Eric Stucky

28 May 2016

Assistance from TheGreatDuck

Implied Integration. In this document, we define a fully formalized notion of the "Implied Integral" which works for a broad variety of symbolic expressions. We show that it is equal to the ordinary integral up to a generalized constant term, and we show that this generalized constant is in some sense optimal.

A Small Observation

The constant term in the indefinite integral is not optional. One may suspect that it is a technicality which can be avoided, producing a genuine transformation between function spaces, by simply setting $C = 0$ when it arises. This is not in fact the case. Suppose we wish to find a canonical antiderivative for $\sec^2(x)\tan(x)$ with this approach. There are two natural ways to do this (and no reason to prefer one over the other):

$$\begin{aligned} \int \sec^2(x)\tan(x)\,dx &= \int \tan(x)\,d\tan(x) &&= \tan^2(x) \\ \int \sec^2(x)\tan(x)\,dx &= \int \sec(x)\,d\sec(x) &&= \sec^2(x) \end{aligned}$$

Of course $\tan^2(x) \neq \sec^2(x)$, so this attempt has failed. Note that this is easily resolved once we put the constants back in: $\tan^2(x) + C = \sec^2(x) + (C - 1)$. The only way to resolve this conflict, without providing an ad-hoc justification to prefer one trigonometric substitution over the other, or putting the constants back in entirely, is to say that the result of the integral is dependent not only on the function, not only on the symbolic expression we use to represent that function, but even is dependent on the method we use to resolve the integral. Of these three options, we believe that replacing the constants is the least repulsive.

Extending this observation: no matter how we define the implied integral, we want the following expressions to be true:

$$\begin{aligned} \int \lfloor x \rfloor \sec^2(x)\tan(x)\,dx &= \lfloor x \rfloor \int \tan(x)\,d\tan(x) &&= \lfloor x \rfloor \tan^2(x) + C \\ \int \lfloor x \rfloor \sec^2(x)\tan(x)\,dx &= \lfloor x \rfloor \int \sec(x)\,d\sec(x) &&= \lfloor x \rfloor \sec^2(x) + D \end{aligned}$$

Note that this time, we have a bigger issue: it does not suffice to take $D = C - 1$; instead we must take $D = C - \lfloor x \rfloor$. Therefore, even permitting arbitrary constants is not sufficient; we must be prepared for C or D to be a *piecewise constant* function. (You may wonder why we do not declare these integrals to be $\lfloor x \rfloor(\mathrm{trig}^2(x) + C)$ instead. To see why this approach is untenable, we suggest the reader try to integrate $\sec^2(x)\tan(x)+1$.

Formal Definitions in a Weak Setting

To motivate the full definition, we begin by considering a class of functions I will call $\mathcal{E}^+_{\text{weak}}$: these are elementary functions and their finite composites, together with the floor function. However, in $\mathcal{E}^+_{\text{weak}}$ the floor function may *not* be composed with other functions on the right. In simpler terms, this means that we may use $\lfloor x\rfloor$, but we may not do anything at all "inside" the floor function, so we cannot have $\lfloor x^2\rfloor$ or $\lfloor \sin(x)\rfloor$, or even $\lfloor x+\frac{1}{2}\rfloor$. However, we may may compose on the *left*, so the functions given by $\sin(x-\lfloor x\rfloor)$ or $\frac{\log(x)}{\lfloor x\rfloor}$ are allowed; these are contained in $\mathcal{E}^+_{\text{weak}}$. Clearly this class does not contain all the functions we wish to work with, but making this restriction for now will allow us to avoid some thorny technicalities.

The key observation is this: essentially, the functions $\mathcal{E}^+_{\text{weak}}$ are functions of *two* variables: x and $\lfloor x\rfloor$. Formally, we may say that $\mathcal{E}^+_{\text{weak}} = \{g : g(x) = \pi(f,x) \text{ for some } f\in\mathcal{E}_2\}$, where $\mathcal{E}_2$ is the set of elementary functions in two variables, say x and t and their finite composites (no floors allowed), and π is the "evaluation map":

$$\begin{aligned}\pi : \mathcal{E}_2\times\mathbb{R} &\to \mathbb{R}\\ (f,x) &\mapsto f(x,\lfloor x\rfloor)\end{aligned}$$

which evaluates the two-variable function f at $t=\lfloor x\rfloor$. We follow this definition up immediately with a caveat: although we wrote things this way to avoid a circular definition, we will usually write $g=\pi(f)$, thinking of π as a "projection map" $L^1(x,t)\to L^1(x)$ by currying π. [If you are not familiar with $L^1(x)$, it is just a very large class of functions, in the variable x, which contains anything reasonable. $L^1(x,t)$ is similar, but for two-variable functions in x and t.]

