General Field Equations around a blackhole

Sadegh Mousavi

Amirkabir university of technology, Tehran, Iran.
Electronic address: sadegh.mousavy@gmail.com

(Dated: March 30, 2016)

After proposition of General Relativity theory by Albert Einstein, at 1914, some scientists tried to solve the field equations of this theory. The first one was Schwarzschild, which his solution leads to the discovery of blackholes. In this research a general compound field construction around a blackhole will be considered using tensor calculations. For this purpose at first some mathematical concepts will be introduced.

PACS numbers: 04.50.Kd, 03.50.De, 41.20.q

1. Introduction

Schwarzschild was the first one who solved the Einstein field equations, and then based on his solution, blackholes was discovered. Gravitational field around a blackhole is described very well by Einstein field equations in General Relativity. But the gravity is not the only field which is exists around a blackhole, and there are some other fields like electromagnetic field around it. In this research we will try to consider a compound field (eg field) around a blackhole, mathematically. For this purpose we need to use both symmetric and antisymmetric tensor calculations, and for better understanding, at first some mathematical concepts will be introduced.

2. Antisymmetric tensors

A particularly important class of tensors of type (0,s) is the class of totally antisymmetric tensors, i.e., covariant tensors which are antisymmetric in every pair of their arguments,

$$T(X_1,\ldots,X_\mu,\ldots,X_\upsilon,\ldots,X_s) = -T(X_1,\ldots,X_\upsilon,\ldots,X_\mu,\ldots,X_s) \tag{1}$$

for all pairs of indices μ and υ and for all X's. Tensor of this kind can be constructed out of a general tensor T of type (0,s) by applying to it the alternating operator A whose effect on it, is to give the linear combination defined by

$$AT(X_1,\ldots,X_s) = \frac{1}{s!}\sum_{\upsilon_1,\ldots,\upsilon_s} \operatorname{sgn}(\upsilon_1,\ldots,\upsilon_s)T(X_{\upsilon_1},\ldots,X_{\upsilon_s}) \tag{2}$$

Where the summation is extended over all s! permutations of the s integers (1,…,s) and $\operatorname{sgn}(\upsilon_1,\ldots,\upsilon_s) = \pm 1$, according as $(\upsilon_1,\ldots,\upsilon_s)$ is an even or an odd permutation of (1,…,s); and equation (2) is to be valid for every $(X_1,\ldots,X_s)$.

It is clear that if T is already totally antisymmetric, the effect of A on it is, simply, to reproduce T. Also, if $s\rangle n$ (the dimension of the vector space) the effect of A on $T(X_1,\ldots,X_s)$ is to reduce it to zero; in other words, there can be no totally antisymmetric tensor of type (0,s) for $s\rangle n$.

Totally antisymmetric tensor of type (0,s) are called s-forms. Since they must vanish when any two of their arguments coincide, it follows that the s-forms span a vector space of dimension $\frac{n!}{s!(n-s)!}$. This space is denoted by $\Lambda^s T_p^*$.

A basis for $\Lambda^s T_p^*$ can be obtained by applying the alternative operator A to the basis elements of the tensor product: $$A(e^{\upsilon_1}\otimes\ldots\otimes e^{\upsilon_s}) \quad (3)$$

The resulting basis elements are written as the exterior or the wedge product of the e^{υ}'s in the manner: $$e^{\upsilon_1}\wedge e^{\upsilon_2}\wedge\ldots\wedge e^{\upsilon_s} \qquad (\upsilon_1\rangle\upsilon_2\rangle\ldots\rangle\upsilon_s) \quad (4)$$

A general s-form can be written as: $$\Omega=\Omega_{\upsilon_1\ldots\upsilon_s}e^{\upsilon_1}\wedge e^{\upsilon_2}\wedge\ldots e^{\upsilon_s} \qquad (\upsilon_1\rangle\upsilon_2\rangle\ldots\rangle\upsilon_s) \quad (5)$$

where the summation is now extended only over strictly descending sequences.

