_2 = \{(0, 0)\}$; to find $x, y, s, r \in [0, 24)$.

$$\begin{aligned} s_1 * s_2 &= 5\, s_1 + 7\, s_2 \\ &= \{(5x, 5y)\} + \{(7s, 7r)\} \\ &= \{(5x + 7s, 5y + 7r)\} \\ &= \{(0, 0)\} \end{aligned}$$

Thus $5x + 7s = 0$ and $5y + 7r = 0$.

Now multiplying both equation by 5 we get $x + 11s = 0$ and
and $y + 11r = 0$

$x = 13s$, $y = 13r$ hence $s_1 = \{(13s, 13r)\}$, $s_2 = \{(s, r)\}$ in G is such that $s_1 * s_2 = 0$. For take $s_1 = 1$ and $r = 2$.

$s_1 = \{(13, 2)\}$ and $s_2 = \{(1, 2)\}$.

$$\begin{aligned} s_1 * s_2 &= 5s_1 + 7s_2 \\ &= \{(65, 10) + (7, 14)\} \\ &= \{(0, 0)\}. \end{aligned}$$

Hence G has MOD pseudo zero divisors other than MOD pseudo nilpotents of order two.

It is observed that $(t, u) = (5, 7)$ is such that $t + u = 0$ (mod 24) only condition is $t^2 = 1$ (mod 24) and $u^2 = 1$ (mod 24).

Thus this has MOD pseudo zero divisors. Take $(t, u) = (13, 7)$ in the above example.

Clearly $13^2 = 1$ (mod 24) and $7^2 = 1$ (mod 24) and this (t, u) contributes to MOD pseudo zero divisors other than MOD pseudo nilpotents of order two.

Consider (t, u) = (13, 11) in the above example.

Clearly $13^2 = 1 \pmod{24}$

and

$11^2 = 1 \pmod{24}$

and

$13 + 11 \equiv 0 \pmod{24}.$

Let $s_1 = \{(a, b)\}$ and $s_2 = \{(c, d)\} \in G$.

$$\begin{aligned} s_1 * s_2 &= 13s_1 + 11s_2 = \{(13a, 13b)\} + \{(11c, 11d)\} \\ &= \{(13a + 11c, 13b + 11d)\} \\ &= \{(0, 0)\}. \end{aligned}$$

Thus

$$13a + 11c = 0$$

$$13b + 11d = 0$$

$$a + 23c = 0$$

$$b + 23d = 0$$

which gives a = c and b = d.

Thus all MOD pseudo zero divisors are only MOD pseudo nilpotents of order two.

Example 3.11: Let G = {Collection of all subsets from the real MOD plane $R_n(20)$, *, (9, 11)} be the MOD real interval groupoid on [0, 20).

Clearly $9^2 \equiv 1 \pmod{20}$ and $11^2 = 1 \pmod{20}$ but $9 + 11 = 0 \pmod{20}$. So G has only MOD pseudo zero divisors of the form which are MOD pseudo nilpotent subsets of order two.

Take (9, 19) = (u, t)

Clearly

$$19^2 = 1 \pmod{20}$$

and

$$9^2 = 1 \pmod{20}$$

but

$$19 + 9 = 8 \pmod{20}.$$

Consider

$$s_1 = \{(a, b)\} \text{ and } s_2 = \{(c, d)\} \in G$$

such that $s_1 * s_2 = \{(0, 0)\}$.

$$\begin{aligned} s_1 * s_2 &= 19s_1 + 9s_2 \\ &= \{(19a, 19b), (9c, 9d)\} \\ &= \{(19a + 9c, 19b + 9d)\} \\ &= \{(0, 0)\}. \end{aligned}$$

$$19a + 9c = 0$$

$$19b + 9d = 0$$

$$a + 11c = 0 \pmod{20}$$

$$b + 11d = 0 \pmod{20}$$

$$a = 9c \text{ and } b = 9d.$$

$s_1 = \{(9c, 9d)\}$ and $s_2 = \{(c, d)\} \in G$ are zero divisors.

Take c = 1 and d =3, then $s_1 = \{(9, 7)\}$ and $s_2 = \{(1, 3)\} \in$ G.

$$\begin{aligned} s_1 * s_2 &= \{(9, 7)\} * \{(1, 3)\} \\ &= \{(19 \times 9, 19 \times 7)\} + \{(9, 7)\} \end{aligned}$$

$= \{(11, 13)\} + \{(9, 7)\} = \{(0, 0)\}.$

Thus has MOD pseudo zero divisors. Also the pair (19,21) will give MOD pseudo zero divisors other than MOD pseudo nilpotents of order two.

In view of this we have the following theorem.

THEOREM 3.2: *Let G = {Collection of all subsets from the real MOD plane R_n (m), *, (t, u)} be the real MOD plane groupoid on [0, m).*

G has MOD pseudo zero divisors which are not MOD pseudo nilpotents of order two only if $t^2 \equiv 1 \pmod{m}$, $u^2 \equiv 1 \pmod{m}$; $t + u \not\equiv 0 \pmod{m}$.

Proof is direct and hence left as an exercise to the reader.

Example 3.12: Let G = {Collection of all subsets from the real MOD plane R_n (12), *, (4, 9)} be the real MOD plane interval subset groupoid.

Consider $s_1 = \{(3, 6), (9, 3), (9, 9), (6, 6), (9, 6)\}$ and
$s_2 = \{(4, 4), (8, 4), (4, 8), (8, 0), (4, 0)\} \in G$.

Clearly $s_1 * s_2 = \{(0, 0)\}$.

Let $s_1 = \{(5, 5)\} \in G$.

$$\begin{aligned} s_1 * s_1 &= 4s_1 + 9s_2 \\ &= \{(8, 8)\} + \{(9, 9)\} \\ &= \{(5, 5)\} = s_1. \end{aligned}$$

Thus this MOD real interval groupoids has idempotents.

Take $p_1 = \{(1, 1)\} \in G$.

$p_1 * p_1 \quad = \{(1, 1)\} * \{(1, 1)\}$

$= \{(4, 4)\} + \{(9, 9)\}$

$= \{(1, 1)\} = p_1$ is an idempotent of G.

Consider $q_1 = \{(3, 3)\} \in$ G;

$q_1 * q_1 \quad = 4q_1 + 9q_1$

$= \{(12, 12)\} + \{(3, 3)\}$

$= \{(3, 3)\} = q_1$ is again an idempotent of G.

Thus all elements $s = \{(a, a)\}$; $a \in Z_{12}$ are subset idempotents of the MOD subset interval groupoid G.

Let $x = \{(9.1, 9.1)\} \in$ G.

$x * x \quad = \{(9.1, 9.1)\} * \{(9.1, 9.1)\}$

$= \{(0.4, 0.4)\} + \{(9.9, 9.9)\}$

$= \{(1.03, 1.03)\} \neq x.$

Thus those subsets $s = \{(s_1, s_1)\}$ with $s_1 \in [0, 12) \setminus Z_{12}$ need not be idempotents of G.

It is observed that for the pair (4, 9), $4^2 = 4$ and

$9^2 = 9$ and $4 + 9 = 1 \pmod{12}$.

Consider $y = \{(3, 8)\} \in$ G

$y * y \quad = \{(3, 8)\} * \{(3, 8)\}$

$= \{(0, 8)\} + \{(3, 0)\}$

$= \{(3, 8)\} = y.$

Thus y is an idempotent of G.

Hence all $z = \{(s_1, s_2)\}$ where $s_1, s_2 \in Z_{12}$ are idempotent subsets of G.

Take $m = \{(3.2, 4.1)\} \in G$

$m * m = 4m + 9m$

$= \{(0.8, 4.4)\} + \{(4.8, 0.9)\}$

$= \{(5.6, 5.3)\} \neq m.$

Thus in general those $x = \{(s_1, s_2)\}$; $s_1, s_2 \in [0, 12) \setminus Z_{12}$ are not idempotents of G.

Now replace (4, 9) by (4, 4) in the example.

Let $x = \{(5, 5)\} \in G.$

$x * x = 4x + 4x$

$= \{(8, 8)\} + \{(8, 8)\}$

$= \{(4, 4)\}$

$\neq x.$

Thus when (4, 9) is replaced by (4, 4) we do not get $x = \{(a, a)\}$ where $a \in Z_{12}$ to be idempotents of G. Consider (9, 9) instead of (4, 9) in the above example.

Let $y = \{(7, 7)\} \in G$

$$y * y = 7y + 7y$$
$$= \{(63, 63)\} + \{(63, 63)\} \pmod{12}$$
$$= \{(6, 6)\} \neq y$$
$$\neq \{(7, 7)\}.$$

Consider $x = \{(1, 1)\} \in G$

$$x * x = \{(9, 9)\} + \{(9, 9)\}$$
$$= \{(6, 6)\} \neq x.$$

Let $a = \{(2, 2)\} \in G$

$$a * a = \{(18, 18)\} + \{(18, 18)\}$$
$$= \{(0, 0)\} \neq a$$

is infact a MOD pseudo nilpotent element of order two.

Let $b = \{(3, 3)\} \in G$

$$b * b = \{(3, 3)\} + \{(3, 3)\}$$
$$= \{(6, 6)\} \neq b.$$

Let $c = \{(4, 4)\} \in G$

$$c * c = \{(0, 0)\}$$
$$\neq c$$

is again a MOD pseudo nilpotent element of order two.

Let d = $\{(5, 5)\} \in G$

$d * d = \{(6, 6)\}$

$\neq d.$

Hence is not an idempotent.

$e = \{(6, 6)\} \in G;$

$e * e = \{(0, 0)\}$

is not an idempotent of G only a MOD pseudo nilpotent element of order two.

Let $f = \{(8, 8)\} \in G$

$f * f = \{(0, 0)\}$

is again a MOD pseudo nilpotent element of order two.

$g = \{(9, 9)\} \in G$ is such that

$g * g = \{(6, 6)\}$

$\neq g$

is not an idempotent of G.

$h = \{(10, 10)\} \in G$ is such that

$h * h = \{(0, 0)\}$

is a MOD pseudo nilpotent of order two and not an idempotent.

$p = \{(11, 11)\} \in G$ is such that

$$\begin{aligned} p * p &= \{(99, 99)\} + \{(99, 99)\} \\ &= \{(3, 3)\} + \{(3, 3)\} \\ &= \{(6, 6)\} \neq p \end{aligned}$$

hence p is not an idempotent.

Thus no $\{(x, x)\} = m \in G$ with $x \in Z_{12}$ is an idempotent for $(9, 9) = (t, u)$ as operator of the groupoid.

Finally we consider an example of this type.

Example 3.13: Let G = {Collection of all subsets from the MOD real plane $R_n(16)$, ∗, (4, 8)} be the MOD subsets interval groupoid.

$$\begin{aligned} \text{Let } s_1 &= \{(3, 3)\} \in G \\ s_1 * s_1 &= \{(12, 12)\} + \{(24, 24)\} = \{(4, 4)\}. \end{aligned}$$

$$\begin{aligned} \text{Let } y &= \{(2, 2)\} \in G \\ y * y &= \{(8, 8)\} + \{(16, 16)\} \\ &= \{(8, 8)\} \neq y. \end{aligned}$$

$$\begin{aligned} \text{Let } a &= \{(4, 4)\} \in G. \\ a * a &= \{(0, 0)\} \end{aligned}$$

is a MOD pseudo nilpotent element of order two.

Let b $= \{(5, 5)\} \in G$.

b $*$ b $= \{(20, 20)\} + \{(40, 40)\}$

$= \{(12, 12)\} \neq b$.

Let c $= \{(6, 6)\}$

c $*$ c $= \{(24, 24)\} + \{(48, 48)\}$

$= \{(8, 8)\} \neq c$.

Let d $= \{(7, 7)\} \in G$

7 $*$ 7 $= \{(28, 28)\} + \{(56, 56)\}$

$= \{(4, 4)\}$.

Thus we see in general if (t, u) is such that $t^2 = 0$ and $u^2 = 0$ then s = $\{(a, a)\}$; $a \in Z_{16}$ need not in general be an idempotent or pseudo nilpotent of order two.

However for m = $\{(8, 8)\}$ and n = $\{(12, 12)\}$ are MOD pseudo nilpotents of order two in G.

Example 3.14: S = {Collection of all subsets from the real MOD plane $R_n(11)$, $*$, (0, 10)} be the MOD subsets real interval groupoid.

Let

$$x = \{(5, 0), (9, 0)\} \in S,$$

x $*$ x $=$ 0x + 10x

$=$ (6, 2).

Let x = {(3, 3), (4, 4)} ∈ S,

x * x = {(8, 8), (7, 7)} ∈ S.

Clearly 10 is a unit in [0, 11) and it is important to study whether S has pseudo nilpotents or pseudo zero divisors.

This study is interesting and innovative for any collection of subsets in a MOD real plane $R_n(m)$ with m a prime and (t, u) = (0, m – 1).

In view of finding those MOD real subset interval groupoid using the MOD plane R_n (m) which has no MOD pseudo zero divisors, no idempotents no MOD pseudo nilpotents, etc happens to be a challenging and an interesting problem.

However a few results in this direction are given.

THEOREM 3.3: *Let S = {Collection of all subsets from the real MOD plane R_n (m); m a prime number, (0, t); t ∈ Z_p, *} be the MOD subsets real interval groupoid.*

S has no MOD pseudo zero divisors.

Proof: Let

x = {(a, b)} and y = {(c, d)} ∈ S.

x * y = {(a, b)} * {(c, d)}

= {(a * c, b * d)}

= {(0 + tc, 0 + td)}

= {(0, 0)} if and only if c = d = 0.

Now if c = d = 0 then y = {(0, 0)} the zero element of S. Thus S has no left MOD pseudo zero divisors or right MOD pseudo zero divisors.

Thus this class of MOD subset groupoids with (0, t); $t \neq 0 \in Z_p$; p a prime can give MOD subsets groupooids with no MOD pseudo zero divisors.

Example 3.15: Let S = {Collectin of all subsets from the real MOD plane R_n (12); (0, 4), $*$} be the subset MOD real interval groupoid.

Let x = {(5, 2), (7, 4), (3, 9), (0, 8), (6, 0)} and

y = {(3, 3), (6, 3), (6, 6), (9, 3), (3, 9), (3, 6), (9, 9)} $\in$ S

x $*$ y = {(5, 2) $*$ (3, 3), (5, 2) $*$ (6, 3), (5, 2) $*$ (6, 6), (5, 2) $*$ (3, 9), (5, 2) $*$ (9, 3), (5, 2) $*$ (3, 6), (5, 2) $*$ (9, 9), …, (6, 0) $*$ (3, 3), (6, 0) $*$ (6, 3), …., (6, 0) $*$ (9, 9)} = {(0, 0)}.

Thus this is only a one sided zero divisors.

However x $*$ y = {(0, 0)} but y $*$ x $\neq$ {0, 0}.

For consider

y $*$ x = {(3, 3) $*$ (5, 2), (3, 3) $*$ (7, 4), …, (9, 9) $*$ (0, 8), …}

$= \{(3 * 5, 3 * 2), (3 * 7, 3 * 4), \ldots, (9 * 0, 9 * 8), \ldots\}$

$= \{(0 + 20, 0 + 8), (0 + 28, 0, 16), \ldots, (0 + 0, 0 + 32), \ldots\}$

$= \{(8, 8), (4, 4), \ldots, (0, 8), \ldots\} \neq \{(0, 0)\}$.

Thus S has MOD pseudo zero divisors. It is to be noted that the MOD subsets real groupoid was built on $R_n(m)$ where m is a composite number having MOD pseudo zero divisors.

Hence the claim.

In view of this we have the following theorem.

THEOREM 3.4: *Let S = {Collection of all subsets from the real MOD plane $R_n(m)$, m a composite number, (0, t); $0 \neq t \in Z_m$ is a divisor of m; *} be the subsets MOD real plane groupoid. S has left MOD pseudo zero divisors which are not right MOD pseudo zero divisors and vice versa.*

Proof is direct and hence left as an exercise to the reader.

Example 3.16: Let S = {Collection of all subsets from the real MOD plane $R_n(12)$, 12 is a composite number (0, 5) where $0 \neq t \in Z_{12}$ is a unit in Z_{12}, $*$} be the MOD subsets interval groupoid.

Consider

$$x = \{(a, b)\} \text{ and } y = \{(c, d)\} \in S.$$

$$\begin{aligned} x * y &= \{(a, b) * (c, d)\} \\ &= \{(a * c, b * d)\} \\ &= \{(0 + 5c, 0 + 5d)\} \\ &= \{(0, 0)\} \text{ if and only if } c = 0 = d. \end{aligned}$$

Thus if $t \in Z_{12}$ is a unit in Z_{12} then S has no right or left MOD pseudo zero divisors.

THEOREM 3.5: *Let S = {Collection of all subsets from the MOD plane R_n (m); m a composite number , (0, t); $t \in Z_m$ is a unit in Z_m, *} be the MOD subset interval real groupoid. S has no right or left MOD pseudo zero divisors.*

Proof: Follows from the simple fact if

$$s_1 = \{(x, y)\} \text{ and } s_2 = \{(a, b)\} \in S$$

$$\begin{aligned} s_1 * s_2 &= \{(x, y)\} * \{(a, b)\} \\ &= \{(x * a, y * b)\} \\ &= \{(0 + at, 0 + bt)\} \end{aligned}$$

$= \{(0, 0)\}$

if and only if a = b = 0 as $t \in Z_m$ is a unit in S.

Example 3.17: Let S = {Collection of all subsets from the MOD real plane $R_n(13)$, (t, 0); $t \in Z_{13} \setminus \{0\}$, $*$} be the subset MOD interval groupoid; S has no left or right MOD pseudo zero divisors.

Example 3.18: Let S = {Collection of all subsets from the MOD real plane $R_n(m)$, (5, 0), $t \in Z_{45} \setminus \{0\}$, $*$} be the subset MOD interval groupoid S has both left and right MOD pseudo zero divisors.

If in the above example (5, 0) is replaced by (0, 5) then the left MOD pseudo zero divisor using (5, 0) will become the right MOD pseudo zero divisor and vice versa.

This is put forth in the form of the result.

THEOREM 3.6: *Let S = {Collection of all subsets from the real MOD plane $R_n(m)$; m a composite number (t, 0) ((0, t)) where t is a zero divisor of Z_m, $*$} be the MOD subset real interval groupoid. Then all left MOD pseudo zero divisors using (t, 0) will be right zero divisors when (0, t) is used instead of (t, 0).*

Proof is direct and hence left as an exercise to the reader.

Now having characterized one of the conditions for the MOD subset real groupoid to have MOD pseudo zero divisors, we now proceed on to define different structures on them.

Example 3.19: Let S = {Collection of all subsets from the MOD plane $R_n(24)$; $*$ (8, 6); t and $u \in Z_{24}$ are MOD pseudo zero divisors of Z_{24}}.

Let

$$A = \{(3, 6), (6, 3), (0, 3), (0, 6)\}$$

and

$$B = \{(4, 0), (8, 4), (4, 4), (8, 8)\} \in S;$$

$$\begin{aligned} A * B &= \{(3, 6) * (4, 0), (3, 6) * (8, 4), (3, 6) * (4, 4), \\ &\quad (3, 6) * (8, 8), \ldots, (0, 6) * (4, 0), (0, 6) * (8, 4), \\ &\quad (0, 6) * (4, 4), (0, 6) * (8, 8)\} \\ &= \{(0, 0)\} \end{aligned}$$

Thus S has MOD pseudo zero divisor.

Clearly B ∗ A is not a MOD pseudo zero divisor for B ∗ A ≠{(0, 0)} for

$$\begin{aligned} B * A &= \{(4, 0) * (3, 6), (8, 4) * (6, 3), (8, 4) * \\ &\quad (0, 3), \ldots, (8, 8) * (0, 6)\} \\ &= \{(4 * 3, 0 * 6), (8 * 6, 4 * 3), (8 * 0, 4 * 3), \\ &\quad \ldots, (8 * 0, 8 * 6)\} \\ &= \{(2, 12), (4, 2), (16, 2), \ldots, (16, 4)\} \\ &\neq \{(0, 0)\}. \end{aligned}$$

Hence the claim.

Next consider the following example.

Example 3.20: Let S = {Collection of all subsets from the real MOD plane $R_n(10)$, (2, 4), ∗} be the MOD subsets interval groupoid.

Consider x = {(5, 5), (5, 0), (0, 5)} ∈ S.

$$x * x = \{(0, 0)\}.$$

It is important and interesting to note that S has MOD pseudo nilpotent element of order two.

However only elements from the subset alone can contribute to MOD pseudo zero divisors.

For x = {(0, 5)} and y = {(5, 5)} is such that

$$x * y = y * x = \{(0, 0)\}.$$

Thus this has two sided MOD pseudo zero divisors.

Example 3.21: Let S = {Collection of all subsets from the real MOD plane $R_n(14)$, (2, 8), ∗} be the MOD subset interval real groupoid.

The one sided MOD pseudo zero divisors of S are

P = {(5, 5), (4, 4)} ∈ S; consider

$$\begin{aligned}
P * P &= \{(5, 5), (4, 4)\} * \{(5, 5), (4, 4)\} \\
&= \{(5, 5) * (5, 5), (5, 5) * (4, 4), (4, 4) * (5, 5), \\
&\quad (4, 4) * (5, 5)\} \\
&= \{(5 * 5, 5 * 5), (5 * 4, 5 * 4), (4 * 5, 4 * 5), \\
&\quad (4 * 4, 4 * 4)\} \\
&= \{(10 + 40, 10 + 40), (10 + 32, 10 + 32), \\
&\quad (8 + 40, 8 + 40), (8 + 32, 8 + 32)\} \\
&= \{(8, 8), (0, 0), (6, 6), (12, 12)\} \\
&\neq \{(0, 0)\}.
\end{aligned}$$

Thus is not a MOD pseudo zero divisor.

However if x = {(5, 5)} and y = {(4, 4)}

$$\begin{aligned}
x * y &= \{(5, 5) * (4, 4)\} = \{(5 * 4, 5 * 4)\} \\
&= \{(10 + 32, 10 + 32)\} \\
&= \{(0, 0)\} \text{ is a zero divisor.}
\end{aligned}$$

$$\begin{aligned} \text{However } y * x &= \{(4, 4)\} * \{(5, 5)\} \\ &= \{(4 * 5, 4 * 5)\} \\ &= \{(8 + 40, 8 + 40)\} \\ &= \{(6, 6)\} \neq \{(0, 0)\}. \end{aligned}$$

Hence x is only a one sided MOD pseudo zero divisor of S.

Consider $x = \{(7, 7), (0, 7), (7, 0)\} \in S$.

Clearly $x = x * x = \{(0, 0)\}$.

Infact x is a MOD pseudo nilpotent element of order two. Thus S has MOD pseudo nilpotent elements of order two as well as left MOD pseudo zero divisors which are not right MOD pseudo zero divisors and vice versa.

Example 3.22: Let S = {Collection of all subsets from the real MOD plane $R_n(22)$, $*$, (2, 4)} be the subset MOD groupoid.

S has right MOD pseudo zero divisors, MOD pseudo nilpotents and left MOD pseudo zero divisors which are not right MOD pseudo zero divisors.

For $x = \{(0, 11), (11, 11), (0, 11)\} \in S$;

$$x^2 = x * x = \{(0, 0)\}.$$

Consider $y = \{(6, 0), (3, 0)\} \in S$,

$$\begin{aligned} y * y &= \{(6, 0), (3, 0)\} * \{(6, 0), (3, 0)\} \\ &= \{(6 * 3, 0), (3 * 6, 0), (6 * 6, 0), (3 * 3, 0)\} \\ &= \{(12 + 12, 0), (6 + 24, 0), (12 + 24, 0), \\ &\qquad (6 + 12, 0)\} \end{aligned}$$

$$= \{(2, 0), (8, 0), (3, 0), (6, 0)\}$$

$$\neq \{(0, 0)\}.$$

Let $x = \{(5, 0)\}$ and $\{(3, 0)\} = y \in S$

$$\begin{aligned} x * y &= \{(5, 0)\} * \{(3, 0)\} \\ &= \{(5 * 3, 0)\} \\ &= \{(10 + 12, 0)\} \\ &= \{(0, 0)\} \text{ is a MOD pseudo zero divisor.} \end{aligned}$$

$$\begin{aligned} y * x &= \{(3, 0)\} * \{(5, 0)\} \\ &= \{(3 * 5, 0)\} \\ &= \{(6 + 20, 0)\} \\ &= \{(4, 0)\} \neq \{(0, 0)\}. \end{aligned}$$

Thus x is only a one sided MOD pseudo zero divisor of S.