Also, we use the notation π to emphasize that this is something like a projection. It is, in particular, a linear operator. We will also be interested in right-inverses of this operator, which, by analogy to bundle constructions, we will call a **jump extension**. To be concrete about it, given a one-variable function g, we say that any f such that $g=\pi(f)$ is a jump extension of g, and we say that a map $\sigma:\mathcal{E}^+_{\text{weak}}\to\mathcal{E}_2$ is a **jump section** if it sends each g to one of its jump extensions. (Note that this implies $\pi\circ\sigma$ is the identity on $\mathcal{E}^+_{\text{weak}}$.)

There is a distinguished jump section which we call the **trivial** jump section σ_0, which has no t-dependence at all. In other words, σ_0 sends every g to the f such that $f(x,t)=g(x)$. However, we usually will be interested in considering jump sections on the "other extreme", which intuitively means that we replace all instances of $\lfloor x\rfloor$ with t in the symbolic representation of the function. But the whole reason for this elaborate edifice is that there is *no way of defining such a thing* by completely function-theoretic methods; if we could just "substitute $\lfloor x\rfloor = t$" then there wouldn't be any need for all this rigamorole.

We now define I_0 to be an "integration" operator, which takes in a one-variable function and puts out a two-variable function (in x and t):

$$\begin{aligned} I_0 : \mathcal{E}_2 &\to L^1(x,t) \\ f &\mapsto \int_0^x f(\xi,t)\,d\xi. \end{aligned}$$

This idea defines a whole family of linear operators I_a where the lower limit of 0 is replaced with a, but it does not really matter for our purposes which a is used, so we will use 0. (The skeptical reader is encouraged to trace the following proofs to verify this independence.)

The Weak Implied Integral

We finally are able to define the implied integral in this weak setting. Through this section, g is a function in $\mathcal{E}^+_{\text{weak}}$. To denote "the implied integral of g" we use the notation

$$\int g\,dJ \qquad\qquad \text{or} \qquad\qquad \int g(x)\,dJ$$

and we provide the following definition: For any jump section σ, the **implied integral associated to** σ is $\pi \circ I_0 \circ \sigma : \mathcal{E}^+_{\text{weak}} \to L^1(\mathbb{R})$, or in other words

$$\int g\,dJ = \int_0^x f\left(\xi, \lfloor x \rfloor\right)\,d\xi,$$

where $f = \sigma(g)$ is the relevant jump extension of g. Let us try to understand why this definition is correct.

If we believe that the point of a jump section is to try to replace $\lfloor x \rfloor$'s with t's, then $\int g\,dJ$ is the operation that first does that, then integrates with respect to the original variable (treating all instances of t as a constant), and finally performs the substitution $t = \lfloor x \rfloor$. So it really does, in this sense, capture the notion of "treating $\lfloor x \rfloor$ as a constant".

However, you may object that the notation is misleading. The definition appears to be sensitive to the exact way in which we chose to "replace $\lfloor x \rfloor$ with t"; in other words, there is some σ-dependence; but this is not reflected in the notation for the implied integral. The justification for this is the following proposition:

Proposition: If σ_1 and σ_2 are jump sections, the implied integral of g associated to σ_1 and the implied integral of g associated to σ_2 differ only by a piecewise constant function, whose discontinuities occur only at integers (but not necessarily at all integers).

Proof: Suppose that G_1 is the implied integral of g associated to σ_1, and G_2 similarly for σ_2. The proposition claims that G_1 and G_2 are piecewise constant on the open intervals between integers; more formally, for any real numbers x, y such that $N < x < y < N+1$, we want to show $G_1(x) - G_2(x) = G_1(y) - G_2(y)$; or equivalently that $G_1(y) - G_1(x) = G_2(y) - G_2(x)$. This is true, since

$$
\begin{aligned}
G_1(x) - G_1(y) &= \int_0^x (\sigma_1 g)\left(\xi, \lfloor x \rfloor\right)\, d\xi - \int_0^y (\sigma_1 g)\left(\xi, \lfloor y \rfloor\right)\, d\xi \\
&= \int_0^x (\sigma_1 g)\left(\xi, N\right)\, d\xi - \int_0^y (\sigma_1 g)\left(\xi, N\right)\, d\xi \\
&= \int_x^y (\sigma_1 g)\left(\xi, N\right)\, d\xi \\
\\
&= \int_x^y g(\xi)\, d\xi \\
\\
&= \int_x^y (\sigma_2 g)\left(\xi, N\right)\, d\xi \\
&= \int_0^x (\sigma_2 g)\left(\xi, N\right)\, d\xi - \int_0^y (\sigma_2 g)\left(\xi, N\right)\, d\xi \\
&= \int_0^x (\sigma_2 g)\left(\xi, \lfloor x \rfloor\right)\, d\xi - \int_0^y (\sigma_2 g)\left(\xi, \lfloor y \rfloor\right)\, d\xi \\
&= G_2(x) - G_2(y)
\end{aligned}
$$