Since interchanging a pair of indices is equivalent to interchanging the corresponding elements in the wedge product, it follows that interchanging the elements in a wedge product must be accompanied by a change of sign; thus $$e^{\upsilon}\wedge e^{\tau}=-e^{\tau}\wedge e^{\upsilon}. \quad (6)$$

In a local coordinate basis, the expression for an s-form is $$\Omega=\Omega_{\upsilon_1\ldots\upsilon_s}dx^{\upsilon_1}\wedge\ldots\wedge dx^{\upsilon_s} \quad (7)$$

Given any p-form Ω^1 and a q-form Ω^2, we can form their wedge product by the rule

$$\Omega^1\wedge\Omega^2=A(\Omega^1\otimes\Omega^2) \quad (8)$$

to obtain a (p+q) form. (it must accordingly vanish identically if $(p+q)\rangle n$).

For, by definition, $$\Omega^1\wedge\Omega^2=(\Omega^1_{\upsilon_1\ldots\upsilon_p}e^{\upsilon_1}\wedge\ldots\wedge e^{\upsilon_p})\wedge(\Omega^2_{\tau_1\ldots\tau_q}e^{\tau_1}\wedge\ldots\wedge e^{\tau_q}) \quad (9)$$

Where $(\upsilon_1,\ldots,\upsilon_p)$ and $(\tau_1,\ldots,\tau_q)$ are strictly descending sequences. Accordingly,

$$\Omega^1\wedge\Omega^2=(-1)^{pq}(\Omega^2_{\tau_1\ldots\tau_q}e^{\tau_1}\wedge\ldots\wedge e^{\tau_q})\wedge(\Omega^1_{\upsilon_1\ldots\upsilon_p}e^{\upsilon_1}\wedge\ldots\wedge e^{\upsilon_p})=(-1)^{pq}\Omega^2\wedge\Omega^1 \quad (10)$$

since each of the q basis elements $e^{\tau_1},\ldots,e^{\tau_q}$ must suffer p interchanges before $\Omega^1\wedge\Omega^2$ can be brought to the form required of $\Omega^2\wedge\Omega^1$.

The derivative of a wedge product can be calculated as following:

$$d(A\wedge B)=d(A_{\upsilon_1\ldots\upsilon_p}dx^{\upsilon_1}\wedge\ldots\wedge dx^{\upsilon_p}\wedge B_{\tau_1\ldots\tau_q}dx^{\tau_1}\wedge\ldots\wedge dx^{\tau_q})$$

$$=\frac{\partial A_{\upsilon_1\ldots\upsilon_p}}{\partial x^{i}}dx^{\mu}\wedge dx^{\upsilon_1}\wedge\ldots\wedge dx^{\upsilon_p}\wedge B_{\tau_1\ldots\tau_q}dx^{\tau_1}\wedge\ldots\wedge dx^{\tau_q}$$

$$+A_{\upsilon_1\ldots\upsilon_p}\frac{\partial B_{\tau_1\ldots\tau_q}}{\partial x^{\mu}}dx^{\mu}\wedge dx^{\upsilon_1}\wedge\ldots\wedge dx^{\upsilon_p}\wedge dx^{\tau_1}\wedge\ldots\wedge dx^{\tau_q}$$

3. Some mathematical concepts

We define the covariant differentiation, which is a type of differentiation which requires that the manifold be endowed with an additional structure. This additional structure is an affine connection, ∇, which assigns to each vector-field X on N a differential operator ∇_x , which maps an arbitrary vector-field ,Y, into a vector-field $\nabla_x Y$. Consistent with these requirements, we impose the conditions,

(a) $\nabla_x Y$ is linear in the argument X, i.e., $\nabla_{fX+gY}Z=f\nabla_x Z+g\nabla_Y Z \qquad (X,Y,Z\in T_0^1)$ (11)

When f and g are any two arbitrary functions defined on N.

