Dirichlet’s proof of Fermat’s Last Theorem for n = 5 is flawed

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Abstract

Dirichlet’s using algorithm is not enough for proving FLT of n = 5.

1 Dirichlet’s proof for n = 5

First, we rewrite a proof in the case z is odd and divisible by 5 (summary only, for details, please see: \(x^5 + y^5 = z^5\) (Dirichlet’s proof) in [1], [2]), which was proven by Dirichlet as follows:

Lemma. if the equation \(x^5 + y^5 = z^5\) is satisfied in integers, then one of the numbers x, y, and z must be divisible by 5 (corollary of Sophie Germain’s theorem)

Since x and y are both odd, their sum and difference are both even numbers.

\[2p = x + y, \quad 2q = x - y\]

Where the non-zero integers p and q are coprime and have different parity (one is even, the other odd). Since \(x = p + q\) and \(y = p - q\),

\[z = 2^{m}5^{n}z'\]

it follows that

\[2^m5^n z' = x^5 + y^5 = (p + q)^5 + (p - q)^5 = 2p(p^4 + 10p^2q^2 + 5q^4)\] (1)

Since 5 divides \(2p(p^4 + 10p^2q^2 + 5q^4)\), then there must be r such that \(p = 5r\)

\[2p(p^4 + 10p^2q^2 + 5q^4) = 2(5r)((5r)^4 + 10(5r)^2q^2 + 5q^4) = 2.5^2r(125r^4 + 50r^2q^2 + q^4)\]

Define three values u, v, t to be the following:

\[t = q^4 + 50r^2q^2 + 125r^4\]

\[u = q^2 + 25r^2\]

\[v = 10r^2\]

And note that \(t = u^2 - 5v^2\)

and t is a fifth power since \(z^5 = 2.5^2r.t\), two factors 2.5^2r, and t are relatively prime, so t is a fifth power and 2.5^2r is a fifth power.

By using the infinite descent, Dirichlet claimed that if t is a fifth power, then there must be a smaller solution.

Setting:

\[u = c(c^4 + 50c^2d^2 + 125d^4)\]

\[v = 5d(c^4 + 10c^2d^2 + 5d^4)\]

now 2.5^2r is a fifth power, so (2.5^2r)^2 is a fifth power

\[(2.5^2r)^2 = 2.5^3.10r^2 = 2.5^3.5d(c^4 + 10c^2d^2 + 5d^4)\]

since gcd2.5^4d, \(c^4 + 10c^2d^2 + 5d^4 = 1\), then 2.5^4d and \(c^4 + 10c^2d^2 + 5d^4\) are fifth power.

In other hand, \(c^4 + 10c^2d^2 + 5d^4 = (c + 5d^2)^2 - 5(2d^2)^2 = u^2 - 5v^6\)

Setting:
Since \(2.5^4d\) is a fifth power, so \((2.5^4d)^2\) is also a fifth power,
\[
(2.5^4d)^2 = 2.5^82^2d^2 = 2.5^8d'(c^4 + 10c^2d^2 + 5d^4)
\]
So \(2.5^8d',\) and \(c^4 + 10c^2d^2 + 5d^4\) are also fifth power. \(c^4 + 10c^2d^2 + 5d^4\) are the same form, and \(d' < d,\) by infinite descent, the original equation \(t = u^2 - 5v^2\) has no solution.

## 2 Dirichlet’s mistake

Dirichlet showed that, there are other ways in which can be a fifth power, but they have the same form as \(u_0 = c(c^4 + 50c^2d^2 + 125d^4)\)
\[
v_0 = 5d(c^4 + 10c^2d^2 + 5d^4)
\]
That means, the other solution will be:
\[
u_i = c_i(c^4 + 50c^2d^2 + 125d^4)
\]
\[
v_i = 5d_i(c^4 + 10c^2d^2 + 5d^4)
\]
Since \(t = u^2 - 5v^2 = (c^2 - 5d^2)^5\), he claimed that if \(c^2 - 5d^2\) has a prime factor, they are the same form as \(c^2 - 5d^2:\)

so all solutions must be the same form as \(u_0, v_0\)

However, this argument is incorrect as below:

The fact that, if \(N\) is not divisible by 5, then \(N = e - 5f\)
, so \(N^5 = (e - 5f)^5 = e(e^2 + 50ef + 125f^2)^2 - 552 f(e^2 + 10ef + 5f^2)^2\)
in other hand, \(N^5 = u_0^5 - 5u_0^5\)

Select*: \(u_0^5 = c(e^2 + 50ef + 125f^2)^2\)
and \(5v_0^5 = 552 f(e^2 + 10ef + 5f^2)^2\)
then \(e\) and \(f\) must be square, \(e = c^2, f = d^2\)
It gives : \(u_0 = c(c^4 + 10c^2d^2 + 125d^4)\)
\[
v_0 = 5d(c^4 + 10c^2d^2 + 5d^4)
\]
and \(N = c^2 - 5d^2\)

However, select* is the only way? There is no proof.

\(N = c^2 - 5d^2\) is from select*, and is not from \(N^5 = u_0^5 - 5u_0^5\)

Note that: \(A_1 = a_1^2 - 5b_1^2, A_2 = a_2^2 - 5b_2^2,\) then:
\[
A = A_1A_2 = (a_1^2 - 5b_1^2)(a_2^2 - 5b_2^2)
\]
\[
A = A_1A_2 = (a_1a_2 + 5b_1b_2)^2 - 5(a_1b_2 + 5a_2b_1)^2
\]
\[
A = A_1A_2 = (a_1a_2 - 5b_1b_2)^2 - 5(a_1b_2 - 5a_2b_1)^2
\]
\(A\) is the same form as \(A_1, A_2\)
but if \(A = a^2 - 5b^2 = A_1A_2\) then \(A_1, A_2\) are not always the same form as \(A\).

In Euler’s proof of FLT for \(n = 3,\) we have seen a similar formula (lemma) such as:
\[
a^2 + 3b^2 = (c^2 + 3d^2)^3
\]
Here: \(a = c(c^2 - 9d^2), b = 3d(c^2 - d^2)\) with gcd\((c,d) = 1,\) and \(c, d\) are none-zero.
Euler also used the technique of infinite descent, but by other way in modified version, unfortunately, his proof is also incorrect [3].
The algorithm above (using by Euler and Dirichlet) is the one way to find a solution of FLT for \(n = 3\) and \(5,\) if a solution is not found by this algorithm, it is not enough to conclude that the equation has no solution in integer.
References


[3] Quang N V, Euler’s proof of Fermat Last’s Theorem for n = 3 is incorrect Vixra:1605.0123v3(NT)

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