In view of this we have the following theorem.

THEOREM 3.7: *Let S = {Collection of subsets from the MOD real plane $R_n(2p)$; p a prime, *, (2, 6)} be the MOD subset real groupoid.*

S has nilpotents and right MOD pseudo zero divisors which are not left zero MOD pseudo divisors and vice versa.

Proof: Consider $M = \{(p, p), (0, p), (p, 0)\} \in S$;
clearly $M^2 = M * M = \{(0, 0)\}$ is a nilpotent element of order two.

Let x = {(a, b)}, there exists y = {(c, d)} such that

$$x * y = \{(0, 0)\} \text{ but } y * x \neq \{(0, 0)\}.$$

Hence the claim.

Now having seen about subset MOD groupoids which has MOD pseudo zero divisors now we proceed onto study idempotents in subset MOD groupoids.

Example 3.23: Let S = {Collection of all subsets from the real MOD plane $R_n(12)$; (4, 3), $*$} be the MOD subset groupoid.

Let x = {(4, 4)} and {(0, 8)} = y ∈ S

$$\begin{aligned} x * y &= \{(4, 4)\} * \{(0, 8)\} \\ &= \{(4 * 0, 4 * 8)\} \\ &= \{(4, 4)\}. \end{aligned}$$

Let z = {(2, 2)} ∈ S,

$$\begin{aligned} z * z &= \{(2, 2)\} * \{(2, 2)\} \\ &= \{(2 * 2, 2 * 2)\} \\ &= \{(8 + 6, 8 + 6)\} \\ &= \{(2, 2)\} \text{ is an idempotent of S.} \end{aligned}$$

Thus these subset groupoids can have non trivial idempotents.

Let a = {(8, 8)} ∈ S is again an idempotent of S.

$$\begin{aligned} \text{For } a * a &= \{(8, 8) * (8, 8)\} \\ &= \{(8 * 8, 8 * 8)\} \end{aligned}$$

$$= \{(32 + 24, 32 + 24)\}$$

$$= \{(8, 8)\} = a.$$

Clearly b = {(10, 10)} $\in$ S is also an idempotent of S.

$$b * b = \{(10, 10)\} * \{(10, 10)\}$$

$$= \{(10 * 10, 10 * 10)\}$$

$$= \{(40 + 30, 40 + 30)\}$$

$$= \{(10, 10)\} = b.$$

Thus b is an idempotent of S.

Example 3.24: Let
S = {All subsets of the MOD plane $R_n(20)$, $*$, (4, 5)} be the MOD subset real interval groupoid.

Let x = {(5, 5)} $\in$ S

$$x * x = \{(5, 5)\} * \{(5, 5)\}$$

$$= \{(5 * 5, 5 * 5)\}$$

$$= \{(20 + 25, 20 + 25)\}$$

$$= \{(5, 5)\} = x.$$

Thus x is an idempotent of S.

Also y = {(10, 10)} $\in$ S is such that

$$y * y = \{(10, 10)\} * \{(10, 10)\}$$

$$= \{(10 * 10, 10 *10)\}$$

$$= \{(40 + 50, 40 + 50)\}$$

$$= \{(10, 10)\} = y.$$

Thus y is also an idempotent of S.

Further $z = \{(15, 15)\} \in S$ is such that

$$z * z = \{(15, 15)\} * \{(15, 15)\}$$

$$= \{(15 * 15, 15 * 15)\}$$

$$= \{(60 + 75, 60 + 75)\}$$

$$= \{(15, 15)\} = z \text{ is an idempotent of S.}$$

In view of all these we can conclude MOD interval subset groupoids have MOD pseudo zero divisors.

It is also observed if the idempotents in Z_m are orthogonal can we have subsets of cardinality greater than or equal to two to be idempotents of S = {Collection of all subsets from the MOD plane $R_n(m)$, $*$, (t, u)}.

This study is also interesting however we have examples of such situation for many m; m a positive non prime integer associated with the MOD plane $R_n(m)$.

It is also important finding subset idempotents in G = {Collection of all subsets from $R_n(m)$, $*$, (t, u); t, u $\in Z_m$} is a difficult job.

However it is left as an open conjecture to find subset idempotents in the MOD subset interval groupoid $G_1^* =$ {Collection of all subsets from $R_n(m)$; $*$, (t, u); t, u $\in R_n(m) \setminus Z_m$} as well as in $G_2^* =$ {Collection of all subsets from $R_n(m)$, $*$, (t, u); t $\in R_n(m) \setminus Z_m$ and u $\in Z_m$} the subset MOD groupoid of $R_n(m)$.

Characterizing idempotents, S-idempotents, units, S-units, MOD pseudo zero divisors, S-MOD pseudo zero divisors nilpotents and S-nilpotents in G_1^* and G_2^* happens to be a challenging open problems/ conjectures for any interested researcher.

Now having seen examples of idempotents here some examples of G_1^* and G_2^* are given.

Example 3.25: Let G_1^* = {All subsets from the MOD plane $R_n(10)$, $*$, (2.5, 4.2)} be the MOD subset interval groupoid.

Let A = {(2, 5), (4, 5)} and B = {(5, 2.5), (5, 6.2)} $\in G_1^*$

$$\begin{aligned} A * B &= \{(2, 5), (4, 5)\} * \{(5, 2.5), (5, 6.2)\} \\ &= \{(2 * 5, 5 * 2.5), (2 * 5, 5 * 6.2), \\ &\quad (4 * 5, 5 * 2.5), (4 * 5, 5 * 6.2)\} \\ &= \{(6, 3), (1, 3), (6, 8.54), (1, 8.54)\} \in S. \end{aligned}$$

This is the way '$*$' operation is performed G_1^*.

$$\begin{aligned} \text{Consider } B * A &= \{(5, 2.5), (5, 6.2)\} * \{(2, 5), (4, 5)\} \\ &= \{(5 * 2, 2.5 * 5), (5 * 2, 6.2 * 5), \\ &\quad (5 * 4, 2.5 * 5), (5 * 4, 6.2 * 5)\} \\ &= \{(0.9, 3.5), (0.9, 7.2), (9.3, 3.5), (9.3, 7.2)\} \\ &\qquad \in G_1^* \end{aligned}$$

Clearly A $*$ B $\neq$ B $*$ A so G_1^* is a non commutative MOD subset groupoid of infinite order.

Finding idempotents or S-idempotents in G_1^* happens to be a challenging one.

Similarly finding units and S-units of G_1^* is a difficult problem.

Finally finding MOD pseudo zero divisors or S-MOD pseudo zero divisors is not a simple problem.

Example 3.26: Let G_2^* = {Collection of all subsets of the MOD plane $R_n(12)$, $*$ (10.5, 0)} be the MOD subset interval groupoid.

Let

$$A = \{(4, 8), (8, 4), (8, 0), (4, 0)\}$$

and

$$B = \{(7.9, 2.8), (4.32, 0.72)\} \in G_2^*.$$

To find $A * B$

$$\begin{aligned} A * B &= \{(4, 8), (8, 4), (8, 0), (4, 0)\} * \{(7.9, 2.8), (4.32, 0.72)\} \\ &= \{(4 * 7.9, 8 * 2.8), (4 * 4.32, 8 * 0.72), (8 * 7.9, 4 * 2.8), (8 * 4.32, 4 * 0.72), \\ &\quad (8 * 7.9, 0 * 2.8), (8 * 4.32, 0 * 0.72), (4 * 7.9, 0 * 2.8), (4 * 4.32, 0 * 0.72)\} \\ &= \{(6, 0), (6, 0), (0, 6), (0, 6), (0, 0), (0, 0), (6, 0), (6, 0)\} \\ &= \{(0, 0), (0, 6), (6, 0)\} \in G_1^* \end{aligned}$$

Now consider

$$A = \{(8, 8), (8, 0), (0, 8)\}$$

and

$$B = \{(9.21, 8.33), (7.1, 10.3), (6.3, 5.4)\} \in G_2^*$$

$$\begin{aligned} A * B &= \{(8, 8), (8, 0), (0, 8)\} * \{(9.21, 8.33), \\ &\quad (7.1, 10.3), (6.3, 5.4)\} \\ &= \{(8 * 9.21, 8 * 8.33), (8 * 9.21, 0* 8.33), \\ &\quad (0 * 9.21, 8 * 8.33), (8 * 7.1, 8 * 10.3), \\ &\quad (8 * 7.1, 0 * 10.3), (0 * 7.1, 8 * 10.3), \\ &\quad (8 * 6.3, 8 * 5.4), (8 * 6.3, 0 * 5.4), \\ &\quad (0 * 6.3, 8 * 5.4)\} \\ &= \{(0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), \\ &\quad (0, 0), (0, 0)\}. \end{aligned}$$

Thus A ∗ B is only a MOD pseudo one sided zero divisors.

However B ∗ A ≠ {(0, 0)}. Hence G_2^* has left MOD pseudo zero divisors which are not right MOD pseudo zero divisors and vice versa.

But if A = {(8, 0), (0, 8), (8, 8)} ∈ G_2^* then A ∗ A = {(0, 0)}, thus A is a nilpotent element of order two.

B = {(0, 8), (8, 8)} is also a nilpotent element of order two in G_2^*.

Example 3.27: Let G_2^* = {Collection of all subsets from the MOD plane $R_n(17)$, ∗, (2, 10.5)} be the interval MOD subset groupoid of infinite order.

Clearly G_2^* is non commutative and is of infinite order. Finding units, MOD pseudo zero divisors and idempotents in G_2^*, happens to be a very difficult task.

Example 3.28: Let G_1^* = {Collection of al subsets from the MOD plane $R_n(20)$, ∗, (10.5, 10.5)} be the MOD subset interval groupoid.

Clearly G_1^* is commutative but is associative and is of infinite order.

Let A = {(0.31, 5)}, B = {(2, 0.7)} and C = {(2.5, 8)} $\in G_1^*$

$$
\begin{aligned}
(A * B) * C &= [\{(0.31, 5) * (2, 0.7)\}] * \{(2.5, 8)\} \\
&= \{(0.31 * 2, 5 * 0.7)\} * \{(2.5, 8)\} \\
&= \{(4.255, 19.85)\} * \{(2.5, 8)\} \\
&= \{(4.255 * 2.5, 19.85 * 8)\} \\
&= \{(10.9275, 12.425)\} \quad \ldots \quad \text{I}
\end{aligned}
$$

Consider

$$
\begin{aligned}
A * (B * C) &= \{(0.31, 5)\} * (\{(2, 0.7)\} * \{(2.5, 8)\}) \\
&= \{(0.31, 5)\} * \{(2 * 2.5, 0.7 * 8)\} \\
&= \{(0.31, 5)\} * \{(7.25, 11.35)\} \\
&= \{(0.31 * 7.25, 5 * 11.35)\} \\
&= \{(19.38, 11.675)\} \quad \ldots \quad \text{II}
\end{aligned}
$$

Clearly I and II are not equal so G_1^* is a non associative interval MOD groupoid.

THEOREM 3.8: *The MOD subset interval groupoid G_2^* is never commutative.*

Proof: By the very definition of G_2^*; G_2^* = {Collection of all subsets from the MOD plane R_n (m), $*$, (t, u) where t $\in$ [0, m) \ Z_m and u $\in$ Z_m} be the MOD subset interval groupoid.

So (t, u) = (u, t) is impossible in G_2^*.

Hence G_2^* is never commutative.

Next MOD subset groupoids using the MOD finite complex modulo integer plane $C_n(m) = \{a + bi_F \mid a, b \in [0, m); i_F^2 = m - 1\}$ is studied. There will be two types in the first type we will use (t, u) to be in Z_m and in the second type t, u $\in C_n(m)$.

All these are illustrated by some examples.

Example 3.29: Let G = {Collection of all subsets of from the MOD complex modulo integer plane $C_n(9)$; $i_F^2 = 8$, (3, 4), $*$} be the simple MOD complex modulo integer interval groupoid.

Clearly $|G| = \infty$.

For $x = \{8 + 6.7i_F\}$ and $y = \{(4.2 + 0.7i_F)\} \in G$;

$$\begin{aligned} x * y &= 3(8 + 6.7i_F) + 4(4.2 + 0.7i_F) \\ &= 24 + 20.1i_F + 16.8 + 2.8i_F \\ &= \{4.8 + 4.9i_F\} \in G. \end{aligned}$$

This is the way $*$ operation is performed on G.

Consider $x = \{3 + 0.5i_F, 0.8i_F, 4\}$ and
$y = \{7i_F, 8, 3.6 + i_F\} \in G$;

$$\begin{aligned} x * y &= \{3 + 0.5i_F, 0.8i_F, 4\} * \{7i_F, 8, 3.6 + i_F\} \\ &= 3\{3 + 0.5i_F, 0.8i_F, 4\} + 4\{7i_F, 8, 3.6 + i_F\} \\ &= \{0 + 1.5i_F, 2.4i_F, 3\} + \{i_F, 5, 0.4 + 4i_F\} \end{aligned}$$

$= \{2.5i_F, 3.4i_F, 3 + i_F, 5 + 1.5i_F, 5 + 2.4i_F, 8, 0.4 + 5.5i_F, 0.4 + 6.4i_F, 3.4 + 4i_F\} \in G.$

It is easily verified $*$ is a non commutative and non associative operation on G.

Example 3.30: Let G = {Collection of all subsets from the MOD complex modulo integer plane C_n (15), $*$, (4, 11)} be the simple complex modulo integer MOD interval groupoid.

Let $S_1 = \{4 + 2i_F, i_F, 3.1\}$ and $S_2 = \{8 + i_F, 7i_F, 0.31\} \in G$.

$$S_1 * S_2 = \{4 + 2i_F, i_F, 3.1\} * \{8 + i_F, 7i_F, 0.31\}$$

$$= \{4 + 2i_F * 8 + i_F, i_F * 8 + i_F, 3.1 * 8 + i_F, 4 + 2i_F * 7i_F, i_F * 7i_F, 3.1 * 7i_F, 4 + 2i_F * 0.31, i_F * 0.31, 3.1 * 0.31\}$$

$$= \{14 + 4i_F, 13, 10.4 + 11i_F, 1 + 10i_F, 6i_F, 12.4 + 2\, i_F, 4.41 + 8i_F, 3.41 + 4i_F, 12.4 + 3.41i_F\} \in G.$$

This is the way $*$ operation is performed on G. It is easily verified $*$ on G both non commutative and non associative.

Example 3.31: Let G = {Collection of all subsets from the complex modulo integer MOD plane $C_n(11)$, $*$, (9, 2)} be the MOD complex modulo integer interval groupoid.

G is non commutative and is of infinite order.

Let $x = \{(0.8, 1)\} \in G$

$$x * x = \{(0.8, 1)\} * \{(0.8, 1)\}$$

$$= \{(0.8 * 0.8, 1 * 1)\}$$

$$= \{(0.8 \times 9 + 2 \times 0.8, 1 * 9, 2 * 1)\}$$

$$= \{(7.2 + 1.6, 9 + 2)\}$$

$$= \{(8.8, 0)\}.$$

Clearly $0.8 \times 9 + 0.8 \times 2$

$$= 7.2 + 1.6$$

$$= 8.8 \neq 0.$$

$$\neq 0.8\ (9 + 2)\ (\text{mod } 11)$$

$$= 0.8 \cdot 0\ (\text{mod } 11)$$

$$= 0.$$

Let $x = \{(3, 2)\} \in G$

$$x * x = \{(3, 2)\} * \{(3, 2)\}$$

$$= \{(3 * 3, 2 * 2)\}$$

$$= \{(3 \times 9 + 3 \times 2, 2 \times 9 + 2 \times 2)\}$$

$$= \{(0, 0)\}.$$

Thus G has only finite number of pseudo nilpotent elements.

All subsets $P = \{(a, b)\} \in G$ where $a, b \in C(Z_{11})$ are such that $P * P = \{(0, 0)\}$.

Clearly if $x = \{(a, b)\}$ where $a, b \in C_n(11) \setminus C(Z_{11})$;

$$x * x \neq \{(0, 0)\}.$$

$$\text{Let } x = \{(8.1, 0.7)\} \in G$$

$$
\begin{aligned}
x * x &= \{(8.1, 0.7)\} * \{(8.1, 0.7)\} \\
&= \{(8.1 * 8.1, 0.7 * 0.7)\} \\
&= \{(72.9 + 16.2, 0.7 \times 9 + 0.7 \times 2)\} \\
&= \{(1.1, 7.7)\} \neq \{(0, 0)\}.
\end{aligned}
$$

In view of this we have the following theorem.

THEOREM 3.9: *Let G = {Collection of all subsets from the complex MOD plane $C_n(m)$, *, (t, u)} be the subset interval complex modulo integer MOD groupoid.*

*x = {(a, b)}; a, b ∈ $C(Z_m)$ is such that x * x = {(0, 0)} if and only if t + u = 0 (mod m); that is x is a MOD pseudo nilpotent subset of G of order two.*

Proof: Let x = {(a, b)}; a, b ∈ $C(Z_m)$;

$$
\begin{aligned}
x * x &= \{(a, b)\} * \{(a, b)\} \\
&= \{(a * a, b * b)\} \\
&= \{(0, 0)\} \text{ if and only if}
\end{aligned}
$$

$ta + ua \equiv 0 \bmod m$ and $tb + ub \equiv 0 \pmod m$.

That is $a (t + u) \equiv 0 \bmod m$
$b (t + u) \equiv 0 \bmod m$ as a, b ∈ $C(Z_m)$.

If $t + u \equiv 0 \pmod m$ then a = b = 0 which is trivial if a ≠ 0 and b ≠ 0 then it forces $t + u \equiv 0 \pmod m$. Hence the claim.

Corollary 3.1: Let G = {Collection of all subsets from the complex modulo integer MOD plane $C_n(m)$, *, (t, u)} be the subset complex modulo integer groupoid.

y = {(a, b)} where a, b ∈ C_n (m)\ C(Z_m) even for t + u ≡ (mod m) are not MOD pseudo nilpotent elements of order two.

Proof: Since if a, b ∈ C_n(m) \ C (Z_m); in y ={(a, b)} are such that

$$\begin{aligned} a * a &= ta + ua \neq 0 \\ &\neq (t + u)a = 0 \\ b * b &= tb + ub \neq (t + u)b \end{aligned}$$

where a, b ∈ C_n(m) \ C(Z_m).

In view of this property we face certain odd situations in case of this type of subset MOD groupoids.

Example 3.32: Let G = {Collection of all subsets from the complex modulo integer MOD plane C_n(6), *, (4, 2)} be the subset complex modulo integer MOD groupoid.

Let x = {(4.3, 2.5)} ∈ G

$$\begin{aligned} x * x &= \{(4.3, 2.5)\} * \{(4.3, 2.5)\} \\ &= \{(4.3 * 4.3, 2.5 * 2.5)\} \\ &= \{(17.2 + 8.6, 10.0 + 5)\} \\ &= \{(7.8, 3)\} \neq \{(0, 0)\} \end{aligned}$$

but 4 + 2 ≡ 0 (mod 6).

Consider y = {(2.5, a)} ∈ G.

Suppose

$$y * y = \{(2.5, a)\} * \{(2.5, a)\}$$

$$= \{(2.5 * 2.5, a * a)\}$$

$$= \{(10 + 5, a * a)\}$$

$$\neq \{(0, 0)\} \text{ for any } a \in C_n(6).$$

$$\text{Let } z = \{(a, a)\} \in G,$$

$$z * z = \{(a, a)\} * \{(a, a)\}$$

$$= \{(a * a, a * a)\}$$

$$= \{(4a + 2a, 4a + 2a)\}$$

$$\neq \{(0, 0)\} \text{ if } a \notin C(Z_6).$$

Let $a = 3.5$

$$a * a = 3.5 \times 4 + 3.5 \times 2$$

$$= 14.0 + 7.0$$

$$= 21 \pmod 6$$

$$= 3 \pmod 6$$

$$\neq 0.$$

Thus $4a + 2a \neq 0 \pmod 6$ if $a \in C([0, m)) \setminus C(Z_m)$ $m = 6$.

Let $y = \{(0.02, 0.212)\} \in G$;

$$y * y = \{(0.02, 0.212)\} * \{(0.02, 0.212)\}$$

$$= \{(0.02 * 0.02, 0.212 * 0.212)\}$$

$$= \{(0.08 + 0.04, 0.848 + 0.424)\}$$

$$= \{(0.12, 1.372)\}$$

$$\neq \{(0, 0)\}.$$

Example 3.33: Let G = {Collection of all subsets from the complex modulo integer C_n (9), $*$, (6, 4)} be the subset complex MOD interval groupoid.

Let $x = \{(3, 2)\} \in G$

$$x * x = \{(3, 2)\} * \{(3, 2)\}$$

$$= \{(3 * 3, 2 * 2)\}$$

$$= \{(3 \times 6 + 3 \times 4, 2 \times 6 + 2 \times 4)\}$$

$$= \{(3, 2)\} = x \in G.$$

Thus x is an idempotent of G.

G has finite number of idempotents.

Example 3.34: Let G = {Collection of all subsets from the complex modulo integer MOD plane $C_n(18)$, $*$, (10, 9)} be the subset interval complex modulo integer MOD groupoid.

Let $x = \{(0.1, 0.12)\} \in G$;

$$x * x = \{(0.1, 0.12)\} * \{(0.1, 0.12)\}$$

$$= \{(0.1 * 0.1, 0.12 * 0.12)\}$$

$$= \{(1 + 0.9, 1.2 + 10.8)\}$$

$$= \{(1.9, 12)\} \neq x.$$

Let $y = \{(3, 2)\} \in G$

$$\begin{aligned} y * y &= \{(3, 2)\} * \{(3, 2)\} \\ &= \{(3 * 3, 2 * 2)\} \\ &= \{(30 + 27, 20 + 18)\} \\ &= \{(57 \pmod{18}, 38 \pmod{18})\} \\ &= \{(3, 2)\} = y. \end{aligned}$$

Thus y is an idempotent of the groupoid.

Example 3.35: Let G = {Collection of subsets from the complex modulo integer $C_n(13)$, $*$, (8, 6)} be the MOD subset complex modulo integer groupoid.

Let $P = \{(7, 4)\} \in G$

$$\begin{aligned} P * P &= \{(7, 4)\} * \{(7, 4)\} \\ &= \{(7 * 7, 4 * 4)\} \\ &= \{(7 \times 8 + 7 \times 6, 4 \times 8 + 4 \times 6)\} \\ &= \{(7, 4)\} = P. \end{aligned}$$

Thus P is an idempotent of G. However Z_{13} has no idempotents.

In view of all these we have the following theorem.

THEOREM 3.10: *Let G = {Collection of all subsets from the MOD complex modulo integer plane $C_n(m)$; $*$, (t, u)} be the complex modulo finite integer MOD subset groupoid. Singleton subsets of the form A = {(r, s)} when r, s $\in C(Z_m)$ are idempotents of G if and only if t, u $\in Z_m \setminus \{0\}$; with t + u = 1 (mod m).*

Proof: Let P = {(r, s)} (s, r ∈ C (Z_m)) be the subset of G.

$$P * P = \{(r, s)\} * \{(r, s)\}$$
$$= \{(tr + ur, ts + us)\}$$
$$= \{((t + u)r, (t + u)s)\}.$$

Now (t + u)r = r if and only if t + u ≡ 1 (mod m) likewise (t + u)s = s if and only if t + u ≡ 1 (mod m). Thus P is an idempotent of G.

Hence the claim.

Example 3.36: Let G = {Collection of all subsets from the MOD complex modulo integer plane $C_n(15)$, ∗, (9, 7)} be the subset MOD complex modulo integer groupoid.

Let A = {(8 + 4i_F)} ∈ G.

$$A * A = \{(8 + 4i_F)\} * \{(8 + 4i_F)\}$$
$$= \{(8 + 4i_F * 8 + 4i_F)\}$$
$$= \{(8 + 4i_F)\}$$
$$= A.$$

Thus G has idempotents. Most of the idempotents in G are of the form A = {(z, x)} where z, x ∈ $C_n(15)$.