(b) $\nabla_x Y$ is linear in the argument Y, i.e., $\nabla_x (Y + Z) = \nabla_x Y + \nabla_x Z \qquad (X, Y, Z \in T_0^1)$ (12)

(c) $\nabla_x f = Xf$ (13) Where f is any function on N, and finally,

(d) $\nabla_x (fY) = (\nabla_x f)Y + f\nabla_x Y$ (14)

It should be noted that, according to equation (13) in a local coordinate basis ($\partial_\tau, \nabla_{\partial_\tau}$), when acting on functions, coincides with partial differentiation with respect to x^τ. With the action of ∇_x on vector fields Y($\in T_0^1$) specified by the rules (a)-(d), we now define the covariant derivative, ∇Y of Y as a tensor field of type (1,1) which maps the contravariant vector-field X to $\nabla_x Y$,i.e., $\nabla Y(X) = \langle \nabla Y, X \rangle = \nabla_x Y$ (15) for every $X \in T_0^1$.

In this notation, we can rewrite equation (14) in the form $\nabla(fY) = df \otimes Y + f\nabla Y$ (16)

To clarify what the assignment of a connection precisely means, it will be useful to rewrite $\nabla_x Y$ relative to some chosen dual basis (e_μ) and (e^υ). Thus, making the use of the rules (a)-(d), we have: $\nabla_x Y = \nabla_x (Y^\upsilon e_\upsilon) = (XY^\upsilon)e_\upsilon + Y^\upsilon \nabla_x e_\upsilon$ (17)

Since $\nabla_x e_\upsilon$, for a particular e_υ, is a tensor field of type (1,0) we must have a representation, in the chosen basis , of the form $\nabla_x e_\upsilon = \omega_\upsilon^\sigma (X) e_\sigma$ (18)

Where ω_υ^σ (depending on σ and υ) are one-forms. Accordingly we may write

$$\nabla_x Y = (XY^\upsilon)e_\upsilon + Y^\upsilon \omega_\upsilon^\sigma (X) e_\sigma \ . \qquad (19)$$

Alternatively, we may also rewrite equation (17) in the form

$$\nabla_x Y = (XY^\upsilon)e_\upsilon + Y^\upsilon \nabla_{x^\tau e_\tau} e_\upsilon = (XY^\upsilon)e_\upsilon + Y^\upsilon X^\tau \nabla_{e_\tau} e_\upsilon \qquad (20)$$

or in conformity with the definition (18), $\nabla_x Y = (XY^\upsilon)e_\upsilon + Y^\upsilon X^\tau \omega_\upsilon^\sigma (e_\tau) e_\sigma$ (21)

Letting $\omega_\upsilon^\sigma (e_\tau) = \omega_{\upsilon\tau}^\sigma$ (22) be the coefficient of e^τ in the expansion of ω_υ^σ in the basis (e^τ), we conclude that a connection ∇ is specified by the n^2 one-forms ω_υ^σ, or equivalently, by the n^3 scalar fields $\omega_{\upsilon\tau}^\sigma$. Returning to equation (19) and rewriting it in the form

$\nabla_x Y = [XY^\upsilon + \omega_\sigma^\upsilon (X) Y^\sigma] e_\upsilon$ (23) We infer that $(\nabla_x Y)^\upsilon = XY^\upsilon + \omega_\sigma^\upsilon (X) Y^\sigma$. (24)

In a local coordinate basis (∂_τ, dx^σ), equation (24) gives:

$$(\nabla_{\partial_\tau} Y)^\upsilon = \partial_\tau Y^\upsilon + Y^\sigma \omega_{\sigma\tau}^\upsilon = Y^\upsilon_{,\tau} + Y^\sigma \omega_{\sigma\tau}^\upsilon \qquad (25)$$

In a local coordinate basis, it is customary to write $\Gamma_{\sigma\tau}^\upsilon$ in place of $\omega_{\sigma\tau}^\upsilon$ (26), so we obtain the standard formula $Y^\upsilon_{;\tau} = Y^\upsilon_{,\tau} + Y^\sigma \Gamma_{\sigma\tau}^\upsilon$ (27)

The definition of covariant derivatives of vector fields can be extended to tensor fields, in general, by requiring that the operation of ∇ satisfies the Leibnitz rule when acting on tensor products.