The first three steps and the last four steps are simply applying the definitions and using standard manipulations of the integral. The fourth equality is the substance of the calculation: it holds by the definition of a jump section: for all $x \le \xi \le y$ we have that $N < \xi < N+1$, which means $N = \lfloor \xi \rfloor$. Therefore, we know that $(\sigma g)(\xi, N) = g(\xi)$ for any jump section σ; in particular $\sigma = \sigma_1$ (in the fourth equality) and $\sigma = \sigma_2$ (in the fifth).

$\square$

The result of this proposition is that although the implied integral may have a σ-dependence, the σ only makes a difference up to a piecewise constant part. But, from the initial observations in the previous section, the implied integral cannot be more well-defined than up to a “piecewise constant of integration”. So indeed, this result shows that the definition is as good as one can hope for, except that we might ask for a larger class of functions that $\mathcal{E}^+_{\text{weak}}$.

Speaking of which, where exactly did we use $\mathcal{E}^+_{\text{weak}}$? We made such a big fuss about it, but it did not seem to come up in the computation at all. In fact, formally it made no difference and we could have gone on without it. However, the rub is in the idea of substituting $t = \lfloor x \rfloor$: if g had been, say $\lfloor x \rfloor + \lfloor x^2 \rfloor$, then when passing to the jump section, we would have been forced to either

- leave $\lfloor x^2 \rfloor$ in the x-part, in which case it would have been integrated normally by I_0 (nope).
- or we would have had to encode it to the t-part, in which case it would have been indistinguishable from $\lfloor x \rfloor$ and π would not have been able to “decode it” properly.

This "problem" has a really easy solution: just use more dummy variables.

A Fully-Formed Implied Integral

Since this is meant to be analogous to the previous two sections, but in more variables, I will stop motivating things and move very quickly through definitions:

By $\mathcal{E}^+$, we mean the elementary functions, the floor function, and their finite composites, with the following technical restriction*: the set of points of continuity must be open. Clearly there are only countably many such functions, so the space of functions $\lfloor \mathcal{E}^+ \rfloor$, which consists of all functions of the form $\lfloor e \rfloor$ for some $e \in \mathcal{E}^+$, is countable. Denote these functions by $\alpha_1, \alpha_2, \alpha_3 \cdots$, and let $\mathcal{E}_\omega$ be the set of elementary functions and their finite composites, but in infinitely many variables $x, t_1, t_2, t_3, \cdots$, which have open sets of continuity (in the product topology)*.

[* I actually suspect this is not a restriction at all; which is to say, I think all $\mathcal{E}^+$ functions have this property. But this is a crucial property and I don't have a proof of my suspicion, so I'll play it safe.]

Define π to be the "evaluation map"

$$\begin{aligned} \pi : \mathcal{E}_\omega \times \mathbb{R} &\to \mathbb{R} \\ (f, x) &\mapsto f(x, \alpha_1(x), \alpha_2(x), \alpha_3(x), \cdots) \end{aligned}$$

But again, we will usually write $g = \pi(f)$, thinking of π as $L^1(x, t_1, t_2, t_3, \cdots) \to L^1(x)$.

Given a one-variable function g, we say that any $f \in \mathcal{E}_\omega$ such that $g = \pi(f)$ is a **jump extension** of g, and we say that a map $\sigma : \mathcal{E}^+ \to \mathcal{E}_\omega$ is a **jump section** if it sends each g to one of its jump extensions. The **trivial** jump section σ_0, is the one sending every g to the f such that $f(x, t_1, t_2, t_3, \cdots) = g(x)$.

Define I_0 to be the "integration operator"

$$\begin{aligned} I_0 : \mathcal{E}_\omega &\to L^1(x, t_1, t_2, \ldots) \\ f &\mapsto \int_0^x f(\xi, t)\, d\xi. \end{aligned}$$

Finally, for any jump section σ, the **implied integral associated to** σ is

$$\pi \circ I_0 \circ \sigma : \mathcal{E}^+ \to L^1(\mathbb{R}),$$

or in other words

$$\int g\, dJ = \int_0^x f\left(\xi, \alpha_1(x), \alpha_2(x), \alpha_3(x), \cdots\right)\, d\xi,$$

where $f = \sigma(g)$ is the relevant jump extension of g.

Again, we have a well-definedness proposition:

Proposition: [I need a couple more hours to work through these details.]