However idempotents of Z_{15} may not in general be idempotents of G.

Now having seen the condition for the existence of idempotents, units and MOD pseudo zero divisors we proceed onto study the corresponding Smarandache structures.

Thus elements of $C_n(m)$ can be denoted by (a, b) where (a, b) = a + bi_F. As per convenience we can denote an element in $C_n(m)$ in any of the two ways.

However it is left as an open conjecture to find for any P = $\{x + yi_F | x, y \in C_n(m) \setminus C(Z_m)\}$ to be idempotents; or for that matter any subset collection in $C_n(m)$ to be idempotent or S-idempotent or unit or S-unit or MOD pseudo zero divisor or S-MOD pseudo zero divisor. Such study is left open for researchers.

Next the study of groupoids in MOD neutrosophic planes is carried out in chapter IV of this book.

However a few problems are suggested for the interested reader.

Problems

1. Obtain any special and interesting feature enjoyed by subset groupoid of the real MOD plane $R_n(m)$.

2. Let G = {Collection of all subsets from $R_n(10)$, *, (8, 7)} be the subset real MOD groupoid.

    i. Prove G is of infinite order.
    ii. Prove G is non commutative.
    iii. Can G have MOD pseudo zero divisors?
    iv. Does G have MOD pseudo zero divisors which are not S-zero divisors?
    v. Does G contain units?
    vi. Can G have S-idempotents?
    vii. Can G have subsets of cardinality greater than two to be zero divisors?
    viii. If A = {(a, b), (c, d), (e, f)} (where a, b, c, d, e, f $\in R_n(10)$) $\in$ G.

    Can for any set of distinct values of a, b, c, …, f be an idempotent?

ix. What are the conditions on $R_n(10)$ to have S-units, S-zero divisors and S-idempotents.

3. Let M = {Collection of all subsets from $R_n(17)$, $*$, (10, 8)} be the subset MOD real groupoid.

   Study questions i to ix of problem 2 for this M.

4. Let N = {Collection of all subsets from real MOD planes $R_n(24)$, $*$, (10.5, 7)} be the subset MOD real groupoid.

   Study questions i to ix of problem 2 for this N.

5. Let T = {Collection of all subsets from the real MOD plane $R_n(12)$, $*$, (4, 0)} be the subset MOD real groupoid.

   Study questions i to ix of problem 2 for this T.

6. Let M = {Collection of all subsets from the real MOD plane $R_n(23)$, $*$, (0.331, 8.1105)} be the MOD real subset groupoid.

   Study questions i to ix of problem 2 for this M.

7. Let P = {Collection of all subsets from the real MOD plane $R_n(24)$, $*$, (0.99235, 0)} be the real MOD subset groupoid.

   Study questions i to ix of problem 2 for this P.

8. Let B = {Collection of all subsets from the real MOD plane $R_n(18)$, $*$, (10, 9)} be the real MOD subset groupoid.

   i. Can B have ideals of finite order?
   ii. Prove B has subgroupoids of finite order.
   iii. Can B have right ideals?

iv. Can left ideal of B be of finite order?
v. Can B have S-ideals?
vi. Can B have S-subgroupoids which are not S-ideals?
vii. How many subgroupoids are there in B of finite order?
viii. Can finite subgroupoids of B be a S-subgroupoid?

9. Let V = {Collection of all subsets from the real MOD plane $R_n(29)$, $*$, (10, 2)} be the subset real MOD groupoid.

   Study question i to viii of problem 8 for this V.

10. Let W = {Collection of all subsets from the real MOD plane $R_n(12)$, $*$, (6, 6)} be the subset real MOD groupoid.

    Study questions i to viii of problem 8 for this W.

11. Let S = {Collection of all subsets from the real MOD plane $R_n(15)$, $*$, (12, 0)} be the subset real MOD groupoid.

    Study questions i to viii of problem 8 for this S.

12. Let T = {Collection of all subsets from the real MOD plane $R_n(6)$, $*$, (3, 0.243)} be the subset real MOD groupoid.

    Study questions i to viii of problem 8 for this T.

13. Let N = {Collection of all subsets from the real MOD plane $R_n(17)$, $*$, (10.333, 0.751)} be the subset real MOD groupoid.

    Study questions i to viii of problem 8 for this N.

14. Let A = {Collection of all subsets from the real MOD plane $R_n(24)$, $*$, (0.11152, 0)} be the real subset MOD groupoid.

    Study questions i to viii of problem 8 for this A.

15. Let D = {Collection of all subsets from the complex modulo integer MOD plane, $C_n(20)$, $*$, (10, 2)} be the subset complex modulo integer MOD groupoid.

    Study questions i to viii of problem 8 for this D.

16. Let E = {Collection of all subsets from the MOD complex modulo integer plane $C_n(23)$, $*$, $(10 + 5i_F, 2 + 0.007i_F)$} be the subset complex modulo integer MOD groupoid.

    Study questions i to viii of problem 8 for this E.

17. Let F = {Collection of all subsets from the MOD complex modulo integer plane $C_n(10)$, $*$, $(5i_F, 6i_F)$} be the subset MOD complex modulo integer groupoid.

    Study questions i to viii of problem 8 for this F.

18. Let G = {Collection of all subsets from the MOD complex plane $C_n(24)$, $*$, $(2 + i_F, 9)$} be the subset MOD complex modulo integer groupoid.

    Study questions i to viii of problem 8 for this G.

19. Let H = {Collection of all subsets from the complex modulo integer MOD plane $C_n(124)$, $*$, $(0, 100 + 20i_F)$} be the subset MOD complex modulo integer groupoid.

    Study questions i to viii of problem 8 for this H.

20. Let I = {Collection of all subsets from the complex modulo integer MOD plane $C_n(101)$, $*$, $(10 + 0.71i_F, 0.332)$} be the subset MOD complex modulo integer plane groupoid.

    Study questions i to viii of problem 8 for this I.

21. Let J = {Collection of all subsets from the complex modulo integer MOD plane $C_n(43)$, $(0.331 + 0.2i_F, 0.115i_F)$} be the subset MOD complex modulo integer plane groupoid.

    Study questions i to viii of problem 8 for this J.

22. Let K = {Collection of all subsets from the complex modulo integer MOD plane $C_n(15)$, $*$, $(5, 12)$} be the subset MOD complex modulo integer groupoid.

    Study questions i to ix of problem 2 for this K.

23. Let L = {Collection of all subsets from the complex modulo integer MOD plane $C_n(11)$, $*$, $(10i_F, 2)$} be the subset MOD complex modulo integer groupoid.

    Study questions i to ix of problem 2 for this L.

24. Let X = {Collection of all subsets from the complex modulo integer MOD plane $C_n(12)$, $*$, $(0.331, 10.32)$} be the subset MOD complex modulo integer plane groupoid.

    Study questions i to ix of problem 2 for this X.

25. Let Y ={Collection of all subsets from the MOD complex modulo integer plane $C_n(115)$, $*$, $(5 + 4i_F, 105 + 79i_F)$} be the subset MOD complex modulo integer plane groupoid.

    Study questions i to ix of problem 2 for this Y.

26. Let Z = {Collection of all subsets from the complex modulo integer MOD plane $C_n(11)$, $*$, $(9 + 4.3i_F, 6.33i_F)$} be the subset complex modulo integer MOD groupoid.

    Study questions i to ix of problem 2 for this Z.

27. Obtain any other special features associated with the subset MOD complex modulo integer place groupoids.

Chapter Four

SUBSET MOD GROUPOIDS USING $R^I_n(m)$, $R_n(m)g$, $R_n(m)h$ AND $R_n(m)k$

In this chapter we study subset MOD groupoids built using $R^I_n(m)$ where $R^I_n(m) = \{a + bI \mid a, b \in [0, m); I^2 = I$, is an indeterminate$\}$ is the MOD neutrosophic plane, $R_n(m)g = \{a + bg \mid a, b \in [0, m); g^2 = 0\}$ is the MOD dual number plane, $R_n(m)h = \{a + bh \mid h^2 = h, a, b \in [0, m)\}$ is the MOD special dual like number plane and $R_n(m)\ k = \{a + bk \mid a, b \in [0, m), k^2 = (m - 1)k\}$ is the MOD quasi special dual number plane. Thus we have six types of MOD plane and already the study of MOD real plane and MOD complex modulo integer planes are already introduced in chapter two and three.

Further groupoids and subset groupoids were built using the MOD real plane $R_n(m)$ and $C_n(m)$ the MOD complex modulo integer groupoids.

Now we first study of MOD neutrosophic groupoids and MOD neutrosophic subset groupoids in the following.

Example 4.1: Let P = $\{R_n^I(24); *, (12, 13)\}$ be the MOD interval neutrosophic groupoid. P has idempotents.

Example 4.2: Let R = $\{R_n^I(13), *, (10, 3)\}$ be the MOD interval neutrosophic groupoid. R has MOD pseudo nilpotents of order two.

Example 4.3: Let R = $\{R_n^I(6), *, (2, 3I)\}$ be the MOD interval neutrosophic groupoid.

Let x = $\{0.3I, 4 + I, 2 + 4I\} \in R$

$$
\begin{aligned}
x * x &= \{0.3I, 4 + I, 4I + 2\} * \{0.3I, 4 + I, 4I + 2\} \\
&= \{0.3I * 0.3I, 0.3I * 4 + I, 0.3I * 4I + 2, 4 + I * 0.3I, 4 + I * 4 + I, 4 + I * 4I + 2, 4I + 2 * 0.3I, \\
&\quad 4I + 2 * 4 + I, 4I + 2 * 4I + 2\} \\
&= \{1.5I, 3.6I, 0.6I, 2 + 2.9I, 4 + 4I, 2 + 2I, \\
&\quad 2.9I + 4, 4 + 5I, 4I + 2\} \in R.
\end{aligned}
$$

This is the way product is performed on R.

Example 4.4: Let R = $\{R_n^I(7), *, (6 + 3I, 0.2 + 0.3I)\}$ be the MOD neutrosophic interval decimal groupoid.

Example 4.5: Let T = $\{R_n^I(20), *, (3I + 9.8I, 0)\}$ be the MOD neutrosophic groupoid.

THEOREM 4.1: *Let G = $\{R_n^I(m), *, (t, u); t, u \in Z_m\}$ be the MOD neutrosophic groupoid. G has non trivial idempotents if and only if $t + u = 1 \pmod m$.*

Proof: Consider A = $\{p + qI\}$, p and q $\in Z_m$ in G.

$$\begin{aligned} A * A &= \{p + qI\} * \{p + qI\} \\ &= \{p + qI * p + qI\} \\ &= \{t\,(p + qI) + u\,(p + qI)\} \\ &= \{(t + u)\,p + (t + u)\,qI\} \\ &= \{p + qI\} \end{aligned}$$

if and only if $t + u \equiv 1 \pmod m$.

Hence the claim.

Corollary 4.1: Let $G = \{R_n^I(m), *, (t, u);\ t, u \in \{a + bI \mid a, b \in Z_m\}\}$ be the MOD neutrosophic groupoid such that $t + u \equiv I \pmod m$. Then $A = \{p + qI \mid p, q \in Z_m\}$ are not idempotents of G.

Proof: Consider

$$\begin{aligned} A * A &= \{p + qI\} * \{p + qI\} \\ &= \{(r + sI)\,(p + qI) + (a + bI) \times (p + qI)\} \end{aligned}$$

$(r + a = 1 (\text{mod } m)$ and $b + s \equiv 1 \pmod m)$

$$\begin{aligned} &= \{rp + spI + rqI + sqI + ap + bIp + aqI + bqI\} \\ &= (r + a)\,p + (sp + rq + sq + bp + aq + bp)\,I \\ &= (r + a)\,p + \{(b + s)\,p + (a + r)\,q + sq + bp\}\,I \\ &= (r + 1)\,p + (p + q + sq + bp)\,I \\ &\neq p + qI \end{aligned}$$

as $q \neq p + q + sq + bp$.

Hence the claim.

However this situation is represented by the following example.

Example 4.6: Let S = $\{R_n^I(10), *, (8I, 2I)\}$ be the MOD neutrosophic groupoid.

$$\text{Let } A = \{5 + 8I\} \in S$$

$$\begin{aligned} A * A &= \{5 + 8I\} * \{5 + 8I\} \\ &= \{5 + 8I * 5 + 8I\} \\ &= \{40I + 64I + 10I + 16I\} \\ &= \{0\} \text{ is a zero divisor.} \end{aligned}$$

However if we take another illustration which has idempotents and not MOD pseudo nilpotents.

Example 4.7: Let G = $\{R_n^I(12), *, (8I, 4I)\}$ be the MOD neutrosophic groupoid.

$$\text{Let } A = \{3 + 10I\} \in G;$$

$$\begin{aligned} A * A &= \{3 + 10I\} * \{3 + 10I\} \\ &= \{24I + 80I + 12I + 40I\} \\ &= \{0\}. \end{aligned}$$

This is again a MOD pseudo zero divisor.

Example 4.8: Let S = $\{R_n^I(13), *, (8I, 6I)\}$ be the MOD neutrosophic groupoid.

Let $A = \{5 + 3I\} \in S$.

$$\begin{aligned} A * A &= \{5 + 3I\} * \{5 + 3I\} \\ &= \{8I\} \neq A. \end{aligned}$$

But $8I + 6I \equiv I \pmod{13}$.

Example 4.9: Let $B = \{ R_n^I(12), *, (1 + 10I, 2I)\}$ be the MOD neutrosophic groupoid.

Let $A = \{3 + 4I\} \in S$

$$\begin{aligned} A * A &= \{3 + 4I\} * \{3 + 4I\} \\ &= \{3 + 4I * 3 + 4I\} \\ &= \{3 + 4I + 30I + 40I + 6I + 8I\} \\ &= \{3 + 4I\} \\ &= A \text{ is an idempotent of } S. \end{aligned}$$

$$\begin{aligned} \text{Let } B &= \{2 + 8I\} \in S \\ B * B &= \{2 + 8I\} * \{2 + 8I\} \\ &= \{2 + 8I * 2 + 8I\} \\ &= \{2 + 8I + 20I + 80I + 4I + 16I\} \\ &= \{2 + 8I\} = B. \end{aligned}$$

Thus B is also an idempotent of S.

Example 4.10: Let T = { $R_n^I(7)$, ∗, (3 + 4I, 5 + 3I)} be the MOD neutrosophic groupoid.

Let x = {5 + 6I} ∈ T

$$
\begin{aligned}
x * x &= \{5 + 6I\} * \{5 + 6I\} \\
&= \{5 + 6I \times 3 + 4I + 5 + 6I \times 5 + 3I\} \\
&= \{15 + 18I + 20I + 24I + 25 + 30I + 15I + 18I\} \\
&= \{5 + 6I\} = x \text{ is an idempotent of T.}
\end{aligned}
$$

Example 4.11: Let M = { $R_n^I(16)$, ∗, (10 + 7I, 7 + 9I)} be the MOD groupoid.

Let A = {6 + 12I} ∈ M

$$
\begin{aligned}
A * A &= \{6 + 12I\} * \{6 + 12I\} \\
&= \{6 + 12I * 6 + 12I\} \\
&= \{60 + 120I + 42I + 84I + 42 + 54I + 84I + 108I\} \\
&= \{6 + 12I\} = A.
\end{aligned}
$$

Thus A is an idempotent in M.

In view of all these we have the following theorem.

THEOREM 4.2: *G = { R_n^I (m), ∗, (a + bI, c + dI)} has MOD neutrosophic idempotent elements if and only if a + c ≡ 1 (mod m) and b + d ≡ 0 (mod m).*

Proof: Let A = {x + yI} where x, y ∈ $\langle Z_m \cup I\rangle$; be an element of G.

$$\begin{aligned}
A * A &= \{x + yI\} * \{x + yI\} \\
&= \{x + yI * x + yI\} \\
&= \{(a + bI)(x + yI) + (x + yI)(c + dI)\} \\
&= \{(ax + bxI + ayI + byI + xc + ycI \\
&\quad + xdI + ydI)\} \\
&= \{(a + c)x + (b + d)yI + (a + c)yI \\
&\quad + (b + d)xI\} \\
&= \{x + yI\} = A
\end{aligned}$$

if and only if $a + c \equiv 1$ and $b + d \equiv 0 \pmod{m}$.

Thus A is an idempotent of G.

It is important to note that if x, y ∈ $\langle [0, m) \cup I\rangle \setminus \langle Z_m \cup I\rangle$ then A will not be an idempotent in general.

This will be illustrated by the following example.

Example 4.12: Let G = $\{R_n^I(11), *, (5 + 3I, 7 + 8I)\}$ be the MOD neutrosophic groupoid.

Let x = 0.31 + 0.16I ∈ G.

$$\begin{aligned}
x * x &= (0.31 + 0.16I) * (0.31 + 0.16I) \\
&= (0.31 + 0.16I) \times (5 + 3I) + (0.31 + 0.16I) \\
&\quad (7 + 8I)
\end{aligned}$$

$$= (1.55 + 0.8I + 0.93I + 0.48I + 2.17 + 1.12I + 1.28I + 2.48I)$$

$$= (1.55 + 8.54I)$$

$$\neq x.$$

Thus x is not an idempotent of G.

Consider $y = 8 + 4I \in G$.

$$y * y = 8 + 4I * 8 + 4I$$

$$= 8 + 4I \times 5 + 3I + 8 + 4I \times 7 + 8I$$

$$= 40 + 20I + 24I + 12I + 56 + 28I + 64I + 32I$$

$$= 8 + 4I$$
$$= y.$$

Thus y is an idempotent in G. Further $5 + 3I + 7 + 8I \equiv 1 \pmod{11}$.

It is seen that in general $x = a + bI$ with $a, b \in \langle[0, 11) \cup I\rangle \setminus \langle Z_{11} \cup I\rangle$ are not idempotents of G.

Next we study pseudo nilpotents in these MOD neutrosophic groupoids.

Example 4.13: Let $G = \{R_n^I(15), *, (8 + 5I, 7 + 10I)\}$ be the MOD neutrosophic groupoid.

Consider $x = 6 + 9I \in G$

$$x * x = 6 + 9I * 6 + 9I$$

$$= 6 + 9I \times 8 + 5I + 6 + 9I \times 7 + 10I$$

$$= \quad 48 + 72I + 30I + 45I + 42 + 63I + 60I + 90I$$

$$= \quad 0 + 0I.$$

Thus x is a MOD pseudo nilpotent element of order two in G.

Consider $v = 0.32 + 4.1I \in G$;

$$v * v \quad = \quad 0.32 + 4.1I * 0.32 + 4.1I$$

$$= \quad 0.32 + 4.1I \times 8 + 5I + 0.32 + 4.1I \times 7 + 10I$$

$$= \quad 2.56 + 32.8I + 1.60I + 20.5I + 2.24 + 2.87I + 3.2I + 41I$$

$$= \quad 4.8 + 11.97I \neq 0.$$

Thus v is not a MOD pseudo nilpotent element of G of order two.

Example 4.14: Let $G = \{ R_n^I(20), *, (10I, 10I)\}$ be the MOD neutrosophic groupoid.

Let $x = 3 + 8I \in G$,

$$x * x \quad = \quad 3 + 8I * 3 + 8I = 10I \times 3 + 8I + 10I \times 3 + 8I$$

$$= \quad 30I + 80I + 30I + 80I$$

$$= \quad 0.$$

Thus x is a MOD pseudo nilpotent element of order two.

Let $y \quad = \quad 6.2 + 4.5I \in G$

$$\begin{aligned} y * y &= 6.2 + 4.5I * 6.2 + 4.5I \\ &= 10I \times 6.2 + 4.5I + 10I \times 6.2 + 4.5I \\ &= 62I + 45I + 62I + 45I \\ &= 14I \neq 0. \end{aligned}$$

Thus y is not a MOD pseudo nilpotent element of order.

Example 4.15: Let G = {R_n^I (16), $*$, (10I + 2, 6I + 15)} be the MOD neutrosophic groupoid.

Let x = 8I + 13 $\in$ G

$$\begin{aligned} x * x &= 8I + 13 * 8I + 13 \\ &= 8I + 13 \times 10I + 2 + 8I + 13 \times 6I + 15 \\ &= 80I + 130I + 16I + 26 + 48I + \\ &\quad 78I + 120I + 195 \\ &= 13 + 8I = x. \end{aligned}$$

Thus x is an idempotent element of G.

Next we study the subset MOD neutrosophic groupoids and subset MOD neutrosophic decimal interval groupoids.

Example 4.16: Let G = {Collection of all subsets from R_n^I (9), $*$, (4, 8)} be the subset MOD neutrosophic interval groupoid.

Let x = {3 + 2I, 4I, 6, 2 + 8I} and y = {2I, 6} $\in$ G.

$$x * y = \{3 + 2I, 4I, 6, 2 + 8I\} * \{2I, 6\}$$

$$= \{3 + 2I * 2I, 4I * 2I, 6 * 2I, 2 + 8I * 2I, 3 + 2I * 6, 4I * 6, 6 * 6, 2 + 8I * 6\}$$

$$= \{12 + 8I + 16I, 16I + 16I, 24 + 16I, 8 + 32I + 16I, 12 + 8I + 48, 16I + 48, 24 + 48, 8 + 32I + 48\}$$

$$= \{3 + 6I, 5I, 6 + 7I, 8 + 3I, 4 + 8I, 7I + 3, 0, 2 + 5I\}.$$

This is the way the operation is performed on G.

Example 4.17: Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(15)$, $*$, $(10I + 4, 5I + 12)$} be the MOD neutrosophic subset interval groupoid.

$$A = \{8I + 11\} \in G.$$

$$A * A = \{8I + 11\} * \{8I + 11\}$$

$$= \{10I + 4 \times 8I + 11 + 5I + 12 \times 8I + 11\}$$

$$= \{80I + 32I + 110I + 44 + 40I + 96I + 55I + 132\}$$

$$= \{11 + 8I\} = A.$$

Thus A is an idempotent of G.

$$\text{Let } B = \{0.3I + 8.4\} \in G$$

$$B * B = \{0.3I + 8.4\} * \{0.3I + 8.4\}$$

$$= 0.3I + 8.4 * 10I + 4 + 0.3I + 8.4 \times 5I + 12$$

$$= 3I + 1.2I + 84I + 33.6 + 1.5I + 12I + 3.6I + 100.8$$

$$= 0.1I + 4.8 \neq B.$$

Thus B is not an idempotent of G.

In view of this we have the following theorem.

THEOREM 4.3: *Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(m)$, (a + bI, c + dI), *} be the subset interval MOD neutrosophic groupoid.*

A = {x + yI / x, y ∈ Z_m} ∈ G is a idempotent if and only if a + c ≡ 1 (mod m) and b + d ≡ 0 (mod m).

Proof: Let A = {x + yI | x, y ∈ Z_m} ∈ G be a subset of G.

$$\begin{aligned} A * A &= \{x + yI\} * \{x + yI\} \\ &= \{x + yI * x + yI\} \\ &= \{(x + yI)(a + bI) + (x + yI \times c + dI)\} \\ &= \{xa + yaI + xbI + ybI + xc + yIc + dIx + ydI\} \\ &= \{x (a + c) + \{xI (b + d) + yI (a + c) + yI (b + d)\}\} \\ &= \{x + yI\} \text{ if and only if } a + c = 1 \text{ and } b + d = 0. \end{aligned}$$

Hence the theorem.

Example 4.18: Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(24)$, *, (10 + 14I, 14 + 10I)} be the subset MOD neutrosophic interval groupoid.

Let P = {8 + 10I}∈ G.

$$\begin{aligned} P * P &= \{8 + 10I\} * \{8 + 10I\} = \{8 + 10I * 10I + 8\} \\ &= \{8 + 10I \times 10 + 14I + 8 + 10I \times 14 + 10I\} \end{aligned}$$

$$= \{80 + 100I + 112I + 140I + 140I + 112 + 100I + 80I\}$$

$$= \{0 + 0I\} = \{0\}.$$

Thus P is a MOD pseudo nilpotent element of order two.