Thus, we require that $\nabla(F \otimes T) = \nabla F \otimes T + F \otimes \nabla T$ (28) where F and T are two arbitrary tensor-fields. An immediate consequence of this requirement is :

$$\begin{aligned}&\nabla_x [T(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s)] = (\nabla_x T)(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s) + T(\nabla_x \omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s) + \ldots \\ &+ T(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_{s-1}, \nabla_x Y_s)\end{aligned} \quad (29)$$

Thus if Ω is a one-form, then for every vector field Y, the foregoing equation gives

$\nabla_x(\Omega(Y)) = (\nabla_x \Omega)(Y) + \Omega(\nabla_x Y)$ (30) or in terms of a local basis (e_μ) and (e^υ), we have $\nabla_x(\Omega_\upsilon(Y^\upsilon)) = (\nabla_x \Omega)_\upsilon Y^\upsilon + \Omega_\upsilon(\nabla_x Y)^\upsilon$ (31)

Now making use of rule (c) and equation (24), we find :

$$\begin{aligned}(\nabla_x \Omega)_\upsilon Y^\upsilon &= (X\Omega_\upsilon) Y^\upsilon + \Omega_\upsilon(XY^\upsilon) - \Omega_\upsilon[XY^\upsilon + Y^\sigma \omega_\sigma^\upsilon(X)] \\ &= (X\Omega_\upsilon) Y^\upsilon - \Omega_\sigma \omega_\upsilon^\sigma(X) Y^\upsilon\end{aligned} \quad (32)$$

We conclude that $(\nabla_x \Omega)_\upsilon = X\Omega_\upsilon - \Omega_\sigma \omega_\upsilon^\sigma(X)$ (33)

or alternatively, $\nabla_x \Omega = [X\Omega_\upsilon - \Omega_\sigma \omega_\upsilon^\sigma(X)] e^\upsilon$ (34)

Specializing this last equation to the case when $\Omega = e^\upsilon$, we obtain the formula

$$\nabla_x e^\upsilon = -\omega_\sigma^\upsilon(X) e^\sigma \quad (35)$$

which is to be contrasted with the earlier formula (18). Equation (35) shows that a knowledge of the n^2 one-forms ω_υ^σ suffices to determine the covariant derivatives of one-forms, as well, once we accept the leibnitz rule for tensor products. Also we may note that in a local coordinate basis, equation (33) gives $\Omega_{\upsilon;\tau} = \Omega_{\upsilon,\tau} - \Omega_\sigma \Gamma_{\upsilon\tau}^\sigma$ (36)

An important result follows from equations (33) and (36) when applied to the one-form df. Since the components of df in a local coordinate basis are $f_{,\upsilon}$, we obtain from equation (36), in this case,

$$f_{,\upsilon;\tau} = f_{,\upsilon,\tau} - f_{,\sigma} \Gamma_{\upsilon\tau}^\sigma \quad (37)$$

and by permuting the indices υ and τ in this equation, we obtain $f_{,\tau;\upsilon} = f_{,\tau,\upsilon} - f_{,\sigma}\Gamma_{\tau\upsilon}^\sigma$ (38)

Returning to equation (29) we now observe that, with aide of equations (23) and (33), we can readily write down the covariant derivative of an arbitrary tensor-field. Thus,

$$A_{\tau;\sigma}^{\mu\upsilon} = A_{\tau,\sigma}^{\mu\upsilon} + A_\tau^{\rho\upsilon}\Gamma_{\rho\sigma}^\mu + A_\tau^{\mu\rho}\Gamma_{\rho\sigma}^\upsilon - A_\rho^{\mu\upsilon}\Gamma_{\tau\sigma}^\rho \quad (39)$$

If we choose a suitable coordinate, it will be seen that our field equations, will be appropriate for infinitely small four dimensional regions. Let x_1, x_2 and x_3 be the space coordinates and x_4 be the time coordinate in an appropriate unit. Here the appropriate unit is the coordinate in which the time unit chosen so that the light speed is equal unit (c=1) in local coordinate. If a rigid rod is chosen, which is given as the unit measure, the coordinates with a given orientation of the coordinates have direct physical meaning in the sense of the theory of relativity. According to relativity theory, the following expression has a value which is independent of the orientation of the local system of coordinates: $ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$.