Let $B = \{10.3 + 1.1I\} \in G$

$$B * B = \{10.3 + 1.1I\} * \{10.3 + 1.1I\}$$

$$= \{10 + 14I \times 10.3 + 1.1I + 14 + 10I \times 10.3 + 1.1I\}$$

$$= 103 + 144.2I + 11I + 15.4I + 103I + 11I + 15.4I + 144.2$$

$$= \{7.2 + 12I\} \neq B.$$

That is B is not MOD pseudo nilpotent.

In view of this we have to make a note of the fact if $A = \{x + yI\}$ where $x, y \in \langle[0, 24) \cup I\rangle \setminus \langle 24 \cup I\rangle$; then A is not a MOD pseudo nilpotent element of G.

Take $L = \{10.3 + 6I\} \in G$

$$L * L = \{10.3 + 6I\} * \{10.3 + 6I\}$$

$$= \{10.3 + 6I * 10.3 + 6I\}$$

$$= \{10.3 + 6I \times 10 + 14I + 10.3 + 6I \times 14 + 10I\}$$

$$= \{103 + 60I + 84I + 144.2I + 103I + 60I + 84I + 144.2\}$$

$$= \{15.2I + 7.2\} \neq \{0\}.$$

So L is not a MOD pseudo nilpotent element of G.

In view of this we have the following theorem.

THEOREM 4.4: *Let G = {Collection of all subset from the MOD neutrosophic plane $R_n^I(m)$, *, (a + bI, c + dI)} be the subset MOD neutrosophic interval groupoid.*

A = {x + yI / x, y ∈ $\langle Z_m \cup I \rangle$} ∈ G is a MOD pseudo nilpotent element of order two if and only if a + c ≡ 0 (mod m) and b + d ≡ 0 (mod m).

Proof: Let A = {x + yI} (x, y ∈ $\langle Z_m \cup I \rangle$) ∈ G

$$\begin{aligned} A * A &= \{x + yI\} * \{x + yI\} \\ &= \{x + yI * x + yI\} \\ &= \{a + bI \times x + yI + c + dI \times x + yI\} \\ &= \{ax + bxI + yaI + byI + cx + dxI + ycI + dyI\} \\ &= \{(a + c)x + (b + d)xI + (b + d)yI + (a + c)yI\} \\ &= \{0\} \end{aligned}$$

if and only if a + c ≡ 0 (mod m) and b + d ≡ 0 (mod n).

Hence the theorem.

It is left as an open conjecture to find MOD pseudo nilpotents as well idempotents in G apart from the one's mentioned in this theorem.

However if the (t, u) of the groupoid is from $\langle [0, m) \cup I \rangle \setminus \langle Z_m \cup I \rangle$ the situation is entirely different.

Consider the following subset decimal interval groupoids.

Example 4.19: Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(12)$, $*$, (0.8, 10.4I)} be the subset MOD neutrosophic interval decimal groupoid.

Let A = {5 + 7I} $\in$ G.

$$\begin{aligned} A * A &= \{5 + 7I\} * \{5 + 7I\} \\ &= \{5 + 7I * 5 + 7I\} \\ &= \{5 + 7I \times 0.8 + 5 + 7I \times 10.4I\} \\ &= \{4 + 5.6I + 52I + 72.8I\} \\ &= \{4 + 10.4I\}. \end{aligned}$$

This is the way operations are performed on G. G is non commutative non associative infinite order subset groupoid.

It is infact a difficult problem to find subsets in G which are idempotents or MOD pseudo nilpotents of order two.

Example 4.20: Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(10)$, $*$, (5, 0.3I)} be the subset MOD neutrosophic interval decimal groupoid.

Let A = {3 + 2I, 4 + 0.2I} and B = {7 + I, 0.3 + 4I} $\in$ G.

$$\begin{aligned} A * B &= \{3 + 2I, 4 + 0.2I\} * \{7 + I, 0.3 + 4I\} \\ &= \{3 + 2I * 7 + I, 4 + 0.2I * 7 + I, 3 + 2I * \\ &\quad 0.3 + 4I, 4 + 0.2I * 0.3 + 4I\} \\ &= \{5 \times 3 + 2I + 0.3I\,(7 + I), 5 \times 4 + 0.2I + \\ &\quad 0.3I \times 7 + I, 5 \times 3 + 2I + 0.3I \times 0.3 + 4I, 5 \times \\ &\quad 4 + 0.2I + 0.3I \times 0.3 + 4I\} \end{aligned}$$

$$= \{5 + 2.1I + 0.3I, I + 2.1I + 0.3I, 5 + 0.09I + 1.2I, I + 0.09I + 1.2I\}$$

$$= \{5 + 2.4I, 3.4I, 5 + 1.29I, 2.29I\} \in G.$$

Clearly if $A = \{3 + 6I\} \in G$

$$A * A = \{3 + 6I\} * \{3 + 6I\}$$

$$= \{3 + 6I * 3 + 6I\}$$

$$= \{5 \times (3 + 6I) + 0.3I (3 + 6I)\}$$

$$= \{5 + 0.9I + 1.8I\}$$

$$= \{5 + 2.7I\} \in G.$$

This is not an idempotent or a nilpotent element of order two.

Example 4.21: Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(11)$, $*$, (1.1, 0)} be the subset interval decimal MOD neutrosophic groupoid.

Let $A = \{5I + 6\} \in G$;

$$A * A = \{5I + 6\} * \{5I + 6\}$$

$$= \{5I + 6 * 5I + 6\}$$

$$= \{1.1 \times 5I + 6 + 0\}$$

$$= \{5.5I + 6.6\} \in G.$$

Let $B = \{10I + 10\} \in G$

$$
\begin{aligned}
B * B &= \{10I + 10\} * \{10I + 10\} \\
&= \{10I + 10 * 10 + 10I\} \\
&= \{1.1 \times 10I + 10 + 0\} \\
&= \{11I + 11\} \\
&= \{0\}.
\end{aligned}
$$

Thus B is a MOD pseudo nilpotent element of order two.

Hence we have MOD pseudo nilpotents of order two.

Let $C = \{10I\} \in G$

$$
\begin{aligned}
C * C &= \{10I\} * \{10I\} \\
&= \{10I * 10I\} \\
&= \{1.1 \times 10I + 0\} \\
&= \{0\}.
\end{aligned}
$$

Thus C is a MOD pseudo nilpotent element of order two.

Let $D = \{10\} \in G$;

$$
\begin{aligned}
D * D &= \{10\} * \{10\} \\
&= \{10 * 10\} \\
&= \{1.1 \times 10 + 0\} \\
&= \{0\}
\end{aligned}
$$

is again a MOD pseudo nilpotent element of order two in G.

Next we proceed on to describe groupoid built using MOD dual number plane.

DEFINITION 4.1: *Let $G = \{R_n(m)g \mid g^2 = 0\}$ be the MOD dual number plane. Define a operation $*$ on G as follows:*

$H = \{R_n(m)g, * (t, u); t, u \in R_n(m)g\}$ is defined as the MOD dual number integer groupoid.

This is described by the following examples.

Example 4.22: Let $H = \{R_n(10)g; *, (9g, 1g + 1)\}$ be the MOD dual number groupoid.

$$A = 8 + 8g \in H;$$

$$\begin{aligned} A * A &= 8 + 8g * 8 + 8g \\ &= 9g\,(8 + 8g) + (1g + 1)\,(8 + 8g) \\ &= 72g + 8g + 0 + 8 + 8g \\ &= 8 + 8g = A. \end{aligned}$$

$$x = 7g + 2 \in H$$

$$\begin{aligned} x * x &= 7g + 2 * 7g + 2 \\ &= 9g\,(7g + 2) + (1g + 1)\,(7g + 2) \\ &= 18g + 2g + 7g + 2 \\ &= 7g + 2 = x. \end{aligned}$$

Example 4.23: Let $G = \{R_n(20)g, *, (10, 11)\}$ be the MOD dual number groupoid.

Let $x = 10.5g + 2 \in G$,

$$x * x = (2 + 10.5g) * (2 + 10.5g)$$
$$= 10\ (2 + 10.5g) + 11\ (2 + 10.5g)$$
$$= 5g + 2 + 15.5g$$
$$= 0.5g + 2.$$

$a = 2 + 3g$ and $b = 8g + 10 \in G$

$$a * b = 2 + 3g * 8g + 10$$
$$= 10\ (2 + 3g) + 11\ (8g + 10)$$
$$= 18g + 10 \in G.$$

Example 4.24: Let $S = \{R_n(14)g \mid g^2 = 0, (1.5g, 0), *\}$ be the MOD dual number groupoid.

Let $x = 8 + 10.1g \in S$

$$x * x = 8 + 10.1g * 8 + 10.1g$$
$$= 1.5g\ (8 + 10.1g)$$
$$= 12g \in S.$$

Let $x = 0.2 + 3g$ and $y = 9 + 0.5g \in S$;

$$x * y = 0.2 + 3g * 9 + 0.5g$$
$$= 1.5g\ (0.2 + 3g) + 0$$
$$= 0.3g \qquad \ldots \qquad \text{I}$$

$$y * x = 9 + 0.5g * 0.2 + 3g$$
$$= 1.5g\ (9 + 0.5g) + 0$$

$$= \quad 13.5g \qquad \ldots \qquad \text{II}$$

Clearly $x * y \neq y * x$ thus S is non commutative groupoid.

Example 4.25: Let G $= \{R_n(7)g, *, (3.5, 0)\}$ be the MOD dual number groupoid.

Let $x = 2 + 2g \in G$;

$$\begin{aligned} x * x &= 2 + 2g * 2 + 2g \\ &= 3.5\,(2 + 2g) + 0 \\ &= 0. \end{aligned}$$

x is a MOD pseudo nilpotent element of order two.

Let $x = 4 + 2g \in G$

$$\begin{aligned} x * x &= 4 + 2g * 4 + 2g \\ &= 3.5 \times (4 + 2g) \\ &= 0 \end{aligned}$$

is again a MOD pseudo nilpotent element of order two.

Example 4.26: Let
$G = \{R_n(21)g;\ g^2 = 0, (10 + 5g, 16g + 12), *\}$ be the MOD dual number groupoid.

$x = 11g + 3 \in G$;

$$\begin{aligned} x * x &= (11g + 3) * (11g + 3) \\ &= (11g + 3) \times 10 + 5g + (16g + 12) \\ &\quad (11g + 3) \end{aligned}$$

$= \quad 110g + 30 + 15g + 132g + 36 + 48g$

$= \quad 3 + 11g = x \in G.$

Thus G is an idempotent of G.

Let $y = 7g + 10 \in G$;

$y * y \quad = \quad 7g + 10 * 7g + 10$

$= \quad (10 + 5g) \times (7g + 10) + (16g + 12) \times (7g + 10)$

$= \quad 70g + 100 + 50g + 120 + 84g + 160g$

$= \quad 220 + 364g$

$= \quad 10 + 7g$

$= \quad y \in G.$

Thus y is an idempotent of G.

In view of this we have the following theorem.

THEOREM 4.5: *Let $G = \{R_n(m)g,\ g^2 = 0\ (t, u);\ t, u \in Z_m g\}$ be the MOD dual number groupoid.*

$x = a + bg$ ($a, b \in Z_m g$) be in G.

$x * x = 0$ if and only if $t + u \equiv 0 \pmod m$.

Proof: Let

$x = a + bg \in G$ if $t = c + dg$ and $u = e + fg$

where $c, d, e, f \in Z_m$.

$$
\begin{aligned}
x * x &= a + bg * a + bg \\
&= t\,(a + bg) + u\,(a + bg) \\
&= (c + dg)\,(a + bg) + (e + fg)\,(a + bg) \\
&= ac + adg + cbg + ea + afg + beg \\
&= a\,(c + e) + [a\,(d + f)g + b\,(c + e)g] \\
&= 0 + 0g \text{ if and only if } c + e = 0 \text{ and } \\
&\quad\; d + f = 0.
\end{aligned}
$$

Example 4.27: Let G = $\{R_n(25)g, *, (14 + 10g, 12 + 15g)\}$ be the MOD dual number groupoid.

Let $x = 12 + 5g \in G$;

$$
\begin{aligned}
x * x &= (12 + 5g)*(12 + 5g) \\
&= (14 + 10g) \times (12 + 5g) + (12 + 15g) \\
&\quad\; (12 + 5g) \\
&= 168 + 120g + 70g + 144 + 60g + 180g \\
&= 312 + 430g \\
&= 12 + 5g = x \in G.
\end{aligned}
$$

Thus x is an idempotent of G.

Example 4.28: Let G = $\{R_n(17)g; g^2 = 0, (15 + 5g, 2 + 12g)\}$ be the MOD dual number groupoid.

Let $x = 11 + 6g \in G$,

$$
\begin{aligned}
x * x &= (11 + 6g) * (11 + 6g) \\
&= (15 + 5g) \times (11 + 6g) + (2 + 12g)\,(11 + 6g)
\end{aligned}
$$

$$= \quad 55g + 165 + 90g + 22 + 12g + 132g$$

$$= \quad 187 + 289g$$

$$= \quad 0 + 0g$$

$$= \quad 0.$$

x is a zero divisor and x is MOD pseudo nilpotent of order two.

$x = 0.3g + 1 \in G$,

$$x * x \quad = \quad (0.3g + 1) * (0.3g + 1)$$

$$= \quad (15 + 5g)\,(0.3g + 1) + (2 + 12g)\,(0.3g + 1)$$

$$= \quad 4.5g + 15 + 5g + 2 + 12g + 0.6g$$

$$= \quad 5.1g.$$

Let $x = 10.3 \in G$;

$$x * x \quad = \quad 10.3 * 10.3$$

$$= \quad (15 + 5g)\,(10.3) + (12g + 2)\,(10.3)$$

$$= \quad 51.5g + 154.5 + 20.6 + 123.6g$$

$$= \quad 175.1 + 175.1g$$

$$= \quad 5.1 + 5.1g.$$

Thus x is not an idempotent or nilpotent element of order two.

Example 4.29: Let $G = \{R_n(16)g,\ g^2 = 0\ (12 + 10g,\ 4 + 6g),\ *\}$ be the MOD dual number groupoid.

$x = 5 + 11g \in G$

$$\begin{aligned} x * x &= (5 + 11g) * (5 + 11g) \\ &= (5 + 11g) \times (12 + 10g) + (5 + 11g) \times \\ &\quad (4 + 6g) \\ &= 60 + 132g + 50g + 20 + 44g + 30g \\ &= 80 + 272g \\ &= 0 + 0g \\ &= 0. \end{aligned}$$

Thus x is a MOD pseudo nilpotent element of order two.

Let $y = 7g \in G$,

$$\begin{aligned} y * y &= 7g * 7g \\ &= (12 + 10g)\, 7g + (4 + 6g)\, 7g \\ &= 84g + 28g \\ &= 112g \\ &= 0. \end{aligned}$$

Thus y is a MOD pseudo nilpotent element of order two.

Let $p = 0.7 + 6.5g \in G$

$$\begin{aligned} p * p &= 0.7 + 6.5g * 0.7 + 6.5g \\ &= (0.7 + 6.5g) \times (12 + 10g) + (0.7 + 6.5g) \\ &\quad \times (4 + 6g) \\ &= 8.4 + 78.0g + 7g + 2.8 + 26.0g + 4.2g \\ &= 11.2 + 115.2g \end{aligned}$$

$$= \quad 11.2 + 3.2g \neq 0.$$

Thus if element are from $[0, 16)g \setminus \langle Z_{16} \cup g\rangle$ then they are not in general MOD pseudo nilpotent elements of G.

Example 4.30: Let $G = \{R_n(15)g, (10.311, 0.22g), *\}$ be the MOD dual number groupoid. It is difficult to find nilpotents and idempotents in G.

In fact finding MOD pseudo zero divisors are also a difficult task.

Example 4.31: Let $G = \{R_n(10)g, g^2 = 0, * (0.12g, 0)\}$ be the MOD dual number groupoid.

Let $x = 0.9g \in G$

$$\begin{aligned} x * x &= 0.9g * 0.9g \\ &= 0.9g \times 0.12g + 0.9g \times 0 \\ &= 0; \end{aligned}$$

thus x a MOD pseudo nilpotent element of order two.

Let $y = 8.5g \in G$;

$$\begin{aligned} y * y &= 8.5g * 8.5g \\ &= 0.12g \times 8.5g + 0 \times 8.5g \\ &= 0 \end{aligned}$$

is again an MOD pseudo nilpotent element of order two.

Thus G has nilpotents even if the sum of $t + u \neq 0$. This is a special case of dual numbers.

Example 4.32: Let G = $\{R_n(19)g, g^2 = 0, *, (0.8g, 18g)\}$ be the MOD dual number groupoid.

All $x = ag \in G$; where $a \in [0, 19)$ are MOD pseudo nilpotents of order two.

THEOREM 4.6: *Let $G = \{R_n(m)g, * (ag, bg)\ a, b \in [0, m)\}$ be MOD dual number groupoid. G has infinite number of MOD pseudo nilpotent elements of order two.*

Proof: All elements of the form $x = dg$; $d \in [0, m)$ are MOD pseudo nilpotent of order two.

$$\begin{aligned} \text{Clearly } x * x &= dg * dg \\ &= dg \times ag + dg * bg \\ &= 0 \text{ (as } g^2 = 0). \end{aligned}$$

The number of such $x \in G$ are infinite in number as the cardinality of [0, m) is infinite.

Example 4.33: Let M = $\{R_n(13)g, g^2 = 0\ (0, 5.8g), *\}$ be the MOD dual number groupoid M has infinite number of MOD pseudo nilpotent elements of order two.

In fact M has infinite number of MOD pseudo zero divisors.

Next we consider another example of MOD dual number groupoids which has infinite number of MOD pseudo zero divisors.

Example 4.34: Let M = $\{R_n(19)g, g^2 = 0\ (7g, 8,4g), *\}$ be the MOD dual number groupoid M has infinite number of MOD pseudo zero divisors.

For take $x = 0.9g$ and $y = 6.8g \in M$

$$\begin{aligned} x * y &= 0.9g * 6.8g \\ &= 0.9g \times 7g + 8.4g + 68g \\ &= 0 \text{ as } g^2 = 0. \end{aligned}$$

Thus $B = \{ag \mid a \in [0, 19)\} \subseteq M$ is such that $x * y = 0$ for every $x, y \in B$.

Hence M has infinite number of MOD pseudo zero divisors.

Take $x = 10 \in M$

$$\begin{aligned} x * x &= 10 \times 7g + 10 \times 84g \\ &= 70g + 84g \\ &= 2g \neq 0. \end{aligned}$$

In view of this we have the following theorem.

THEOREM 4.7: *Let $S = \{R_n(m)g \mid g^2 = 0, *, (ag, bg)\}$ where $a, b \in [0, m)$ be the MOD dual number groupoid.*

S has infinite number of MOD pseudo zero divisors which are zero divisors.

Proof: Consider the subgroupoid $B = \{ag \mid a \in [0, m)\} \subseteq S$; every pair distinct or otherwise is such that $x * y = 0$.

Hence every element is nilpotent of order two and product of every pair is zero.

Thus S has infinite number of zero divisors only in this case as $g^2 = 0$.

Now we give examples of subset MOD dual number groupoids.

Example 4.35: Let S = {Collection of all subsets from the MOD dual number plane $R_n(6)g$; $*$, (4, 3)}. S is a subset MOD dual number groupoid of infinite order.

$$A = \{(3.1, 2), (4, 0)\} \text{ and } B = \{(3, 2)\} \in S;$$

$$\begin{aligned} A * B &= \{(3.1, 2), (4, 0)\} * \{(3, 2)\} \\ &= \{(3.1, 2) * (3, 2), (4, 0) * (3, 2)\} \\ &= \{(4 \times 3.1 + 3 \times 3, 4 \times 2 + 3 \times 2), \\ &\quad (4 \times 4 + 3 \times 3, 0 \times 4 + 3 \times 2)\} \\ &= \{(3.4, 2), (3, 0)\} \in S. \end{aligned}$$

This is the way product operation is performed on the subset MOD dual number groupoid.

$$\text{Let } x = \{(2, 1)\} \in S$$

$$\begin{aligned} x * x &= \{(2, 1)\} * \{(2, 1)\} \\ &= \{(2 * 2, 1 * 1)\} \\ &= \{(2 \times 4 + 2 \times 3, 1 \times 4 + 1 \times 3)\} \\ &= \{(2, 1)\} = x. \end{aligned}$$

Thus $x \in S$ is an idempotent element of S.

$$\text{Consider } y = \{(3.2, 0.8)\} \in S$$

$$\begin{aligned} y * y &= \{(3.2, 0.8)\} * \{(3.2, 0.8)\} \\ &= \{(3.2 * 3.2, 0.8 * 0.8)\} \end{aligned}$$

$$= \{(3.2 \times 4 + 3 \times 3.2,\ 0.8 \times 4 + 3 \times 0.8)\}$$

$$= \{(4.4,\ 5.6)\} \neq y.$$

Thus all those $x \in S \setminus \{(a, b) \mid a, b \in Z_6(g) = \langle Z_6 \cup g\rangle\}$ are not in general idempotents.

Example 4.36: Let S = {Collection of all subsets from the MOD dual number plane $R_n(11)$, $*$, (5, 7)} be the subset MOD dual number groupoid.

Let $A = \{9g + 3\} \in S$

$$\begin{aligned} A * A &= \{9g + 3\} * \{9g + 3\} \\ &= \{9g + 3 * 9g + 3\} \\ &= \{5\ (9g + 3) + 7\ (9g + 3)\} \\ &= \{45g + 15 + 63g + 21\} \\ &= \{9g + 3\} = A \text{ is an idempotent.} \end{aligned}$$

Example 4.37: Let S = {Collection of all subsets from the MOD dual number plane $R_n(12)g$, $*$, $(10g + 7,\ 2g + 6)$} be the subset MOD dual number groupoid.

Let $A = \{9g + 5\} \in S$;

$$\begin{aligned} A * A &= \{9g + 5\} * \{9g + 5\} \\ &= \{10g + 7 \times 9g + 5 + 9g + 5 \times 2g + 6\} \\ &= \{63g + 50g + 35 + 10g + 54g + 30\} \\ &= \{9g + 5\} = A. \end{aligned}$$

Thus A is an idempotent of S. In fact S has only finite number of idempotents.

Now consider

$$B = \{10.3 + 0.5g\} \in S$$

$$\begin{aligned} B * B &= \{10.3 + 0.5g\} * \{10.3 + 0.5g\} \\ &= \{10.3 + 0.5g \times 10g + 7,\ 10.3 + 0.5g \times 2g + 6\} \\ &= \{103g + 3.5g + 72.1 + 20.6g + 3g + 61.8\} \\ &= \{10.1g + 1.9\} \neq B. \end{aligned}$$

So if B is a subset of S having entries as decimals then B is not an idempotent.

In view of this we prove the following theorem.

THEOREM 4.8: *Let S = {Collection of all subsets from the MOD dual number plane $R_n(m)g$; $g^2 = 0$, (a, b); a, b $\in R_n(m)g$} be the subset MOD dual number interval groupoid.*

$A = \{x + yg \,/\, x, y \in Z_m\} \in S$ are idempotents if and only if $a + b \equiv 1 \pmod m$.

Proof: Let

$$A = \{x + yg \mid x, y \in Z_m\} \in S;$$

$$\begin{aligned} A * A &= \{x + yg\} * \{x + yg\} \\ &= \{x + yg * x + yg\} \end{aligned}$$

$$= \{x + yg \times t + ug + x + yg \times x + s + vg\}$$

where $a = t + ug$ and $b = s + vg$; $t, u, s, v \in Z_m$.

$$A * A = \{xt + ygt + ugx + xs + ysg + xvg\}$$

$$= \{x (t + s) + y (t + s)g + x (u + v) g\}$$

$$= \{x + yg\}$$

$$= A$$

if and only if $t + s \equiv 1 \bmod m$ and $u + v = 0 \pmod m$ that is $a + b \equiv 1 \pmod m$.

Hence the theorem.

Note if $A = \{x + yg\}$ with one of x or y as a decimal $A * A \neq A$.

Consider the subset groupoids of MOD dual numbers which has infinite number of MOD pseudo nilpotents as well as finite number of nilpotents.

Example 4.38: Let S = {Collection of all subsets from the MOD dual number plane $R_n(10)g$, $g^2 = 0$, $(6 + 2g, 4 + 8g)$} be the subset MOD dual number interval groupoid.

Let $x = \{4 + 7g\} \in S$

$$x * x = \{4 + 7g\} * \{4 + 7g\}$$

$$= \{4 + 7g * 4 + 7g\}$$

$$= \{4 + 7g \times 6 + 2g + 4 + 8g \times 4 + 7g\}$$

$$= \{24 + 42g + 8g + 16 + 32g + 2g\}$$

$$= \{0 + 0g\} \in S.$$

Thus x is a MOD pseudo nilpotent element of order two.

Consider $y = \{0.8g + 1.2\} \in S$,

$$\begin{aligned} y * y &= \{0.8g + 1.2\} * \{0.8g + 1.2\} \\ &= \{0.8g + 1.2 * 0.8g + 1.2\} \\ &= \{6 + 2g \times 0.8g + 1.2 + 4 + \\ &\qquad\qquad 8g \times 0.8g + 1.2\} \\ &= \{4.8g + 7.2 + 2.4g + 4.8 + 6.4g + 3.2g\} \\ &= \{2 + 6.8g\} \neq \{0\}. \end{aligned}$$

Thus if the subset has decimal entries certainly it will not yield MOD pseudo zero divisors or nilpotents.

Example 4.39: Let S = {Collection of all subsets from the MOD dual number plane $R_n(18)g$; (5g, 13g), $*$} be the subset MOD dual number interval groupoid.

Let $A = \{10 + 3g\} \in S$

$$\begin{aligned} A * A &= \{10 + 3g\} * \{10 + 3g\} \\ &= \{10 + 3g * 10 + 3g\} \\ &= \{5g \times 10 + 3g + 13g \times 10 + 3g\} \\ &= \{50g + 130g\} \\ &= \{0\}. \end{aligned}$$

Thus A is a nilpotent element of order two.

Let x = {10g} and y = {7.32g} $\in$ S.

$$\begin{aligned} x * y &= \{10g\} * \{7,32g\} \\ &= \{10g * 7.32g\} \\ &= \{10g \times 5g + 7.32g \times 13g\} \\ &= \{0\}. \end{aligned}$$

Thus x is a zero divisor in S.

Also x is a nilpotent element of S.

Infact S has infinite number of zero divisors and nilpotents for this (t, s) = (5g, 13g).

In view of this we have the following theorem.

THEOREM 4.9: *Let S = {Collection of all subsets from the MOD dual number plane $R_n(m)g$, $g^2 = 0$, (a, b); *} be the subset MOD dual number interval groupoid.*

$A = \{x + yg \,/\, x, y \in Z_m\} \subseteq S$ are nilpotent elements of order two if and only if $a + b \equiv 0 \pmod m$.

Proof is similar to earlier cases hence left as an exercise to the reader.

Example 4.40: Let S = {Collection of all subsets from the MOD dual number plane $R_n(20)$ (g); $g^2 = 0$, (0.4g, 1.2g), $*$} be the MOD subset dual number groupoid.

Let x = {3.8g} $\in$ S;

$$\begin{aligned} x * x &= \{3.8g\} * \{3.8g\} \\ &= \{3.8g * 3.8g\} \end{aligned}$$

$$= \{0.4g \times 3.8g + 1.2g \times 3.8g\}$$

$$= \{0\}.$$

Thus x is a nilpotent element of S.

Let $y = \{5 + 2g\} \in S$;

$$y * y = \{5 + 2g\} * \{5 + 2g\}$$

$$= \{5 + 2g * 5 + 2g\}$$

$$= \{0.4g \times 5 + 2g + (5 + 2g)\ 1.2g\}$$

$$= \{2g + 6g\} = \{8g\} \neq 0.$$

Thus y is not a nilpotent element of S.

In view of all these we have the following theorem.

THEOREM 4.10: *Let S = {Collection of all subsets from the MOD dual number plane $R_n(m)g$; $g^2 = 0$; (ag, bg) / a, b ∈ [0, m), ∗} be the subset MOD dual number groupoid. Every subset z in B = {Collection of all subsets form ag; a ∈ [0, m)} ⊆ S are such that z ∗ z = 0 and z ∗ x = 0 for every x ∈ B.*

Proof follows from the simple fact $g^2 = 0$.

We will first illustrate this situation by an example or two.

Example 4.41: Let S = {Collection of all subsets from the MOD dual number plane $R_n(9)g$, g^2, (6.3g, 4g), ∗} be the MOD dual number subset interval groupoid.

Let A = {0.7g, 2g, 2.5g, 3.7772g, 6.015g, 4g} and

B = {6g, 6.72g, 8.19g, 0.19g, 0.18g, 1.8g, 4.14g} ∈ S.

Clearly $A * B = \{0\}$.

Thus the set A annihilates B and vice versa.

Further $A * A = \{0\}$ and $B * B = \{0\}$.

Example 4.42: Let S ={Collection of all subsets from the MOD dual number plane $R_n(17)g$; $g^2 = 0$ (10.222g, 0.458g), $*$} be the subset MOD dual number interval groupoid.

Let M = {Collection of all subsets from B = {ag | a $\in$ [0, 17)}} $\subseteq$ S. Every subset in M is a zero divisor and infact a nilpotent element of order two. Further $M * M = \{0\}$.

In view of this we define the notion of zero square subgroupoids of a MOD subset groupoid.

DEFINITION 4.2: *Let S = {Collection of all subsets from the MOD dual number plane $R_n(m)g$, $g^2 = 0$, $*$ (t, u); t, u $\in R_n(m)g$} be the MOD subset dual number interval groupoid. If H is a subgroupoid of S such that $H * H = \{0\}$, then H is defined as the zero square subgroupoid.*

It is important to keep on record that the same definition holds good for any general groupoid G need not even be MOD.

Example 4.43: Let M = {a + bg | a, b $\in Z_{48}$, $g^2 = 0$, $*$, (4g, 10g)} be the dual number groupoid.

Let P = {ag | a $\in Z_{48}$} $\subseteq$ M be the subgroupoid of M. Clearly $P * P = \{0\}$ that P is a zero square subgroupoid of M.

Example 4.44: Let M = {a + bg | a, b $\in R_n(24)g$, $g^2 = 0$, (10g, 4g), $*$} be the MOD dual number groupoid.

T = {ag | a $\in$ [0, 24)} $\subseteq$ M is the subgroupoid of M which is a zero square subgroupoid as $T * T = \{0\}$.

It is left as an open problem to find groupoids G such that G $*$ G = {0}. That is zero square groupoids.

Next the study of groupoids built using MOD special dual like numbers is carried out.

Example 4.45: Let S = {$R_n(16)g$, $g^2 = g$, (4, 5), $*$} be the MOD special dual like number groupoid.

Let x = 6 + 8g and y = 7 + 10g $\in$ S.

$$\begin{aligned} x * y &= 6 + 8g * 7 + 10g \\ &= 4\,(6 + 8g) + 5\,(7 + 10g) \\ &= 24 + 32g + 35 + 50g \\ &= 11 + 2g \in S. \end{aligned}$$

This is the way product is performed on S.

Example 4.46: Let S = {a + bh | $h^2 = h$, a, b $\in$ [0, 6); (2 + 3h, 1 + 4h), $*$} be the MOD special dual like number groupoid.

For x = 5 + 2h and y = 1 + 4h $\in$ S we find x $*$ y and y $*$ x

$$\begin{aligned} x * y &= (5 + 2h) * (1 + 4h) \\ &= 5 + 2h \times 2 + 3h + 1 + 4h \times 1 + 4h \\ &= 10 + 4h + 15h + 6h + 1 + 4h + 4h + 16h \\ &= 5 + h \qquad \ldots \text{ I} \end{aligned}$$

Consider

$$\begin{aligned} y * h &= (1 + 4h) * 5 + 2h \\ &= 1 + 4h \times 2 + 3h + 5 + 2h \times 1 + 4h \end{aligned}$$

$$= 2 + 8h + 3h + 12h + 5 + 2h + 20h + 8h$$

$$= 1 + 5h \qquad \ldots \qquad \text{II}$$

Clearly I and II are distinct hence S is a non commutative MOD special dual like number groupoid.

Example 4.47: Let $S = \{a + bh \mid h^2 = h, a, b \in [0, 12), *, (1.1, 0.8)\}$ be the MOD special dual like number groupoid.

Let $x = 7 + 0.5h$ and $y = 0.2 + 4h \in S$;

$$\begin{aligned} x * y &= (7 + 0.5h) * (0.2 + 4h) \\ &= 7 + 0.5h \times 1.1 + 0.2 + 4h \times 0.8 \\ &= 7.7 + 0.35h + 0.16 + 3.2h \\ &= 7.86 + 3.55h \in S. \end{aligned}$$

This is the way $*$ operation is performed on S.

Clearly $*$ operation on S is non associative.

Take $z = 0.3 \in S$ and use $x * y$ obtained above.

$$\begin{aligned} (x * y) * z &= (7.86 + 3.55h) * 0.3 \\ &= 7.86 + 3.55h \times 1.1 + 0.3 \times 0.8 \\ &= 8.646 + 3.905h + 0.24 \\ &= 8.886 + 3.905h \qquad \ldots \quad \text{I} \end{aligned}$$

Consider

$$\begin{aligned} x * (y * z) &= x * ((0.2 + 4h) * 0.3) \\ &= x * (0.2 + 4h \times 1.1 + 0.3 \times 0.8) \end{aligned}$$

$$= \quad x * (0.22 + 0.44h + 0.24)$$

$$= \quad (7 + 0.5h) * (0.46 + 0.44h)$$

$$= \quad 7 + 0.5h \times 1.1 + 0.46 + 0.44h \times 0.3$$

$$= \quad 7.7 + 0.55h + 0.138 + 0.132h$$

$$= \quad 7.838 + 0.682h \qquad \ldots \quad \text{II}$$

Clearly I and II are distinct so this MOD special dual like number groupoid is non associative.

Example 4.48: Let
$S = \{a + bh \mid h^2 = h, a, b \in [0, 16), (0.5h, 6h), *\}$ be the MOD special dual like number groupoid.

Let $x = 0.3 + 6.2h$ and $y = 7.5 + 1.5h \in S$.

$$x * y \quad = \quad 0.3 + 6.2h * 7.5 + 1.5h$$

$$= \quad 0.3 + 6.2h \times 0.5h + 7.5 + 1.5h \times 6h$$

$$= \quad 0.15h + 3.10h + 45.0h + 9h$$

$$= \quad 9.25h$$

This is the way product is found.

Example 4.49: Let S = {Collection of a + bh with a, b $\in$ [0, 13), $h^2 = h$, $*$, (11, 2)} be the MOD special dual like number groupoid.

Let $x = 5 + 3h \in S$;

$$x * x \quad = \quad 5 + 3h * 5 + 3h$$

$$= \quad 5 + 3h \times 11 + 5 + 3h \times 2$$

$$= 55 + 33h + 10 + 6h$$

$$= 0.$$

Thus x is a MOD pseudo nilpotent element of order two.

Infact $A = \{x + yh \mid x, y \in Z_{13}, h^2 = h\} \subseteq S$ is a subgroupoid of S and every element in A is MOD pseudo nilpotent of order two.

Let $a = 2 + 4h$ and $b = 5h + 10 \in A$,

$$\begin{aligned} a * b &= (2 + 4h) * (5h + 10) \\ &= 2 + 4h \times 11 + 5h + 10 \times 2 \\ &= 22 + 44h + 10h + 20 \\ &= 3 + 3h \in A. \end{aligned}$$

Clearly $x * y \neq 0$ in general for all $x, y \in A$ $x \neq y$.

Example 4.50: Let
$S = \{a + bh \mid h^2 = h, a, b \in [0, 10), *, (4 + 2h, 6 + 8h)\}$ be the MOD special dual like number groupoid.

Let $x = 5 + 3h$ and $y = 2 + 8h \in S$;

$$\begin{aligned} x * y &= (5 + 3h) * (2 + 8h) \\ &= 5 + 3h \times 4 + 2h + 2 + 8h \times 6 + 8h \\ &= 20 + 12h + 10h + 6h + 12 + 64h + \\ &\quad 48h + 16h \\ &= 2 + 8h \neq 0. \end{aligned}$$

Consider

$$x * x = (5 + 3h) * (5 + 3h)$$

$$= (5 + 3h) \times (4 + 2h) + (5 + 3h) \times (6 + 8h)$$

$$= 20 + 12h + 10h + 6h + 30 + 18h + 24h + 40h$$

$$= 0.$$

Thus x is a nilpotent element of order two. Infact S has only finite number of MOD pseudo nilpotent elements of order two.

In view of these examples the following theorem is true.

THEOREM 4.11: *Let $S = \{a + bh \mid a, b \in [0, m), h^2 = h, *, (x, y); x, y \in Z_m(h)\}$ be the MOD special dual like number groupoid. Every $A = t + uh$; $(t, u \in Z_m)$ in S is a MOD pseudo nilpotent element of order two if and only if $x + y \equiv 0 \pmod m$.*

Proof: As in case of earlier results.

However it remains as an open problem to find MOD special dual like number groupoids that has zero square subgroupoid.

Example 4.51: Let $S = \{a + bh \mid a, b \in [0, 9), h^2 = h, (4 + 3h, 6 + 6h)\}$ be the special MOD dual like number groupoid.

Let $x = 2 + 3h \in S$.

$$x * x = 2 + 3h * 2 + 3h$$

$$= 2 + 3h \times 4 + 3h + 2 + 3h \times 6 + 6h$$

$$= 8 + 12h + 6h + 9h + 12 + 18h + 12h + 18h$$

$$= 2 + 3h = x.$$

Thus x is an idempotent of S. S has finitely many idempotents.

Let $y = 0.3 + 7h \in S$

$$\begin{aligned} y * y &= 0.3 + 7h * 0.3 + 7h \\ &= 0.3 + 7h \times 4 + 3h + 0.3 + 7h \times 6 + 6h \\ &= 1.2 + 28h + 0.9h + 21h + 1.8 + 42h \\ &= 3 + 7h \neq y. \end{aligned}$$

Example 4.52: Let $S = \{a + bh \mid a, b \in [0, 12), h^2 = h, (10 + 3h, 3 + 9h)\}$ be the MOD special dual like number groupoid. S has idempotents.

$x = 9.1 + 0.8h \in S$

$$\begin{aligned} x * x &= 9.1 + 0.8h \times 9.1 + 0.8h \\ &= 9.1 + 0.8h \times 10 + 3h + 9.1 + 0.8h \times 3 + 9h \\ &= 91 + 8h + 27.3h + 2.4h + 27.3 + 2.4h + 81.9h + 7.2h \\ &= 10.3 + 9.2h \neq x. \end{aligned}$$

x is not an idempotent.

In view of all this we have the following theorem.

THEOREM 4.12: *Let $S = \{a + bh \mid h^2 = h, a, b \in [0, m), *(t, u); t, u \in Z_m(g)\}$ be the MOD special dual like number groupoid.*

Every $x = a + bh \in S$ $a, b \in Z_m$ is an idempotent if and only if $t + u \equiv 1 \pmod m$.

Proof: Follows from the earlier results.

Example 4.53: Let S ={a + bh | h^2 = h, a, b ∈ [0, 19); ∗, (th, uh); t, u ∈ [0, m)} be the MOD special dual like number groupoid.

Let x = a + bh and y = c + dh ∈ S;

$$\begin{aligned} x * y &= (a + bh) * (c + dh) \\ &= (a + bh)\, th + (c + dh)\, uh \\ &= ath + bth + cuh + duh \\ &= eh;\ e \in [0, m). \end{aligned}$$

For instance x = 5.1 + 3.8h and y = 7 + 6.9h ∈ S;

$$\begin{aligned} x * y &= (5.1 + 3.8h) * (7 + 6.9h) \\ &= (5.1 + 3.8h)\, th + (7 + 6.9h)\, uh \\ &= kh\ (k \in [0, m)). \end{aligned}$$

Example 4.54: Let S = {a + bh | h^2 = h, a, b ∈ [0, 10); ∗, (5h, 0.2h)} be the MOD special dual like number groupoid.

Let x = 8 + 0.7h and y = 0.7 + 6h ∈ S;

$$\begin{aligned} x * y &= (8 + 0.7h) * (0.7 + 6h) \\ &= 5h\,(8 + 0.7h) + 0.2h\,(0.7 + 6h) \\ &= 40h + 3.5h + 0.14h + 1.2h \\ &= 4.84h. \end{aligned}$$

Example 4.55: Let S = {a + bg | a, b ∈ [0, 10), g^2 = g, ∗, (6g, 7 + 2g)} be the MOD special dual like number groupoid.

Let $x = 5 + 1.2g$ and $y = 0.4 + 6g \in S$

$$
\begin{aligned}
x * y &= (5 + 1.2g) * (0.4 + 6g) \\
&= 5 + 12g \times 6g + 0.4 + 6g \times 7 + 2g \\
&= 30g + 72g + 2.8 + 42g + 0.8g + 12g \\
&= 6.8g + 2.8.
\end{aligned}
$$

This is the way operation is done on S.

Next examples of subset MOD special dual like number groupoids.

Example 4.56: Let S = {Collection of all subsets from $R_n(9)h$; $h^2 = h$; $*$, (3, 6)} be the subset special dual like number groupoid.

Let $x = \{4 + 5g\} \in S$,

$$
\begin{aligned}
x * x &= \{4 + 5g\} * \{4 + 5g\} \\
&= 4 + 5g \times 3 + (4 + 5g) \times 6 \\
&= 12 + 15g + 24 + 30g \\
&= \{0\}.
\end{aligned}
$$

x is a zero divisor.

Example 4.57: Let S = {Collection of all subsets from $R_n(12)h$; $h^2 = h$, $*$, $(8 + 5h, 4 + 7h)$} be the subset special dual like number groupoid.

Let $A = \{10 + 4h\}$ and $B = \{7 + 6h\} \in S$;

$$
\begin{aligned}
A * B &= \{10 + 4h\} * \{7 + 6h\} \\
&= \{10 + 4h * 7 + 6h\} \\
&= 10 + 4h \times 8 + 5h + 7 + 6h \times 4 + 7h \\
&= 80 + 32h + 50h + 20h + 28 + 24h \\
&\quad + 49h + 42h \\
&= \{0 + 5h\} \neq \{0\}.
\end{aligned}
$$

$$
\begin{aligned}
A * A &= \{10 + 4h\} * \{10 + 4h\} \\
&= \{10 + 4h * 10 + 4h\} \\
&= 10 + 4h \times 8 + 5h + 10 + 4h \times 4 + 7h \\
&= 80 + 32h + 50h + 20h + 40 + 16h + \\
&\quad 70h + 28h \\
&= \{0 + 0h\} \text{ is a nilpotent element of S.}
\end{aligned}
$$

Example 4.58: Let S = {Collection of all subsets from $R_n(15)h$, $h^2 = h$, $*$, $(12 + 4h, 3 + 11h)$} be the subset special dual like number groupoid.

Let $x = \{10 + 8h\} \in S$.

$$
\begin{aligned}
x * x &= \{10 + 8h\} * \{10 + 8h\} \\
&= \{10 + 8h * 10 + 8h\} \\
&= \{10 + 8h \times 12 + 4h + 3 + 11h \times \\
&\quad 10 + 8h\}
\end{aligned}
$$

$$= \{120 + 96h + 32h + 40h + 30 + 24h + 88h + 110h\}$$

$$= \{0 + 0h\} \text{ is a nilpotent element.}$$

Example 4.59: Let M = {Collection of all subsets from $R_n(20)h$, $h^2 = h$, $*$, $(11 + 18h, 9 + 2h)$} be the subset special dual like number groupoid.

Let $x = \{4 + 8h\} \in M$

$$\begin{aligned} x * x &= \{4 + 8h\} * \{4 + 8h\} \\ &= \{4 + 8h * 4 + 8h\} \\ &= \{4 + 8h \times 11 + 18h + 4 + 8h \times 9 + 2h\} \\ &= \{44 + 88h + 72h + 144h + 36 + 72h + 8h + 16h\} \\ &= \{0 + 0h\} \text{ is a nilpotent element of M.} \end{aligned}$$

In view of all these we have the following theorem.

THEOREM 4.13: *Let S = {Collection of all subsets from $R_n(m)h$; $h^2 = h$, $*$, (t, u); $t, u \in Z_m(h)$} be the subset MOD special dual like number groupoid. $A = \{x + yh \mid x, y \in Z_m\} \in S$; is a pseudo nilpotent if and only if $t + u \equiv 0 \pmod m$.*

Proof: Follows as in case of other MOD planes.

Example 4.60: Let S = {Collection of all subsets from $R_n(12)h$, $h^2 = h$, $*$, $(0.3h, 0.8 + 0.7h)$} be the subset MOD special dual like number groupoid. S has no MOD pseudo zero divisors and idempotents.

Thus if (s, t) where s and t are decimals then finding MOD pseudo zero divisors, MOD pseudo nilpotents and idempotents happens to be a difficult one.

Next we can develop the MOD special quasi dual number groupoids.

This is illustrated by some examples.

Example 4.61: Let G = {a + bk | a, b ∈ [0, 9), k^2 = 8k; (4, 2), ∗} be the MOD special quasi dual number groupoid.

Let x = 6 + 0.2k; y = 0.7 + 4k ∈ G;

$$
\begin{aligned}
x * y &= (6 + 0.2k) * (0.7 + 4k) \\
&= (6 + 0.2k)\, 4 + (0.7 + 4k)\, 2 \\
&= 24 + 0.8k + 1.4 + 8k \\
&= 7.4 + 8.8k.
\end{aligned}
$$

This is the way operation is performed on G.

Example 4.62: Let
G = {a + bk | a, b ∈ [0, 12); k^2 = 11k, (4, k), ∗} be the MOD special quasi dual number groupoid.

Let x = 8 + 6k and y = 2 + k ∈ G

$$
\begin{aligned}
x * y &= (8 + 6k) * (2 + k) \\
&= 8 + 6k \times 4 + (2 + k) \times k \\
&= 32 + 24k + 2k + 11k \\
&= 8 + k \in G.
\end{aligned}
$$

This is the way operation on G is performed.

Example 4.63: Let G = {a + bk | a, b ∈ [0, 14), k^2 = 13k, (0.3k, 10k), ∗} be the MOD special quasi dual like number groupoid.

Let $x = 5 + 9k$ and $y = 10 + 3k \in G$;

$$\begin{aligned} x * y &= (5 + 9k) * (10 + 3k) \\ &= 5 + 9k \times 0.3k + 10 + 3k \times 10k \\ &= 1.5k + 2.7 \times 13k + 100k + 30 \times 13k \\ &= 1.5k + 35.1k + 100k + 390k \\ &= 8.6k \in S. \end{aligned}$$

This is the way operations are performed on G.

Example 4.64: Let
$G = \{a + bk \mid a, b \in [0, 21); k^2 = 20k, (10k, 11k); *\}$ be the MOD special dual like number groupoid.

Let $x = 8 + 4k \in G$.

$$\begin{aligned} x * x &= 8 + 4k * 8 + 4k \\ &= 8 + 4k \times 10k + 8 + 4k \times 11k \\ &= 80k + 40 \times 20 \times k + 88k + 44k \times 20 \\ &= 0. \end{aligned}$$

x is a MOD pseudo nilpotent element of G of order two.

Let $x = 0.8 + 1.1k \in G$.

$$\begin{aligned} x * x &= 0.8 + 1.1k * 0.8 + 1.1k \\ &= 0.8 + 1.1k \times 10k + 0.8 + 1.1k \times 11k \\ &= 8k + 11k \times 20 + 8.8k + 12.1 \times 20k \end{aligned}$$

$$= \quad 8k + 220k + 8.8k + 242k$$

$$= \quad 478.8k \pmod{21}$$

$$= \quad 16.8k \neq 0.$$

x is not a MOD pseudo nilpotent element of G.

Example 4.65: Let
$G = \{a + bk \mid k^2 = k, a, b \in [0, 16); (0.3k, 4), *\}$ be the MOD special dual like number groupoid.

Let $x = 10 + 2.1k$ and $y = 0.7 + 9k \in G$

$$\begin{aligned} x * y &= 10 + 2.1k * 0.7 + 9k \\ &= 10 + 2.1k \times 0.3k + 0.7 + 9k \times 4 \\ &= 3k + 0.63 + 3 \times 15k + 2.8 + 36k \\ &= 2.8 + 0.45k. \end{aligned}$$

Finding idempotents and nilpotents in case of these groupoids happens to be a difficult problem.