Let *ds* be the magnitude of linear element pertaining to points of the four-dimensional continuum in infinite proximity. To the mentioned linear element or to the two infinitely proximate point events, there are correspond definite differentials dx_1, dx_2, dx_3, dx_4. In this system the dx_υ represented here by definite linear homogeneous expression of the dx_σ : $dx_\upsilon = \sum_\sigma \alpha_{\upsilon\sigma} dx_\sigma$ inserting these expressions in above equation, we obtain: $ds^2 = \sum_{\tau\sigma} g_{\sigma\tau} dx_\sigma dx_\tau$ where $g_{\sigma\tau}$ are functions of x_σ. These are independent from the orientation and the state of motion of the local system of the coordinates. *ds* is independent of any particular choice of coordinates.

If it is possible to choose a system of coordinate in the finite region in such a way that the $g_{\mu\upsilon}$ has constant values:

$$g_{\mu\upsilon} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{40}$$

We will see that a free material point moves, relatively to this system, with uniform motion on a straight line. But if we choose a new space-time coordinates x_1, x_2, x_3, x_4, the $g_{\mu\upsilon}$ in the new system will not be constant, but functions of space and time, and the motion of free material point will be a curvilinear non-uniform motion. We must interpret this motion as motion under the influence of the compound eg field. So we find the occurrence of an eg field connected with the space-time variables of $g_{\mu\upsilon}$.So the $g_{\mu\upsilon}$ representing the eg field at the same time define the metrical properties of the space-time.

There is a simple relationship between the coordinate basis vectors $h_\mu = (\hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4)$ and the coordinate system of u^μ where $\mu = 1, 2, 3, 4$. We must have: $ds^2 = h_\mu . h_\upsilon du^\mu du^\upsilon$ (41) and $h_\mu . h_\upsilon = g_{\mu\upsilon}$ where $g_{\mu\upsilon}$ is the fundamental tensor.

We have: $g^{\alpha\tau} R^{\mu}_{\upsilon\sigma\tau} = R^{\mu\alpha}_{\upsilon\sigma}$ where $R^{\mu}_{\upsilon\sigma\tau}$ is the Riemann tensor.

Now we write: $$h_\alpha R^{\mu\alpha}_{\upsilon\sigma} = R^{\mu\alpha}_{\upsilon\sigma\alpha} \tag{42}$$

By contracting (42), two times, we find: $R^{\mu\alpha}_{\upsilon\mu\alpha} = R_\upsilon$, where R_υ is the curvature four vector.

We define $S_{\mu\upsilon}$ using the following wedge product: $$S_{\mu\upsilon} = R_\mu \wedge h_\upsilon \tag{43}$$

Now we define the eg tensor as following:

$$\frac{\partial S_{\mu\upsilon}}{\partial x^\sigma} + h_\mu \left(\frac{\partial \Gamma^\tau_{\upsilon\tau}}{\partial x^\sigma} - \frac{\partial \Gamma^\tau_{\upsilon\sigma}}{\partial x^\tau} + \Gamma^\kappa_{\upsilon\tau}\Gamma^\tau_{\sigma\kappa} - \Gamma^\kappa_{\upsilon\sigma}\Gamma^\tau_{\tau\kappa} \right) - \Gamma^\tau_{\mu\sigma} S_{\tau\upsilon} - \Gamma^\tau_{\sigma\upsilon} S_{\mu\tau} \tag{44}$$

4. The eg field equation in the absence of matter

The mathematical importance of the above mentioned eg tensor is that, If there is a coordinate system with reference to which the $g_{\mu\nu}$ are constant, then all components of the eg tensor will vanish . If we choose any new system of coordinates, the $g_{\mu\nu}$ will not be constant, but the transformed components of the eg tensor will still vanish in the new system. Relatively to this system, all components of the eg tensor vanish in any other system of coordinates. Thus the required equations of the matter-free eg field must in any case be satisfied if all components of the eg tensor vanish.