Next we proceed on to study MOD subsets groupoids of the special quasi dual like numbers.

This is illustrated by some examples.

Example 4.66: Let S = {Collection of all subsets from the special quasi dual number MOD plane $R_n(13)k$, $k^2 = 12k$, $*$, (10, 3)} be the MOD subset special quasi dual like number groupoid.

Let P = {7 + 10k} $\in$ S,

$$\begin{aligned} P * P &= \{7 + 10k\} * \{7 + 10k\} \\ &= \{7 + 10k * 7 + 10k\} \\ &= 7 + 10k \times 10 + 7 + 10k \times 3 \\ &= 70 + 100k + 21 + 30k \\ &= \{0\}. \end{aligned}$$

Thus P is a MOD pseudo nilpotent element of order two.

Let x = {12 + 5k} $\in$ S;

$$\begin{aligned} x * x &= \{12 + 5k\} * \{12 + 5k\} \\ &= \{12 + 5k * 12 + 5k\} \\ &= \{12 + 5k \times 10 + 12 + 5k \times 3\} \\ &= \{120 + 50k + 36 + 15k\} \\ &= \{0\} \end{aligned}$$

is again a MOD pseudo nilpotent element of order two.

Let x = {0.7 + 1.2k} $\in$ S

$$\begin{aligned} x * x &= \{0.7 + 1.2k\} * \{0.7 + 1.2k\} \\ &= \{0.7 + 1.2k * 0.7 + 1.2k\} \\ &= 0.7 + 1.2k \times 10 + 0.7 + 1.2k \times 3 \\ &= 7 + 12k + 2.1 + 3.6k \\ &= 9.1 + 2.6k. \end{aligned}$$

So if the subset has decimal entries then $x * x \neq \{0\}$.

Example 4.67: Let S = {Collection of all subsets from the special quasi dual number MOD plane $R_n(31)k$, $k^2 = 30k$, (0.3k, 1.6k), $*$} be the subset MOD special quasi dual like number groupoid.

Let $x = \{7 + 10k\} \in S$.

$$\begin{aligned} x * x &= \{7 + 10k\} * \{7 + 10k\} \\ &= \{7 + 10k * 7 + 10k\} \\ &= \{7 + 10k \times 0.3k + 7 + 10k \times 1.6k\} \\ &= \{2.1k + 3 \times 30k + 11.2k + 16 \times 30k\} \\ &= \{25.3k\}. \end{aligned}$$

Finding MOD pseudo nilpotent elements in S happens to be a difficult problem.

Example 4.68: Let S = {Collection of all subsets from the MOD special quasi dual number plane $R_n(17)k$, $k^2 = 16k$, $*$, (10k, 7k)} be the MOD subset special quasi dual like number groupoid.

Let $x = \{8 + 7k\} \in S$;

$$\begin{aligned} x * x &= \{8 + 7k\} * \{8 + 7k\} \\ &= \{8 + 7k * 8 + 7k\} \\ &= \{8 + 7k \times 10k + 8 + 7k \times 7k\} \\ &= \{80k + 70 \times 16k + 56k + 49 \times 16k\} \\ &= \{0\}. \end{aligned}$$

Thus x is a nilpotent of order two.

Infact S has only finite number of MOD pseudo nilpotent elements of order two.

Study of properties of subset MOD special quasi dual like number groupoids can be carried out with appropriate changes as in case of subset MOD dual numbers groupoids or subset MOD special dual like number groupoids.

This task is left as an exercise to the reader.

Some problems in this direction are suggested.

Problems

1. Obtain some special features associated with MOD neutrosophic interval groupoid.

2. Let G = {a + bI | a, b $\in R_n^I$ (23), $*$, (10, 13)} be the MOD neutrosophic interval groupoid.

   i. Prove G has zero MOD pseudo divisors.
   ii. Can G have MOD pseudo zero divisors other than nilpotents of order two?
   iii. Can G have idempotents?
   iv. Find S- MOD pseudo zero divisors if any in G.
   v. Can G have S-units?
   vi. Find units if any that are not S-units.
   vii. Find subgroupoids of G.
   viii. Can G have ideals of finite order?
   ix. Can G have S-ideals of finite order?
   x. Obtain some important and interesting features associated with G.

3. Let G = {a + bI | a, b $\in R_n^I$ (28), $*$, (14, 15)} be the MOD neutrosophic groupoid.

   Study questions i to x of problem 2 for this G.

4. Let H = {a + bI | a, b ∈ $R_n^I(40)$,∗, (16I, 10I)} be the MOD neutrosophic groupoid.

   Study questions i to x of problem 2 for this H.

5. Let K = {a + bI | a, b ∈ $R_n^I(24)$, ∗, (10.31, 0.52I) } be the MOD neutrosophic groupoid.

   Study questions i to x of problem 2 for this K.

6. Let L = {a + bI | a, b ∈ $R_n^I(35)$, ∗, (0, 0.3215) } be the MOD neutrosophic groupoid.

   Study questions i to x of problem 2 for this L.

7. Let T = {a + bI | a, b ∈ $R_n^I(141)$, ∗, (6.7703I, 0) } be the MOD neutrosophic groupoid.

   Study questions i to x of problem 2 for this T.

8. Let W = {Collection of all subsets from the neutrosophic MOD plane $R_n^I(189)$,∗, (21, 168) be the MOD neutrosophic subset groupoid.

   i. Find ideals and subgroupoids of W
   ii. Can ideals of W be of finite order?
   iii. Prove W has subgroupoids of finite order.
   iv. Can W have subgroupoids of infinite order?
   v. Find all S- MOD pseudo subgroupoids of finite order.
   vi. Can W have S-zero divisors?
   vii. Can W have infinite number of MOD pseudo zero divisors?
   viii. Can W have S-units?
   ix. Find units which are not be S-units.
   x. Find any other interesting property associated with W.

xi. Can subsets of cardinality greater than one MOD pseudo S-units?
xii. If A is an idempotent can $|A| > 1$?
xiii. Let B be a zero divisor in W can $|B| > 1$?

9. Obtain some special features enjoyed by subset MOD neutrosophic interval groupoids.

10. Let M = {Collection of all subsets from the MOD neutrosophic plane R_n^I (15),∗, (10, 5I)} be the subset MOD neutrosophic interval groupoid.

    Study questions i to xiii of problem 8 for this M.

11. Let N = {Collection of all subsets from the MOD neutrosophic plane R_n^I (12), ∗, (6.5I, 5.5I)} be the subset MOD neutrosophic interval groupoid.

    Study questions i to xiii of problem 8 for this M.

12. Let B = {Collection of all subsets from the MOD neutrosophic plane R_n^I (13), ∗, (10 + 3I, 4 + 10I)} be the subset MOD neutrosophic interval groupoid.

    Study questions i to xiii of problem 8 for this B.

13. Let V = {Collection of all subsets from the MOD neutrosophic plane R_n^I (12), ∗, (10I, 2I)} be the subset MOD neutrosophic interval groupoid.

    Study questions i to xiii of problem 8 for this V.

14. Let D = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(47)$, ∗, (0.337, 40.03I)} be the subset MOD neutrosophic interval groupoid.

    Study questions i to xiii of problem 8 for this D.

15. Can MOD subset neutrosophic interval groupoids have MOD pseudo zero square subgroupoids?

16. Let M = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(6)$, ∗, (0.0004I, 0.002)} be the subset MOD neutrosophic interval groupoid.

    Study questions i to xiii of problem 8 for this M.

17. Let S = {a + bg | a, b ∈ $R_n(12)$, $g^2 = 0$, (10, 2), ∗} be the MOD dual number groupoid.

    i. Find all nilpotent elements of S.
    ii. Can S have infinite number of MOD pseudo zero divisors?
    iii. Does S contain S- MOD pseudo zero divisors?
    iv. Can S have idempotents?
    v. Can S have zero square subgroupoids?
    vi. Obtain some special features enjoyed by S.
    vii. Can S have units?
    viii. Can S have right ideals which are not left ideals and vice versa?
    ix. Can S have S-ideals?
    x. Can S have S nilpotents?
    xi. Can S have S-subgroupoids of infinite order?

18. Let W = {a + bg | $g^2 = 0$, a, b ∈ $R_n(24)$, ∗, (13, 12)} be the MOD dual number groupoid.

Study questions i to xi of problem 17 for this W.

19. Let S = $\{a + bg \mid g^2 = 0, a, b \in R_n(10)g, (4.2, 3), *\}$ be the MOD dual number groupoid.
    Study questions i to xi of problem 17 for this S.

20. Let
    M = $\{a + bg \mid g^2 = 0, a, b \in R_n(16), *, (5 + 10I, 11 + 6I)\}$ be the MOD dual number groupoid.

    Study questions i to xi of problem 17 for this M.

21. Let T = $\{a + bI \mid a, b \in R_n(15), g^2 = 0, (0.5I, 10 + 4I), *\}$ be the MOD dual number groupoid.

    Study questions i to xi of problem 17 for this T.

22. Let N = {Collection of all subsets from the MOD dual number plane $R_n(10)g$; $g^2 = 0$ (4.5, 5.5), *} be the subset MOD interval dual number groupoid.

    i. Find all the special features enjoyed by N
    ii. Can N have nilpotents?
    iii. If s $\in$ N is a MOD pseudo zero divisor can the subset s have more than one element?
    iv. Find right ideals of N which are not left ideals of N.
    v. Can N have S-ideals?
    vi. Can N have ideals of finite order?
    vii. Can N have subset subgroupoids of finite order?
    viii. When will N have MOD pseudo zero square subset subgroupoids?
    ix. Can N have idempotents?
    x. Is it possible for N to have S-idempotents?

23. Let E = {Collection of all subsets from the MOD dual number plane $R_n(18)g$, $g^2 = 0$, (11, 7), $*$} be the MOD subset dual number interval groupoid.

    Study questions i to x of problem 22 for this E.

24. Let F = {Collection of all subsets from the MOD dual number plane $R_n(6)g$, $g^2 = 0$, (3, 4), $*$} be the subset MOD dual number groupoid.

    Study questions i to x of problem 22 for this F.

25. Let D = {Collection of all subsets from the MOD dual number plane $R_n(11)g$, $g^2 = 0$, $*$, (9 + 5g, 3 + 6g)} be the subset MOD dual number groupoid.

    Study questions i to x of problem 22 for this D.

26. Let H = {Collection of all subsets from the MOD dual number plane $R_n(10)g$, $g^2 = 0$, (0.41g, 5.2g), $*$} be the MOD dual number subset groupoid.

    Study question i to x of problem 22 for this H.

27. Compare the groupoids given in problems 22, 23, 24, 25 and 26. Find the similarities and difference between them.

28. Let S = {a + bh | $h^2 = h$, a, b $\in R_n(20)h$, (10, 11), $*$} be the MOD special dual like number groupoid.

    Study questions i to xi of problem 17 for this S.

29. Let $V = \{a + bh \mid a, b \in R_n(11)h, h^2 = h, *, (8, 3)\}$ be the MOD special dual like number groupoid.

    Study questions i to xi of problem 17 for this V.

30. Let $W = \{a + bh \mid a, b \in R_n(15)h, h^2 = h, *, (0.311, 11.215)\}$ be the MOD special dual like number groupoid.

    Study questions i to xi of problem 17 for this W.

31. Let $M = \{a + bh \mid h^2 = h, a, b \in R_n(24)h, *, (12, 13h)\}$ be the MOD special dual like number groupoid.

    Study questions i to xi of problem 17 for this M.

32. Let $S = \{a + bh \mid h^2 = h, a, b \in R_n(15)h, (10 + 3h, 6 + 12h), *\}$ be the MOD special dual like number groupoid.

    Study questions i to xi of problem 17 for this S.

33. Let $T = \{a + bh \mid a, b \in R_n(148)h, h^2 = h, (0.331 + 0.2h, 0), *\}$ be the MOD special dual like number groupoid.

    Study questions i to xi of problem 17 for this T.

34. Let $Z = \{a + bh \mid a, b \in R_n(24)h, h^2 = h, (0.5h, 6.3h), *\}$ be the MOD special dual like number groupoid.

    Study questions i to xi of problem 17 for this Z.

35. Compare and contrast the groupoids given in problems 28, 29, 30, 31, 32, 33 and 34 with each other.

36. Let S = {Collection of all subsets from the MOD special dual like number plane $R_n(16)h$, $*$, (10, 6), $h^2 = h$} be the MOD subset special dual like number groupoid.

    Study questions i to x of problem 22 for this S.

37. Let A = {Collection of all subsets from the MOD special dual like number plane, $R_n(10)h$, $h^2 = 0$, (4, 7), $*$} be the subset MOD special dual like number groupoid.

    Study questions i to x of problem 22 for this A.

38. Let B = {Collection of all subsets from the MOD special dual like number plane, $R_n(7)h$, $h^2 = h$, $*$, (3 + 4h, 2 + 5h)} be the subset MOD special dual like number groupoid.

    Study questions i to x of problem 22 for this B.

39. Let C = {Collection of all subsets from the MOD special dual like number plane, $R_n(12)h$, $*$, $h^2 = h$, (7h, 6h)} be the subset MOD special dual like number groupoid.

    Study questions i to x of problem 22 for this C.

40. Let D = {Collection of all subsets from the MOD special dual like number plane, $R_n(16)h$, $*$, $h^2 = h$, (10 + h, 11 + 7h)} be the subset MOD special dual like number groupoid.

    Study questions i to x of problem 22 for this D.

41. Let E = {Collection of all subsets from the MOD special dual like number plane, $R_n(20)h$, $h^2 = h$, (10.331, 0.738), $*$} be the subset MOD special dual like number groupoid.

    Study questions i to x of problem 22 for this E.

42. Let F = {Collection of all subsets from the MOD special dual like number plane, $R_n(23)h$, $h^2 = h$, (0.115h, 9.32h), $*$} be the subset MOD special dual like number groupoid.

    Study questions i to x of problem 22 for this F.

43. Let G = {Collection of all subsets from the MOD special dual like number plane, $R_n(16)h$, $h^2 = h$, (4.885, 10h), $*$} be the subset MOD special dual like number groupoid.

    Study questions i to x of problem 22 for this G.

44. Compare all the subsets MOD special dual like number groupoids in problems 36 to 43 with each other.

    Study the special and distinct features enjoyed by them.

45. Prove given a MOD real plane $R_n(m)$ (m fixed positive integer one can construct infinite number of groupoids and subset groupoids.

46. Let Z = {a + bk | a, b $\in$ $R_n(10)k$, $k^2 = 9k$, $*$, (6, 5)} be the MOD special quasi dual number groupoid.

    Study question i to xi of problem 17 for this Z.

47. Let Y = {a + bk | a, b ∈ $R_n(24)k$, $k^2 = 23k$, ∗, (10 + 5k, 14 + 7k)} be the MOD special quasi dual number groupoid.

    Study question i to xi of problem 17 for this Y.

48. Let X = {a + bk | a, b ∈ $R_n(43)k$, $k^2 = 42k$, ∗, (10 + 3k, 14 + 27k)} be the MOD special quasi dual number groupoid.

    Study question i to xi of problem 17 for this X.

49. Let W = {a + bk | a, b ∈ $R_n(15)k$, $k^2 = 14k$, ∗, (0.377 + 0.5k, 8.77 + 4.2k)} be the MOD special quasi dual number groupoid.

    Study question i to xi of problem 17 for this W.

50. Let
    V = {a + bk | a, b ∈ $R_n(21)k$, $k^2 = 20k$, (10.31k, 5), ∗} be the MOD special quasi dual number groupoid.

    Study question i to xi of problem 17 for this V.

51. Let
    U = {a + bk | a, b ∈ $R_n(18)k$, $k^2 = 17k$, (0, 10.37k), ∗} be the MOD special quasi dual number groupoid.

    Study question i to xi of problem 17 for this U.

52. Compare and contrast the MOD groupoids in problems 46 to 51.

53. Let T = {Collection of all subsets from the MOD special quasi dual like number plane $R_n(12)k$, $k^2 = 11k$, (10, 4), $*$} be the subset MOD special quasi dual number groupoid.

    Study questions i to x of problem 22 for this T.

54. Let S = {Collection of all subsets from the MOD special quasi dual like number plane $R_n(41)k$, $k^2 = 40k$, (5k, 39k), $*$} be the subset MOD special quasi dual number groupoid.

    Study questions i to x of problem 22 for this S.

55. Let R = {Collection of all subsets from the MOD special quasi dual like number plane $R_n(40)k$, $k^2 = 39k$, (14 + 34k, 27 + 6k), $*$} be the subset MOD special quasi dual number groupoid.

    Study questions i to x of problem 22 for this R.

56. Let Q = {Collection of all subsets from the MOD special quasi dual like number plane $R_n(27)k$, $k^2 = 26k$, (16k, 0.331), $*$} be the subset MOD special quasi dual like number groupoid.

    Study questions i to x of problem 22 for this Q.

57. Let P = {Collection of all subsets from the MOD special quasi dual like number plane $R_n(18)k$, $k^2 = 17k$, (0, 16.004), $*$} be the subset MOD special quasi dual like number groupoid.

    Study questions i to x of problem 22 for this P.

58. Let O = {Collection of all subsets from the MOD special quasi dual like number plane $R_n(16)k$, $k^2 = 15k$, $(10.31k, 0)$, $*$} be the subset MOD special quasi dual like number groupoid.

    Study questions i to x of problem 22 for this O.

59. Compare the subset groupoids in problems 53 to 58. Enumerate the similarities and dissimilarities in them. Distinguish the special features enjoyed by each of the subset MOD special quasi dual like number groupoids.

Chapter Five

MOD NON-ASSOCIATIVE RINGS AND PSEUDO NON-ASSOCIATIVE SEMIRINGS

In this chapter MOD non-associative rings and pseudo semirings of several different MOD planes are introduced. The notions and notations from earlier chapters are used.

DEFINITION 5.1: *Let G = {(a, b) | a, b ∈ $R_n(m)$, *, (s, t)} be the real MOD groupoid. R any commutative ring with unit.*

$$RG = \left\{ \sum_i a_i g_i \right| \text{ i runs over finite index } g_i \in G;\ a_i \in R\}$$

under usual '+' and '×' is defined as a MOD real groupoid ring which is a non-associative ring.

We shall give examples of them.

Example 5.1: Let G = {(a, b) | a, b ∈ [0, 20), *, (10, 11)} be the MOD real groupoid. R be the field of reals; RG is the MOD real non-associative ring of G over R.

Let x = 5(3, 0.1) + 2(1, 5) and y = 8(3, 0.1) + 12(0, 3) $\in$ RG.

$$\begin{aligned} x + y &= 5(3, 0.1) + 2(1, 5) + 8(3, 0.1) + 12(0, 3) \\ &= 13\,(3, 0.1) + 2(1, 5) + 12\,(0, 3) \in \text{RG}. \\ x \times y &= [5(3, 0.1) + 2(1, 5)] \times [8\,(3, 0.1) + 12(0, 3)] \\ &= 40\,[(3, 0.1) * (3, 0.1)] + 16[(1,5) * (3, 0.1)] \\ &\quad + 60\,[(3, 0.1) * (0, 3)] + 24\,[(1, 5) * (0, 3)] \\ &= 40\,(3, 2.1) + 16\,(3, 11.1) + 60\,(10, 14) + \\ &\quad 24\,(10, 3) \in \text{RG}. \end{aligned}$$

Let a = 5 (0.3, 1.2),
b = 7 (0.3, 1.2) and c = 5(0.1, 0.2) $\in$ RG

$$\begin{aligned} (a+b) \times c &= [5(0.3, 1.2) + 7(0.3, 1.2)] \times 5\,(0.1, 0.2)] \\ &= 12\,(0.3, 1.2) \times 5(0.1, 0.2) \\ &= 60\,(4.1, 14.2) \qquad \ldots \qquad \text{I} \end{aligned}$$

$a \times c + b \times c$

$$\begin{aligned} &= 5\,(0.3, 1.2) \times 5\,(0.1, 0.2) \\ &\quad + 7\,(0.3, 1.2) \times 5\,(0.1, 0.2) \\ &= 25\,(4.1, 14.2) + 35\,(4.1, 14.2) \\ &= 60\,(4.1, 14.2) \qquad \ldots \qquad \text{II} \end{aligned}$$

So I and II are identical hence for this a, b and c the distributive law is true.

Example 5.2: Let G = {(a, b) | a, b $\in$ [0, 10), *, (6.91, 0)} be the MOD real groupoid.

Let S = Z be the ring of integers SG be the MOD non-associative ring. Clearly SG is non-associative and non-commutative and is of infinite order.

Next we give examples of MOD neutrosophic non-associative ring using MOD neutrosophic groupoids G.

Example 5.3: Let G = {a + bI | a, b ∈ R_n^I (15), *, (10,6)} be the MOD neutrosophic groupoid. Q be the ring of rationals, QG be the groupoid ring. QG is defined as the MOD neutrosophic non-associative ring.

Let α = 5(2 + 3I) + 7(0.5 + 0.8I) and β = (10 (0.1 + 9I) ∈ QG.

$$\alpha\beta = [5 (2 + 3I) + 7(0.5 + 0.8I)] \times 10(0.1 + 9I)$$

$$= 50 (2 + 3I) * (0.1 + 9I) + 70 (0.5 + 0.8I) * (0.1 + 9I)$$

$$= 50 (5.6 + 9I) + 70 (5.6 + 9.8I)$$

$$= 50 (5.6 + 9I) + 70 (5.6 + 9.8I) \quad \ldots \quad \text{I}$$

Consider

$$\beta\alpha = 10 (0.1 + 9I) \times [5 (2 + 3I) + 7(0.5 + 0.8I)]$$

$$= 50 (0.1 + 9I) * (2 + 3I) + 70 (0.1 + 9I) * (0.5 + 0.8I)$$

$$= 50 (13 + 3I) + 70 (4 + 4.8I) \quad \ldots \quad \text{II}$$

Clearly I and II are distinct. Thus this MOD neutrosophic ring is non-associative and non-commutative.

Example 5.4: Let G = {a +bI | a + bI ∈ R_n^I (10), *, (6, 4)} be the MOD neutrosophic groupoid. Let C be the complex field CG be the groupoid ring. CG is the MOD neutrosophic complex non-associative ring.

Let $\alpha = (9 + 4i)(4 + 8I) + (10 - 5i)(1 + 2I)$ and $\beta = (7i)(3 + 2I) \in CG$.

$$\alpha\beta = [(9 + 4i)(4 + 8I) + (10 - 5i)(1 + 2I)] \times 7i(3 + 2I)$$

$$= (63i - 28)(6 + 6I) + (70i + 35)(8 + 0I).$$

This is the way operations are performed on CG.

Example 5.5: Let $G = \{a + bI \mid a + bI \in R_n^I(15), (0.31, 0.8I), *\}$ be the MOD neutrosophic groupoid. R be the field of reals RG be the MOD neutrosophic non-associative ring.

Example 5.6: Let $G = \{a + bI \mid a, b \in R_n^I(27), * (5I, 6.8I)\}$ be the MOD neutrosophic groupoid. Z_{16} be the ring of modulo integers $Z_{16}G$ is the defined as the MOD neutrosophic non-associative ring.

Next we proceed onto give examples the notion of MOD complex modulo integer non-associative ring.

Example 5.7: Let $G = \{a, bi_F \mid a + bi_F \in C_n(12);\ i_F^2 = 11, (11, 1), *\}$ be the MOD complex modulo integer groupoid. Z_{43} be the ring of modulo integers. $Z_{43}G$ be the MOD complex modulo integer non-associative ring.

Example 5.8: Let $G = \{a + bi_F \mid a + bi_F \in C_n(18);\ i_F^2 = 17, (10 + 2i_F, 9i_F, *\}$ be the MOD complex modulo integer groupoid.

$S = C_n(10)$ be the ring of complex modulo integers SG is the MOD complex modulo integer non-associative ring.

Let $x = (1.6 + 4i_F)(2 + 3i_F)$ and $y = (5 + 8i_F)(3 + 10i_F) \in SG$

$$x \times y = (1.6 + 4i_F)\,(2 + 3i_F) \times (5 + 8i_F)\,(3 + 10i_F)$$

$$= (1.6 + 4i_F) \times (5 + 8i_F)\,[(2 + 3i_F) * (3 + 10i_F)$$

$$= (8 + 20i_F + 12.8i_F + 32 \times 9)\,(14 + 7i_F)]$$

$$= (6 + 2.8i_F)\,(14 + 7\,i_F).$$

This is the way product operation is performed on SG. Clearly SG is an infinite non-associative, non-commutative ring.

Example 5.9: Let
G = $\{a + bi_F \mid a + bi_F \in C_n(15),\ i_F^2 = 14, *, (10, 2.4i_F)\}$ be the MOD complex modulo integer groupoid. Let S = $\langle R \cup I\rangle$ be the real neutrosophic ring. SG be the MOD complex modulo integer neutrosophic non-associative ring.

Let $x = (5I + 10)\,(5 + 4i_F)$ and $y = (8 + 12I)\,(10 + 10i_F) \in SG$

$$x \times y = (5I + 10)\,(5 + 4i_F) \times (8 + 12I) \times (10 + 10i_F)$$

$$= (5I + 10) \times (8 + 12I)\,[(5 + 4i_F) * (10 + 10i_F)]$$

$$= (40I + 80 + 60I + 120I)\,(50 + 40i_F + 24i_F + 24 \times 14)$$

$$= (80 + 220I)\,(11 + 4i_F) \in SG.$$

This is the way product operation is performed on SG. SG is a non-associative ring.

Example 5.10: Let
G = $\{a + bg \mid a + bg \in R_n(10)g,\ g^2 = 0, (4, 7), *\}$ be the MOD dual number groupoid. R = $\langle Q \cup I\rangle$ be the neutrosophic ring. RG is the MOD dual number neutrosophic non-associative ring.

Let x = (5 + 3I) (2 + 0.2g) and y = (7 + 10I) (0.8 + 5g) $\in$ RG.

$$\begin{aligned} x \times y &= (5 + 3I)(2 + 0.2g) \times (7 + 10I)(0.8 + 5g) \\ &= (5 + 3I)(7 + 10I)[(2 + 0.2t) * (0.8 + 5g)] \\ &= (35 + 21I + 50I + 30I)[4(2 + 0.2g) + 7(0.8 + 5g)] \\ &= (35 + 101I)(3.6 + 5.8g) \in RG. \end{aligned}$$

This is the way operation of $\times$ is performed on RG.

It is easily verified RG is non-commutative and non-associative.

Example 5.11: Let G = {a + bg | a + bg $\in$ $R_n(15)g$; $g^2 = 0$, (5g, 10g), *} be the MOD dual number groupoid S = C be the field of complex numbers SG be the groupoid ring. SG is the MOD dual number non-associative ring of infinite order.

If x = (10 + 3i) (4g) + (2 + 7i) 2g and y = (7 + 5i) 6g $\in$ SG

$$\begin{aligned} \text{Now } x \times y &= ((10 + 3i) 4g + (2 + 7i)2g) \times (7 + 5i) 6g \\ &= (10 + i)(7 + 5i)(4g * 6g) + ((2 + 7i)(7 + 5i) \\ &\quad (2g * 6g) = 0. \end{aligned}$$

Thus x is a zero divisor of SG.

Infact SG has infinite number of zero divisors.

Let x = (10 + 3i) (4 + 8g) $\in$ SG;

$$x \times x = (10 + 3i)(4 + 8g) \times (10+3i)(4 + 8g)$$

$$= \quad (10 + 3i)^2 [(4 + 8g) * (4 + 8g)]$$
$$= \quad (100 + 60i - 9) [5(4 + 8g) + 10(4 + 8g)$$
$$= \quad (91 + 60i) (0)$$
$$= \quad 0$$

is a nilpotent element of order two.

Let

$$x = (10 + 4i) (0.2 + 6.1g) \in SG,$$

$$x \times x \quad = \quad (10 + 4i) (0.2 + 6.1g) * (10 + 4i) (0.2 + 6.1g)$$

$$= \quad (10 + 4i)^2 (0.2 + 6.1g * 0.2 + 6.1g)$$

$$= \quad (100 + 80i - 16) (g + 0 + 2g + 61g^2)$$

$$= \quad (84 + 80i) (3g)$$

$$\neq \quad 0.$$

Thus all elements in SG are not nilpotent. SG has only a finite number of nilpotent elements. In view of this we have the following theorem.

THEOREM 5.1: *Let $G = \{a + bg \mid a + bg \in R_n(m)g, g^2 = 0, (tg, sg); t, s \in \langle Z_m \cup g\rangle, *\}$ be the MOD dual number groupoid. S any commutative ring with unit SG the MOD non-associative dual number groupoid ring.*

i) *Every $x=ag \in S$, $a \in S$ is such $x^2 = 0$.*
ii) *Every $x \in \{pg \mid p \in S\} = A$ is such that $x \times y = 0$ for every $y \in A$.*
iii) *If $x = p(a + bg)$; $a, b \in Z_m$; $p \in S$ in SG is such that $x^2 = 0$ if and only if $t + s \equiv 0 \ (mod\ m)$.*

The proof is considered as a matter of routine and is left as an exercise to the reader.

Example 5.12: Let G = {a + bh | h^2 = h, a + bh ∈ $R_n(12)$h, (10, 3), *} be the MOD special dual like number groupoid. S = Z the ring of integers. SG be the groupoid ring SG is the non-associative MOD special dual like number ring of infinite order.

Let x = 18 (5 + 3h) + 2(3 + 7h) ∈ ZG.

$$x \times x = (18\,(5 + 3h) + 2(3 + 7h)) \times (18\,(5 + 3h) + 2(3 + 7h))$$

$$= 18[(5 + 3h) * (5 + 3h)] + 2 \times 18\,(3 + 7h) * (5 + 3h) + 18 \times 2\,(5 + 3h) * (3 + 7h) + 2^2\,((3 + 7h) * (3 + 7h))$$

$$= 18^2\,(50 + 30h + 15 + 9h) + 36\,(30 + 70h + 15 + 9\,h) + 36\,(50 + 30h + 9 + 21h) + 4\,(30 + 70h + 9 + 21h)$$

$$= 18^2\,(5 + 3h) + 36\,(9 + 7h) + 36(11 + 3h) + 4(3 + 7h)$$

This is the way the product operation is performed on SG.

Let x = 20 (10 + 3h) ∈ SG

$$x \times x = 20\,(10 + 3h) * 20(10 + 3h)$$

$$= 400\,((10 + 3h) * (10 + 3h))$$

$$= 400\,(100 + 30h + 30 + 9h)$$

$$= 400\,(10 + 3h).$$

If x is to be an idempotent then all we need to do is take the coefficient of $10 + 3h$ to be 1.

Thus in view of this if

$x = 6 + 7h \in$ SG then

$$x \times x = 6 + 7h * 6 + 7h$$

$$= 60 + 70h + 18 + 21h$$

$$= 78 + 91h$$

$$= 6 + 7h = x.$$

Thus x is an idempotent of SG.

All $x \in \{a + bh \mid a, b \in Z_{12}\}$ are idempotents of SG.

Example 5.13: Let $G = \{a + bh \mid h^2 = h, a + bh \in R_n(19)\,h, (10 + 8h, 9 + 11h), *\}$ be the MOD special dual like number groupoid. $S = Z_7$ be the ring of modulo integers. SG be the groupoid ring of G over S.

SG has MOD pseudo zero divisors.

In view of all these we have the following theorem.

THEOREM 5.2: *Let*
$G = \{a + bh \mid h^2 = h, (t, s), * a + bh \in R_n(m)h\}$ be the MOD special dual like number groupoid. R any ring RG be the groupoid ring that is MOD non-associative ring, RG has MOD pseudo nilpotents of order two if and only if $t + s \equiv 0 \pmod{m}$.

Proof as in case of earlier theorems.

We now proceed onto describe MOD special quasi like dual number non-associative rings by some examples.

Example 5.14: Let
G = {a + bk | a + bk ∈ $R_n(12)$ k; k^2 = 11k, (10, 2), *} be the MOD special quasi dual number grouopoid.

S = Z be the ring of integers. ZG be the MOD special quasi dual like number non-associative ring. Let x = 10 + 7k ∈ SG

$$x \times x = (10 + 7k) * (10 + 7k)$$

$$= 100 + 70k + 20 + 14\,k$$

$$= \{0\}.$$

Thus x ∈ SG is nilpotent of order two.

Finding idempotents if any in SG is left as an exercise to the reader.

Example 5.15: Let G = {a + bk | a + bk ∈ $R_n(18)$ k, k^2 = 17k, (10 + 9k, 9 + 9k), *} be the MOD special quasi dual number groupoid. Z_{12} = S be the ring of modulo integers. SG be the MOD special quasi dual non-associative ring. SG is of infinite order. SG has both idempotents as well as nilpotents of order two.

$$\text{Let } x = 6\,(7 + 8.3k) \in SG$$

$$x \times x = 0,$$

that is x is nilpotent of order two. Infact SG has infinite number of nilpotent elements of order two.

Let y = 6 (0.333 + 6.775 k) $\in$ SG; y $\times$ y = 0 is again a nilpotent of order two.

Let a = 4(5 + 8k) $\in$ SG.

$$
\begin{aligned}
a \times a &= 4(5 + 8k) \times 4(5 + 8k) \\
&= 16(5 + 8k) * (5 + 8k) \\
&= 16(10 + 9k \times 5 + 8k + 9 + 9k \times 5 + 8k) \\
&= 16(50 + 45\,k + 80\,k + 72 \times 17\,k + 45 + \\
&\quad 45k + 72k + 72 \times 17k) \\
&= 16\,(5 + 8k) = a.
\end{aligned}
$$

Thus a is an idempotent of SG.

Further b = 9(5 + 8k) $\in$ SG is also an idempotent of SG.Infact we can find several such idempotents but however they are only finite in number.

Example 5.16: Let G = {a + bk | a + bk $\in$ $R_n(20)$k, k^2 = 19k, (10 + 10k, 11 + 10k), *} be the MOD special quasi dual number grouopid. Z = S be the ring of integers. SG be the groupoid ring SG is a MOD special quasi dual number non-associative ring.

SG has only finite number of idempotents but it is a difficult job to find zero divisors.

a = (8k + 10) $\in$ SG is an idempotent.

In view of all these the following theorems.

THEOREM 5.3: *Let*
$G = \{a + bk \mid a + bk \in R_n(m)k, k^2 = (m - 1)k, *, (t, u)\}$ be the MOD special quasi dual numbers $S = Z_n$ (MOD composite number having MOD pseudo zero divisors and zero divisors) be the ring. SG the groupoid ring. That is the MOD non-associative special quasi dual number ring.

i) SG has infinite number of zero divisors.
ii) SG has idempotents if and only if $t + u = 1 (mod\ m)$.
iii) SG has MOD pseudo nilpotents if and only if $t + u \equiv 0$ (mod m).

Proof: If a, b $\in Z_m$ is such that a . b = 0.

Then SG has infinite number of zero divisors as if
x = a (p + qk) and r = b (r + sk)
where p + qk, r + sk $\in R_n(m)k$; since cardinality of $R_n(m)k$ is infinite SG has infinite number of zero divisors.

Hence (i) is true.

Proof of (ii): All elements of the form x = a + bk, a, b $\in Z_m$ are such that $x \times x = x$ in $\langle Z_m \cup k \rangle$ if and only if t, u $\in$ $t + u \equiv 1$ (mod m).

Proof of (iii): x = a + bk; a, b $\in Z_m$ are such that $x^2 = 0$ if and only if t, m $\in \langle Z_m \cup k \rangle$ and $t + u \equiv 0$ (mod m).

Example 5.17: Let $G = \{a + bk \mid a + bk \in R_n(20)k, *, k^2 = 19k, (5, 10)\}$ be the MOD special quasi dual like number groupoid.

R = Z the ring of integers RG the groupoid ring is the MOD special dual quasi number non-associative ring.

Finding zero divisors and idempotents happens to be a difficult task.

Clearly $x = 12\,(4 + 8k) \in RG$

$$x \times x = 12\,(4 + 8k) \times 12\,(4 + 8k)$$
$$= 144\,[(4 + 8k) * (4 + 8k)]$$
$$= 144\,(20 + 40k + 40 + 80k)$$
$$= 0.$$

Thus there are zero divisors.

Let $y = 5\,k \in SG$

$$y \times y = 5k \times 5k$$
$$= 5 \times 5k + 10 \times 5\,k$$
$$= 5\,k + 10k$$
$$= 15k.$$

Study in this direction is an open problem.

Finding ideals, subrings happens to be a matter of routine.

Let us now introduce the notion of subset MOD non-associative rings with examples.

Example 5.18: Let M = {collection of all subsets from the MOD real plane $R_n(90)$; *, (45, 46)} be the subset MOD real groupoid. R be the real plane. RM be the MOD non-associative subset ring.

Let x = {(20, 14)} $\in$ RM

$$x \times x = \{(20, 14)\} \times \{(20, 14)\}$$

$$= \{(20 * 20, 14 * 14)\}$$

$$= \{(20 \times 45 + 20 \times 46, 14 \times 46 + 14 \times 45)\}$$

$$= \{20, 14\} = x.$$

Thus x is an idempotent of RM. Infact RM has finite number of idempotents.

Example 5.19: Let M = {collection of all subsets from the $R_n(15)$, *, (12, 3)} be the MOD subset real groupoid. R = {a + bI | a, b $\in$ R} be the real neutrosophic ring. RM be the subset MOD real neutrosophic non-associative ring.

$$\text{Let } x = \{(4,8)\} \in RM;$$

$$x * x = \{(4,8)\} * \{(4,8)\}$$

$$= \{(4 * 4, 8 * 8)\}$$

$$= \{(48+12, 96 + 24)\}$$

$$= \{(0, 0)\} \text{ is a nilpotent element of order.}$$

RM has infinite number of nilpotents.

Let $x = \{a\,(3,2)\} \in$ RM ($a \in \langle R \cup I \rangle$);

$$\begin{aligned} x \times x &= \{(a\,(3,2)\} \times \{a\,(3,2)\} \\ &= \{a^2(3,2) * (3,2)\} \\ &= \{a^2\,(3 * 3,\ 2 * 2)\} \\ &= \{a^2\,(0,\ 0)\}; \end{aligned}$$

is not a MOD pseudo nilpotent as

$$a^2\,(0, 0) \neq (0, 0) = (g_0, g_0);$$

Example 5.20: Let G = {Collection of all subsets from the real MOD plane $R_n(24)$, * (3, 22)} be the MOD subset real groupoid. S = $C(Z_{18})$ be the finite complex modulo integer ring SG be the subset non-associative real MOD ring.

SG has finite number of idempotents as well as infinite number of zero divisors.

Let x = {(10, 3)} be in SG.

$$\begin{aligned} x \times x &= \{(10,\ 3)\} * \{(10,\ 3)\} \\ &= \{(10 * 10,\ 3 * 3)\} \\ &= \{(30 + 220,\ \ 9 + 66)\} \\ &= \{(10,\ 3)\} \\ &= x. \end{aligned}$$

Thus x is an idempotent of SG.

Let y = {9 (3.21, 5.315)} and z = {4 (9.31, 8.334)} $\in$ SG,

$$y \times z = \{(0, 0)\}.$$

In fact SG has infinite number of zero divisors.

Example 5.21: Let G = {(a, b) | (a, b) $\in$ $R_n(47)$; $*$, (3.11, 12.715)} be the MOD groupoid.

Let M = {Collection of subsets of G} be the MOD subset real groupoid R = {$\langle Z_{12} \cup I \rangle$} be the neutrosophic ring RG be the MOD subset non-associative ring.

In fact RG has infinite number of zero divisors.

For A = {(6 + 6I) (x, y)} $\in$ RG where (x, y) $\in$ $R_n(47)$ and B = {(4 + 4I) (a, b)} where (a, b) $\in$ $R_n(47)$. It is easily verified.

A $\times$ B = {(0, 0)}. Such zero divisors in RG are infinite in number.

Hence the claim.

Example 5.22: Let G = {Collection of all subsets from the MOD complex plane $C_n(26)$, $*$, $(10i_F + 16, 16i_F + 11)$, $*$} be the subset MOD complex modulo integer groupoid. R = $\langle Z_{12} \cup I \rangle$ be the neutrosophic ring. RG be the MOD non-associative finite complex modulo integer ring. RG has finite number of idempotents but infinite number of zero divisors.

Example 5.23: Let G = {Collection of all subsets from the MOD complex plane $C_n(20)$, (10, 10), $*$} be the MOD subset complex modulo integer groupoid. S = {$\langle Z_{12} \cup I \rangle$} be the neutrosophic integer ring SG be the MOD subset non-associative finite complex neutrosophic integer ring.

SG is commutative and has finite number of zero divisors.

Let $x = \{(0.8, 0.4)\} \in SG$

$$\begin{aligned} x * x &= \{(0.8, 0.4)\} * \{(0.8, 0.4)\} \\ &= \{(0.8 * 0.8, 0.4 * 0.4)\} \\ &= \{(8 + 8, 4 + 4)\} \\ &= \{(16, 8)\} \neq \{0\}. \end{aligned}$$

(Here $(0.8, 0.4) = 0.8 + 0.4i_F$)

Let $x = \{(9 + 12i_F)\} \in SG$

$$\begin{aligned} x * x &= \{(9 + 12i_F)\} * \{(9 + 12i_F)\} \\ &= \{(9 + 12i_F) * (9 + 12i_F)\} \\ &= \{(90 + 120i_F + 90 + 120i_F)\} \\ &= \{0\}. \end{aligned}$$

Thus SG has zero divisors but the number of zero divisors is only finite in number.

Example 5.24: Let G = {Collection of all subsets from the complex MOD plane $C_n(17)$, $i_F^2 = 16$, (10, 7)} be the subset groupoid of complex finite modulo integers.

Let $S = \{\langle Z_{12} \cup g\rangle \mid g^2 = 0\}$ be the dual number ring of modulo integers. SG be the subset MOD complex finite modulo integer non-associative ring.

Let $x = \{4\,(0.3 + 0.8i_F)\}$ and $y = \{3\,(0.7 + 6.9i_F)\} \in SG$.

Clearly $x \times y = \{(0)\}$ is a zero divisor.

All subsets x = {4 (a + bi_F)} and y = {3 (a + bi_F) + 6 (c + di_F) + 9 (e + fi_F), 3 (a′ + b′i_F), 6 (t + ui_F)} ∈ SG are such that

x × y = {0}.

In fact SG has infinite number of zero divisors.

However SG has only finite number of idempotents.

In view of this we have the following theorem.

THEOREM 5.4: *Let G = {Collection of all subsets from the finite complex modulo integer MOD plane $C_n(m)$, $i_F^2 = m - 1$, (t, s); t, s ∈ $C(Z_m)$; *} be the MOD subset complex modulo integer groupoid. S = {Z_s} (or $\langle Z_s \cup I\rangle$ or $\langle Z_s \cup g\rangle$; $g^2 = 0$ or $\langle Z_s \cup h\rangle$; $h^2 = h$; or $\langle Z_s \cup k\rangle$; $k^2 = (s - 1)\, k$ or $C(Z_s)$; s a composite number and Z_s has zero divisors) be the ring. SG be the MOD subset groupoid non-associative ring.*

i. SG has idempotents if and only if (t + s) ≡ 1 (mod m)
ii. SG has infinite number of zero divisor.

Proof is left as an exercise to the reader.

Example 5.25: Let G = {Collection of all subsets from the MOD neutrosophic plane, $R_n^I(12)$, *, (7 + 8I, 6 + 4I)} be the subset MOD neutrosophic groupoid.

S = Z_{16} be the ring of modulo integers.

SG be the MOD subset non-associative neutrosophic ring.

SG has idempotents which are only finite in number and SG has infinite number of zero divisors and nilpotent elements of order 2 and four.

Example 5.26: Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(13)$, ∗, (10 + 3I, 3 + 10I)} be the subset MOD neutrosophic groupoid. S = C be the complex field. SG be the MOD subset non-associative neutrosophic ring.

SG has only finite number of MOD pseudo zero divisors.

For x = {7 + 2I} ∈ SG

$$\begin{aligned} x \times x = &\ \{7 + 2I\} \times \{7 + 2I\} \\ = &\ \{7 + 2I * 7 + 2I\} \\ = &\ \{70 + 21I + 6I + 20I + 21 + 70I + 6I + 20I\} \\ = &\ \{0\} \text{ is a zero divisor in SG.} \end{aligned}$$

Example 5.27: Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(124)$, (12.443 + 0.3I, 0.8 + 9.224I), ∗} be the MOD subset neutrosophic groupoid.

S = {$\langle Z_{24} \cup g\rangle$; $g^2 = 0$} be the ring of modulo integer dual numbers. SG be the MOD subset neutrosophic non-associative ring.

SG has infinite number of zero divisors.

Example 5.28: Let G = {Collection of all subsets from the MOD dual number plane $R_n(12)g$; $g^2 = 0$, (10 + g, 2g + 11), ∗} be the MOD subset dual number groupoid.

S = $\langle Z_{18} \cup I\rangle$ be the neutrosophic modulo integer ring SG be the MOD subset dual number non-associative ring.

SG has infinite number of zero divisors.

Example 5.29: Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(42)$, (10 + 3I, 32 + 39I), ∗} be the subset MOD neutrosophic groupoid. S = R be the real plane RG be the MOD subset neutrosophic non-associative groupoid.

RG has only finite number of MOD pseudo nilpotents of order two.

Example 5.30: Let G = {Collection of all subsets from the dual number MOD plane $R_n(18)g$, $g^2 = 0$, (10. 9), $*$} be the MOD subset dual number groupoid. S = Z_{148} be the ring of modulo integers. SG be the MOD subset non-associative ring.

SG has finite number of idempotents but infinite number of zero divisors.

Example 5.31: Let G = {Collection of all subsets from the MOD dual number plane $R_n(15)g$, $g^2 = 0$, (5g, 3.115g), $*$} be the MOD subset groupoid.

S = Z_{12} be the ring of modulo integers. SG be the MOD subset non-associative ring. SG has infinite number of zero divisors and SG has a subring which is a zero square ring.

Example 5.32: Let G = {Collection of all subsets from the dual number MOD plane $R_n(19)g$, (10.31g, 0.0015g), $*$} be the MOD subset dual number groupoid. S = R the field of reals and RG be the subset non-associative groupoid ring. RG has proper subrings which are zero square rings.

Example 5.33: Let G = {Collection of all subsets from the MOD special dual like number plane $R_n(12)h$, $h^2 = h$, (10h + 10, 2 + 2h), $*$} be the MOD subset groupoid. S = {$\langle Z_{48} \cup I \rangle$} be the ring. SG be the MOD subset non-associative ring. SG has infinite number of zero divisors.

Example 5.34: Let G = {Collection of all subsets from the special dual like number plane $R_n(43)h$, $h^2 = h$, (10h, 3.3h), $*$} be the subset MOD groupoid. S = Z_{144} be the ring of modulo integer.

SG be the MOD subset non-associative groupoid. SG has infinite number of zero divisors.

THEOREM 5.5: *Let G= {Collection of subsets from $R_n(m)g$, $g^2 = 0$, (t, u), *} be the subset MOD groupoid S = Z_s (or $\langle Z_s \cup I\rangle$ or $C(Z_s)$ or $\langle Z_s \cup g\rangle$, $g^2 = 0$ or $\langle Z_s \cup h\rangle$; $h^2 = h$ or $\langle Z_s \cup k\rangle$, $k^2 = (s-1)k$) be the ring of integers s a composite number SG be the subset non-associative ring.*

i. SG has infinite number of zero divisors.

ii. SG has finite number of idempotents if and only if t, s ∈ $\langle Z_m \cup g\rangle$; $t + s \equiv 1 \pmod m$.

Proof is a matter of routine with appropriate changes hence left as an exercise to the reader.

Example 5.35: Let G = {Collection of all subsets from the special quasi dual MOD plane $R_n(20)k$, $k^2 = 19k$, (19, 1), ∗} be the subset MOD special quasi dual number groupoid. S = Z_{64} be the ring of modulo integer SG be the MOD subset non-associative ring.

SG has infinite number of zero divisors. Whether SG has units or idempotents happens to be a challenging problem.