So using expression (44), the equations of matter-free eg field, are:

$$\frac{\partial S_{\mu\upsilon}}{\partial x^{\sigma}}+h_{\mu}\left(\frac{\partial\Gamma^{\tau}_{\upsilon\tau}}{\partial x^{\sigma}}-\frac{\partial\Gamma^{\tau}_{\upsilon\sigma}}{\partial x^{\tau}}+\Gamma^{\kappa}_{\upsilon\tau}\Gamma^{\tau}_{\sigma\kappa}-\Gamma^{\kappa}_{\upsilon\sigma}\Gamma^{\tau}_{\tau\kappa}\right)-\Gamma^{\tau}_{\mu\sigma}S_{\tau\upsilon}-\Gamma^{\tau}_{\sigma\upsilon}S_{\mu\tau}=0 \qquad (45)$$

5. The Equation of the Geodetic Line

Let Y represent a contravariant vector-field. We must consider its variation along a curve λ on N. The change δY in Y caused by a displacement along λ resulting from an increment δt in t (which parameterizes λ) is given (in a local coordinate system) by,

$$(\delta Y)^{\upsilon}=Y^{\upsilon}_{,\tau}\frac{dx^{\tau}(\lambda(t))}{dt}\delta t \qquad (46)$$

In Euclidean geometry and in a Cartesian system of coordinates, one would say that Y is 'parallely propagated' along λ if $\delta Y=0$. In a general differentiable manifold with a connection, one defines, analogously, that a vector Y is parallely propagated along λ, if

$$(DY)^{\upsilon}=(\nabla_{\partial_\tau}Y)^{\upsilon}\frac{dx^{\tau}(\lambda(t))}{dt}\delta t=Y^{\upsilon}_{;\tau}\frac{dx^{\tau}(\lambda(t))}{dt}\delta t=0 \qquad (47)$$

or, alternatively, if $$(Y^{\upsilon}_{,\tau}+Y^{\sigma}\Gamma^{\upsilon}_{\sigma\tau})\frac{dx^{\tau}(\lambda(t))}{dt}\delta t=0\ . \qquad (48)$$

In other words, for parallel propagation of Y along λ, we require that

$$(\delta Y)^{\upsilon}=-Y^{\sigma}\Gamma^{\upsilon}_{\sigma\tau}\frac{dx^{\tau}(\lambda(t))}{dt}\delta t\ . \qquad (49)$$

In particular, for the tangent vector to the curve λ, $\frac{dx^{\upsilon}(\lambda(t))}{dt}$ parallely propagated along λ,

$$\delta\left(\frac{dx^{\upsilon}(\lambda(t))}{dt}\right)=-\Gamma^{\upsilon}_{\sigma\tau}\frac{dx^{\sigma}(\lambda(t))}{dt}\frac{dx^{\tau}(\lambda(t))}{dt}\delta t\ . \qquad (50)$$

A curve λ on N is said geodesic if the tangent vector to λ, parallely propagated, remains a multiple of itself. This condition for λ to be a geodesic, is, clearly,

$$\frac{dx^{\upsilon}(\lambda(t))}{dt}-\Gamma^{\upsilon}_{\sigma\tau}\frac{dx^{\sigma}(\lambda(t))}{dt}\frac{dx^{\tau}(\lambda(t))}{dt}\delta t=[1-\phi(t)\delta t]\left[\frac{dx^{\upsilon}(\lambda(t))}{dt}+\frac{d^2x^{\upsilon}(\lambda(t))}{dt^2}\delta t\right] \qquad (51)$$