Example 5.36: Let G = {Collection of all subsets from the MOD plane of special quasi dual numbers $R_n(17)k$, $k^2 = 16k$, (10 + 5k, 8 + 12k), ∗} be the MOD subset groupoid. S = $\langle R \cup I\rangle$ be the neutrosophic ring. SG be the MOD subset non-associative ring. SG has finite number of non-trivial idempotents.

It is important to note that there is no relation between zero divisors and MOD pseudo zero divisors.

For $x \times y = 0$; 0 is in the ring.

For $x \times y = (0, 0) = (g_0\ g_0)$; $g_0 \in Z_n$ of the groupoid is the MOD pseudo zero divisors.

Next we proceed on to give some problem for the reader.

Problems

1. What are the special features enjoyed by MOD non-associative rings?

2. Can these rings be constructed other than using MOD groupoids?

3. Can a MOD non-associative ring be without zero divisors?

4. Show there exists a class of MOD non-associative rings which has infinite number of MOD pseudo nilpotent elements.

5. What are the special feature enjoyed by the groupoid ring, RG which is a MOD non-associative ring where R = C; complex field and G = $\{(a, b) \mid (a, b) \in R_n(29), (10, 13), *\}$?

6. Let G = $\{(a, b) \mid (a, b) \in R_n(42), (10, 32), *\}$ be the MOD real plane groupoid. S = Z_{10} be the ring of modulo integers. SG be the groupoid ring which is a non-associative MOD ring.

   i. Prove SG is non-commutative.
   ii. Prove SG has MOD pseudo nilpotents of order two.
   iii. Prove SG has infinite number of zero divisors.
   iv. Find S-ideals if any in SG.
   v. Is SG a S-ring?
   vi. Can SG have S-zero divisors?
   vii. Can SG have S-idempotents?
   viii. Can SG have ideals of infinite order?
   ix. Can SG enjoy any other special property than those mentioned?

7. Let $G = \{(a, b) \mid (a, b) \in R_n(27), (15, 13), *\}$ be the MOD real groupoid. R = Z the ring of integers. ZG be the groupoid ring, which is a non-associative MOD ring.

   Study questions i to ix of problem 6 for this ZG.

8. Let $G = \{(a + bg) \mid a + bg \in R_n(12)g; g^2 = 0, (10 + g, 4 + 8g), *\}$ be the MOD dual number groupoid. S = Q be the ring of rationals. SG be the MOD non-associative ring.

   Study questions i to ix of problem 6 for this SG.

9. Let $G = \{a + bg \mid a + bg \in R_n(13)g, g^2 = 0, (10, 0.3g), *\}$ be the MOD dual number groupoid. $S = Z_{15}$ be the ring of modulo integers. SG be the MOD non-associative dual number ring.

   Study questions i to ix of problem 6 for this SG.

10. Let $G = \{a + bg \mid a + bg \in R_n(18)g, g^2 = 0, (16, 3), *\}$ be the MOD dual number groupoid. S = C the field of complex integers. SG be the MOD non-associative ring.

    Study questions i to ix of problem 6 for this SG.

11. Let $G = \{a + bg \mid a + bg \in R_n(17)g, g^2 = 0, (10 + 5g, 8 + 12g), *\}$ be the MOD dual number groupoid. Let $S = C(Z_{12})$ be the finite complex modulo integer ring SG be the MOD non-associative dual number ring.

    Study questions i to ix of problem 6 for this SG.

12. Let $G = \{a + bi_F \mid a + bi_F \in C_n(24), i_F^2 = 23, (10, 12), *\}$ be the MOD complex modulo integer groupoid. Let S = Q be the field of rationals. SG be the MOD non-associative complex modulo integer ring.

    Study questions of i to ix of problem 6 for this SG.

13. Let G = {a + bi_F| a + $bi_F \in$ $C_n(29)$, i_F^2 = 28, (10 + i_F, 19 + $28i_F$), $*$} be the MOD complex modulo integer groupoid. Let S = Z_{40} be the ring of modulo integers. SG be the MOD complex modulo integer non-associative ring.

    Study questions of i to ix of problem 6 for this SG.

14. Let G = {a + bi_F| a + $bi_F \in$ $C_n(12)$, i_F^2 = 11, (10 + $3i_F$, 3 + $9i_F$), $*$} be the MOD complex modulo integer groupoid. S = $C(Z_{15})$ be the complex modulo integer ring. SG be the MOD complex modulo integer non-associative ring.

    Study questions of i to ix of problem 6 for this SG.

15. Let G = {a + bI| a + bI$\in$ R_n^I (14), (10, 4), $*$} be the MOD neutrosophic groupoid. S = R be the field of reals. SG be the groupoid ring that is non-associative MOD neutrosophic ring.

    Study questions of i to ix of problem 6 for this SG.

16. Let G = {a + bI | a + bI $\in$ R_n^I(19), (5 + 8I, 14 + 11I), $*$} be the MOD neutrosophic groupoid. S = Z_{15} be the ring of modulo integers. SG be the MOD neutrosophic non-associative ring.

    Study questions of i to ix of problem 6 for this SG.

17. Let G = {a + bI | a + bI $\in$ R_n^I(10), (8 + 2I, 0.5I), $*$} be the MOD neutrosophic groupoid. S = Z_{45} be the ring of modulo integers. SG be the MOD non-associative neutrosophic ring.

    Study questions of i to ix of problem 6 for this SG.

18. Let G = {a + bI | a + bI $\in$ R_n^I(143), (10.331, 6.534I), $*$} be the MOD neutrosophic groupoid. S = C(25) be the complex

modulo integer ring. SG be the MOD non-associative neutrosophic ring.

Study questions of i to ix of problem 6 for this SG.

19. Let G = {a + bh|bh + a ∈ $R_n(12)h$, $h^2 = h$, (5, 10), ∗} be the MOD special dual like number groupoid. S = R be the field of reals. SG be the MOD non-associative special dual like number ring.

    Study questions of i to ix of problem 6 for this SG.

20. Let G = {a + bh | bh + a ∈ $R_n(24)h$, $h^2 = h$, (10, 0.33h), ∗} be the MOD special dual like number groupoid. S = $\langle Z_{40} \cup I\rangle$ be the neutrosophic ring.

    Study questions of i to ix of problem 6 for this SG.

21. Let G = {a + bh | bh + a ∈ $R_n(23)h$, $h^2 = h$, (10h + 3, 6.33h + 0.715), ∗} be the MOD special dual like number groupoid. R = $\langle Z_{45} \cup g\rangle$ where $g^2 = 0$ is a dual number. RG be the MOD special dual like number non-associative ring.

    Study questions of i to ix of problem 6 for this RG.

22. Let G = {a + bh |a + bh ∈ $R_n(20)h$, $h^2 = h$, (12h + 5, 10.3h + 7.4), ∗} be the MOD special dual like number groupoid. R = $\langle Z_{14} \cup I\rangle$ be the neutrosophic ring. RG be the MOD non-associative special dual number neutrosophic groupoid.

    Study questions of i to ix of problem 6 for this RG.

23. Let G = {a + bh | a + bh ∈ $R_n(25)h$, $h^2 = h$, (10, 15), ∗} be the MOD special quasi dual like number groupoid. S = $\langle Z_{12} \cup k\rangle$ $k^2 = 11k$ be the special quasi dual like number ring. SG be the MOD non-associative special dual like number ring.

    Study questions of i to ix of problem 6 for this SG.

24. Let $G = \{a + bk| a + bk \in R_n(43)k, k^2 = 42k, (25, 18), *\}$ be the MOD special quasi dual like number groupoid. R = C be the complex field. RG be the groupoid ring which is a MOD non-associative special quasi dual number ring.
Study questions of i to ix of problem 6 for this RG.

25. Let $G = \{a + bk | a + bk \in R_n(24)k, k^2 = 23k, (10 + 2k, 14 + 22k), *\}$ be the MOD special quasi dual number groupoid. $R = C(Z_{42})$ be the finite complex modulo integer ring. RG be the MOD non-associative special quasi dual like ring.

    Study questions of i to ix of problem 6 for this RG.

26. Let $G = \{a + bk | a + bk \in R_n(23)k, k^2 = 22k, (10 + 5k, 0.932 + 6.315k), *\}$ be the MOD special dual quasilike number groupoid. $R = \langle Z \cup I\rangle$ be the neutrosophic ring. RG be the MOD non-associative special quasi dual number ring.

    Study questions of i to ix of problem 6 for this RG.

27. Let $G = \{a + bk | a + bk \in R_n(10)k, k^2 = 9k, (5 + 6k, 6 + 4k), *\}$ be the MOD special dual like number groupoid. Let $R = \langle Q \cup g\rangle; g^2 = 0$ be the dual number rational ring. RG be the non-associative MOD special dual number ring.

    Study questions of i to ix of problem 6 for this RG.

28. Let $G = \{a + bk | a + bk \in R_n(15)k, k^2 = 14k, (10.35, 10.35k), *\}$ be the MOD special quasi dual number groupoid. Let $S = C(Z_9)$ be the finite complex modulo integer ring. SG be the MOD non-associative like ring.

    Study questions of i to ix of problem 6 for this RG.

29. Let $G = \{a + bk | a + bk \in R_n(10)k, k^2 = 9k, *\}$ be the MOD special quasi dual number groupoid. $R = \langle Z_{46} \cup I\rangle$ be the neutrosophic ring. RG be the MOD non-associative ring.

Study questions of i to ix of problem 6 for this RG.

30. Let G = {Collection of all subsets from the real MOD plane $R_n(15)$, (10, 5), ∗} be the MOD subset groupoid. S = R be the field of reals. RG be the MOD subset non-associative ring.

    i. Find any interesting feature enjoyed by RG.
    ii. Study questions i to ix of problem 6 for this RG.
    iii. Can RG has infinite number of MOD pseudo zero divisors?

31. Let G = {Collection of all subsets from the MOD real plane $R_n(29)$, (10, 20), ∗} be the MOD subset groupoid. S = Z_{12} be the ring of modulo integers. SG be the MOD subset non-associative ring.

    i. Study questions i to ix of problem 6 for this SG.
    ii. Prove SG has only finite number of idempotents.
    iii. Prove SG has infinite number of zero divisors.

32. Let G = {Collection of all subsets from the MOD neutrosophic $R_n^I(22)$, (11 + I, 2 + I), ∗} be the MOD subset neutrosophic groupoid. Let S = C the complex number field. SG be the groupoid ring which is the MOD subset non-associative ring.

    Study questions i to ix of problem 6 for this SG.

33. Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(48)$, (10.334I, 0.3848), ∗} be the subset MOD neutrosophic groupoid. Let S = $C(Z_{40})$ be the ring of complex modulo integers. SG be the MOD subset non-associative ring.

    Study questions i to ix of problem 6 for this SG.

34. Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(43)$, (10.3I, 40.3I), ∗} be the subset

MOD neutrosophic groupoid. S = Q be the ring of rational. SG be the subset MOD neutrosophic non-associative ring.

Study questions i to ix of problem 6 for this SG.

35. Let G = {Collection of all subsets from the MOD neutrosophic plane $R_n^I(42)$, (10I, 32I), $*$} be the subset MOD neutrosophic groupoid. S = $\langle Q \cup g \rangle$; $g^2 = 0$ be the ring of rational dual numbers. SG be the MOD subset non-associative ring.

    Study questions i to ix of problem 6 for this SG.

36. Let G = {Collection of subsets from the MOD complex plane $C_n(20)$, $(15i_F, 5i_F)$, $*$} be the MOD subset complex groupoid. R = Z be the ring of integers. RG be the MOD subset non-associative ring.

    Study questions i to ix of problem 6 for this RG.

37. Let G = {Collection of all subsets from the MOD complex plane $C_n(15)$, $(10 + 5i_F, 5 + 10i_F)$, $*$} be the MOD complex subset groupoid. S = Z_{12} be the ring of integers. SG be the subset MOD non-associative ring.

    Study questions i to ix of problem 6 for this SG.

38. Let G = {Collection of all subsets from the MOD complex plane $C_n(17)$, $i_F^2 = 16$, $(10, 7i_F)$, $*$} be the MOD subset groupoid of complex modulo integers. S = Z_7 be the ring of modulo integers. SG be the MOD subset non-associative ring.

    Study questions i to ix of problem 6 for this SG.

39. Let G = {Collection of all subsets from the MOD dual number plane $R_n(10)g$, $g^2 = 0$, $(4, 6g)$, $*$} be the subset MOD

dual number groupoid. S = Z the ring of integers. SG be the subset MOD non-associative ring.

Study questions i to ix of problem 6 for this SG.

40. Let G = {Collection of all subsets from the MOD dual number plane $R_n(19)g$, $g^2 = 0$, (10g, 5.3g), $*$} be the MOD subset dual number groupoid. S = Z_{12} the ring of modulo integers. SG be the MOD subset non-associative dual number ring.

    Study questions i to ix of problem 6 for this SG.

41. Let G = {Collection of all subsets from the MOD subset dual number plane $R_n(20)g$, $g^2 = 0$, (10g, 11.458g), $*$} be the MOD subset dual number groupoid. R = $C(\langle Z_{20} \cup I\rangle)$ be the finite complex modulo neutrosophic integer ring. RG be the MOD subset groupoid ring which is a non-associative ring.

    i. Study questions i to ix of problem 6 for this RG.

    ii. Show there exists subrings in RG which are a zero square ring.

42. Obtain some special and unique features enjoyed by MOD subset non-associative neutrosophic ring where the ring is Z_n (n a prime).

43. Study problem 42 if Z_n is replaced by $\langle Z_n \cup g\rangle$; $g^2 = 0$.

44. Study problem 42 if Z_n is replaced by $\langle Z_n \cup h\rangle$; $h^2 = h$.

45. Let G = {Collection of all subsets from the MOD special dual like number plane $R_n(24)h$, $h^2 = h$, (10h, 15h), $*$} be the MOD subset groupoid of special dual like numbers. S = C be the complex field. SG be the MOD subset non-associative ring.

    Study questions i to ix of problem 6 for this SG.

46. Let G = {Collection of all subsets from the special dual like number plane $R_n(24)h$, $h^2 = h$, (10, 15); $*$} be the MOD subset special dual like number groupoid. S = Z_{24} be the ring of integers. SG be the MOD subset non-associative ring.

    i. Study questions i to ix of problem 6 for this SG.
    ii. Prove SG has infinite number of zero divisors.

47. Compare in problems 45 and 46. Which of the rings has more number of zero divisors?

48. Does there exists a MOD subset groupoid ring which has infinite number of idempotents?

49. Does there exists any MOD subset non-associative ring in which every zero divisor is a S-zero divisor?

50. Let G = {Collection of all subsets from the MOD special quasi dual number plane $R_n(24)k$, $k^2 = 23k$, (10k, 14k + 1); $*$} be the MOD subset groupoid. S = $\langle Z_{12} \cup g \rangle$; $g^2 = 0$ be the ring SG be the non-associative subset MOD ring.

    i. Study questions i to ix of problem 6 for this SG.
    ii. Prove SG has infinite number of zero divisor.
    iii. Can SG have S-zero divisors?

FURTHER READING

1. Albert, A.A., Non-associative algebra I, II, Ann. Math. (2), 43, 685-707, (1942).

2. Birkhoff, G. and Bartee, T.C. Modern Applied Algebra, Mc-Graw Hill, New York, (1970).

3. Bruck, R. H., A survey of binary systems, Springer-Verlag, (1958).

4. Bruck, R.H, Some theorems on Moufang loops, Math. Z., 73, 59-78 (1960).

5. Castillo J., The Smarandache Semigroup, International Conference on Combinatorial Methods in Mathematics, II Meeting of the project 'Algebra, Geometria e Combinatoria', Faculdade de Ciencias da Universidade do Porto, Portugal, 9-11 July 1998.

6. Chang Quan, Zhang, Inner commutative rings, Sictiuan Dascue Xuebao (Special issue), 26, 95-97 (1989).

7. Chein, Orin and Goodaire, Edgar G., Loops whose loop rings in characteristic 2 are alternative, Comm. Algebra, 18, 659-668 (1990).

8. Chein, Orin, and Goodaire, Edgar G., Moufang loops with unique identity commutator (associator, square), J. Algebra, 130, 369-384 (1990).

9. Chein, Orin, and Pflugfelder, H.O., The smallest Moufang loop, Arch. Math., 22, 573-576 (1971).

10. Chein.O, Pflugfelder.H.O and Smith.J.D.H, (eds), Quasigroups and loops: Theory and applications, Sigma Series in Pure Maths, Vol. 8, Heldermann Verlag, (1990).

11. Fenyves.F, Extra loops II: On loops with identities of Bol-Moufang types, Publ. Math. Debrecen, Vol.16, 187-192 (1969).

12. Goodaire, E.G., and Parmenter, M.M., Semisimplicity of alternative loop rings, Acta. Math. Hung, 50. 241-247 (1987).

13. Hall, Marshall, Theory of Groups. The Macmillan Company, New York, (1961).

14. Herstein., I.N., Topics in Algebra, Wiley Eastern Limited, (1975).

15. Ivan, Nivan and Zukerman. H. S., Introduction to number theory, Wiley Eastern Limited, (1984).

16. Kepka.T, A construction of commutative Moufang loops and quasi modulus, Comment Math. Univ. Carolin., 27 (3), 499-518 (1986).

17. Lang, S., Algebra, Addison Wesley, (1967).

18. Michael.K.Kinyon and Phillips.J.D, Commutants of Bol loops of odd order, (2002). http://lanl.arxiv.org/pdf/math.GR/0207119

19. Raul, Padilla, Smarandache Algebraic Structures, Smarandache Notions Journal, 9, 36-38 (1998).

20. Singh, S.V., On a new class of loops and loop rings, Ph.D. thesis IIT (Madras), guided by Vasantha. W.B., (1994).

21. Smarandache Florentin, Multi structures and Multi spaces, (1969) www.gallup.unm.edu/~smarandache/transdis.txt

22. Smarandache, Florentin, (editor), Proceedings of the First International Conference on Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, (2001).

23. Smarandache, Florentin, A Unifying Field in Logics: Neutrosophic Logic, Preface by Charles Le, American Research Press, Rehoboth, 1999, 2000. Second edition of the Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, Gallup, (2001).

24. Smarandache, Florentin, Neutrosophic Logic—Generalization of the Intuitionistic Fuzzy Logic, Special Session on Intuitionistic Fuzzy Sets and Related Concepts, International EUSFLAT Conference, Zittau, Germany, 10-12 September 2003.

25. Smarandache, Florentin, Special Algebraic Structures, in Collected Papers, Abaddaba, Oradea, 3, 78-81 (2000).

26. Solarin, A.R.T., and Sharma B.L., On the identities of Bol-Moufang type, Kyungpook Math. J., 28, 51-62 (1988).

27. Tim Hsu, Class 2 Moufang loops small Frattini Moufang loops and code loops, (1996). http://lanl.arxiv.org/pdf/math.GR/9611214

28. Vasantha Kandasamy, W.B., Biloops, U. Sci. Phy. Sci., 14, 127-130 (2002).

29. Vasantha Kandasamy, W.B., Fuzzy subloops of some special loops, Proc. 26th Iranian Math. Conf., 33-37 (1995).

30. Vasantha Kandasamy, W.B., Groupoids and Smarandache groupoids, American Research Press, Rehoboth, (2002). http://www.gallup.unm.edu/~smarandache/Vasantha-Book2.pdf

31. Vasantha Kandasamy, W.B., Muti Fuzzy differentiation in Fuzzy Polynomial rings, J. of Inst. Math and Comp. Sci., 9, 171-173, (1996).

32. Vasantha Kandasamy, W.B., On a new class of Jordan loops and their loop rings, J. Bihar Math. Soc., 19, 71-75 (1999).

33. Vasantha Kandasamy, W.B., On decimal and Fractional Calculus, Ultra Sci. of Phy. Sci., 6, 283-285 (1994).

34. Vasantha Kandasamy, W.B., On Fuzzy Calculus over fuzzy Polynomial rings, Vikram Mathematical Journal, 22 (2002) 557-560.

35. Vasantha Kandasamy, W.B., On Muti differentiation of decimal and Fractional Calculus, The Mathematics Education, 38 (1994) 220-225.

36. Vasantha Kandasamy, W.B., On ordered groupoids and groupoid rings, J. Math. Comp. Sci., 9, 145-147 (1996).

37. Vasantha Kandasamy, W.B., On Partial Differentiation of Decimal and Fractional calculus, J. of Inst. Math and Comp. Sci., 13, 287-290 (2000).

38. Vasantha Kandasamy, W.B., On Quasi loops, Octogon, 6, 63-65 (1998).

39. Vasantha Kandasamy, W.B., On Smarandache Cosets, (2002). http://www.gallup.unm.edu/~smaranandache/pseudo ideals.pdf

40. Vasantha Kandasamy, W.B., Smarandache groupoids, (2002). http://www.gallup.unm.edu/~smarandache/Groupoids.pdf

41. Vasantha Kandasamy, W.B., Smarandache Loops, American Research Press, Rehoboth, NM, (2002). http://www.gallup.unm.edu/~smarandache/Vasantha-Book4.pdf

42. Vasantha Kandasamy, W.B., Smarandache loops, Smarandache Notions Journal, 13, 252-258 (2002). http://www.gallup.unm.edu/~smarandache/Loops.pdf

43. Vasantha Kandasamy, W.B., Smarandache Semigroups, American Research Press, Rehoboth, NM, (2002). http://www.gallup.unm.edu/~smarandache/Vasantha-Book1.pdf

44. Vasantha Kandasamy, W.B., Some Applications of the Decimal Derivatives, The Mathematics Education, 34, 128-129, (2000).

45. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD planes, EuropaNova, (2015).

46. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., MOD Functions, EuropaNova, (2015).

47. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Natural Neutrosophic numbers and MOD Neutrosophic numbers, EuropaNova, (2015).

48. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Algebraic structures on MOD planes, EuropaNova, (2015).

49. Vasantha Kandasamy, W.B., and Singh S.V., Loops and their applications to proper edge colouring of the graph K_{2n}, Algebra and its applications, edited by Tariq et al., Narosa Pub., 273-284 (2001).

50. Vasantha Kandasamy, W.B., and Smarandache, F., N-algebraic structures and S-N-algebraic structures, Hexis, Phoenix, Arizona, (2005).

51. Vasantha Kandasamy, W.B., and Smarandache, F., Fuzzy Interval Matrices, Neutrosophic Interval Matrices and their Applications, Hexis, Phoenix, (2006).

52. Vasantha Kandasamy, W.B., and Smarandache, F., Neutrosophic algebraic structures and neutrosophic N-algebraic structures, Hexis, Phoenix, Arizona, (2006).

53. Vasantha Kandasamy, W.B., and Smarandache, F., Smarandache Neutrosophic algebraic structures, Hexis, Phoenix, Arizona, (2006).

54. Vasantha Kandasamy, W.B., and Smarandache, F., Finite Neutrosophic Complex Numbers, Zip Publishing, Ohio, (2011).

55. Vasantha Kandasamy, W.B., and Smarandache, F., Dual Numbers, Zip Publishing, Ohio, (2012).

56. Vasantha Kandasamy, W.B., and Smarandache, F., Special dual like numbers and lattices, Zip Publishing, Ohio, (2012).

57. Vasantha Kandasamy, W.B., and Smarandache, F., Special quasi dual numbers and Groupoids, Zip Publishing, Ohio, (2012).

58. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures using [0, n), Educational Publisher Inc, Ohio, (2013).

59. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures using Subsets, Educational Publisher Inc, Ohio, (2013).

60. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures on the fuzzy interval [0, 1), Educational Publisher Inc, Ohio, (2014).

61. Vasantha Kandasamy, W.B., and Smarandache, F., Algebraic Structures on Real and Neturosophic square, Educational Publisher Inc, Ohio, (2014).

62. Vasantha Kandasamy, W.B., and Smarandache, F.,, Algebraic Structures on the fuzzy unit square and neutrosophic unit square, Educational Publisher Inc, Ohio, (2014).

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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The notion of non-associative structures are introduced on the MOD planes. For a MOD plane $R_n(m)$, m is fixed, one can build infinitely many MOD groupoids all of which are of infinite order.

Several open problems are proposed in this book regarding the new concept.

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