Where $\phi(t)$ is some function of t. In the limit $\delta t\to 0$, the equation for geodesic becomes,

$$\frac{d^2x^{\upsilon}}{dt^2}+\Gamma^{\upsilon}_{\sigma\tau}\frac{dx^{\sigma}}{dt}\frac{dx^{\tau}}{dt}=\phi(t)\frac{dx^{\upsilon}}{dt} \qquad (52)$$

It can be readily verify that if we reparameterize the curve λ by the variable

$$s=\int^{t}dt''\exp\left\{\int^{t''}dt'\phi(t')\right\}\ , \qquad (53)$$

Equation (52) becomes $$\frac{d^2x^{\upsilon}}{ds^2}+\Gamma^{\upsilon}_{\sigma\tau}\frac{dx^{\sigma}}{ds}\frac{dx^{\tau}}{ds}=0 \qquad (54)$$

6. The General Form of the field Equations

The field equations (eq.45), which are obtained for matter free space-time, are to be compared with the field equation $\nabla^2\varphi = 0$ of Newton's theory or $R_{\mu\upsilon} = 0$ of Einstein gravity field equations in vacuum. We require the equation corresponding to Poisson's equation: $\nabla^2\varphi = 4k\pi\rho$ or $R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R = -kT_{\mu\upsilon}$ of Einstein general form of the gravitational field equation. For this purpose we define $T_{\mu\nu\sigma}$ as following: $T_{\mu\upsilon\sigma} = g_{\alpha\upsilon}(kT^{\alpha}_{\sigma} + k'T'^{\alpha}_{\sigma})h_{\mu}$ (55)

Where k and k' are two constants related to the gravity and electromagnetism respectively, and $T_{\upsilon\sigma} = g_{\alpha\upsilon}T^{\alpha}_{\sigma}$ is the energy-momentum tensor and T'^{α}_{σ} is the electromagnetic energy tensor.

Thus instead of eq.45 we write:

$$\frac{\partial S_{\mu\upsilon}}{\partial x^{\sigma}} + h_{\mu}\left(\frac{\partial\Gamma^{\tau}_{\upsilon\tau}}{\partial x^{\sigma}} - \frac{\partial\Gamma^{\tau}_{\upsilon\sigma}}{\partial x^{\tau}} + \Gamma^{\kappa}_{\upsilon\tau}\Gamma^{\tau}_{\sigma\kappa} - \Gamma^{\kappa}_{\upsilon\sigma}\Gamma^{\tau}_{\tau\kappa}\right) - \Gamma^{\tau}_{\mu\sigma}S_{\tau\upsilon} - \Gamma^{\tau}_{\sigma\upsilon}S_{\mu\tau} - \frac{1}{2}g_{\upsilon\sigma}h_{\mu}R = -T_{\mu\upsilon\sigma} \quad (56)$$

Where R is the Ricci scalar. We can find out that, the electromagnetic energy tensor can be written as :

$$g_{\alpha\upsilon}T'^{\alpha}_{\sigma}\ h_{\mu} = h_{\mu}T'_{\upsilon\sigma} = T'_{\mu\upsilon\sigma} = h_{\mu}\left(-F_{\sigma\alpha}F^{\alpha}{}_{\upsilon} + \frac{1}{4}g_{\sigma\upsilon}F_{\alpha\beta}F^{\alpha\beta}\right) \quad (57)$$

where $F_{\alpha\beta}$ is the electromagnetic tensor.

Therefore we have found the required general form of the field equations around a blackhole.

References:

1- A.Einstein. *The foundation of the General theory of relativity*.Doc.30, 1914.

2- Paul Adrien Maurice Dirac, *General Theory of Relativity*, Princeton university press, 1975.

3- Subrahmanyan, Chandrasekhar,*The mathematical theory of blackholes*, Newyork,Oxford university press, 1983.

4- M.P.Hobson, G.P.Efstrathiou and A.N.Lasenby. *General relativity*, Cambridge university, 2006