Curry's non-Paradox and Its False Definition

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Abstract Curry's paradox is generally considered to be one of the hardest paradoxes to solve. However, it is shown here that the solution is however trivial and the paradox turns out to be no paradox at all. Reviewing the starting point of the paradox, it is concluded that it amounts to a false definition or assertion and therefore it is to be expected, as opposed to being paradoxical, to arrive to a false conclusion. Despite that fact that verifying the truth value of the first statement of the paradox is trivial, mathematicians and logicians have failed to do so and merely assumed that it is true. Taking this into consideration that it is false, the paradox is however dismissed. This conclusion puts to rest an important paradox that preoccupies logicians and points out the importance of verifying one's assumptions.

Curry's Paradox

Curry's paradox, named after its discoverer, Haskell B. Curry,[4, 5] is one of the hardest paradoxes in logic.[2]. For related work on this topic see Beall[1] and Rogerson[7]. The paradox can be presented succinctly as follows:

1. $Q \longleftrightarrow (Q \longrightarrow Y)$, by definition or assumption. For effect, Y is taken to be false.
2. $Q \longrightarrow (Q \longrightarrow Y)$, derived from step 1 by biconditional elimination.
3. $Q \longrightarrow Y$, derived from step 2 using contraction.
4. Q, derived from step 1 and 3 by modus ponens.
5. Y, derived from step 3 and 4 by modus ponens.

Therefore, the contradiction lies in the fact that we derived that Y is true even though we chose Y to be false.

While it is generally believed that axiomatic theory avoids the paradox, it has been shown that it cannot avoid the powerset version of Curry's paradox (PCP).[3]

Classical Solutions

It is generally claimed that axiomatic set theory (such as Zermelo-Fraenkel set theory, ZFC) avoids Curry's paradox by replacing the axiom of unrestricted comprehension by the standard axioms of axiomatic set theory. The solution, the axiomatization of set theory, was put in place in order to avoid Russell's paradox and was extended to Curry's paradox as well. Particularly, it is the axiom of specification (also called the axiom of separation) that excludes large collections (or proper classes) such as the set of all sets from Russell's paradox: $\exists S \forall x[x \in S \longleftrightarrow x \in Z \wedge \varphi(x)]$. The requirement that x belongs to Z makes S a subset of Z and thus it only allows subsets to be constructed.

It is here suggested that the axioms don't succeed in avoiding the paradox. One reason is that collection defined as

$$S = \{\, x \mid x \in S \longrightarrow Y \,\} \tag{1}$$

(where Y is false) is not a large set (as it's the case with the set of of all sets from Russell's paradox). Instead, it's the empty set because there is no x that belongs to S for which Y is true. The empty set is a subset of any set and thus the axiom of separation is satisfied. For any Z we have $S \subset Z$ because $S = \varnothing$. Just as the empty set being a subset of any set is vacuously true, $x \in S \longleftrightarrow x \in Z \wedge \varphi(x)$ is also satisfied vacuously: any x that belongs to S (that is, none) also belong to Z and $\varphi(x)$ is true.

But even if the axiom of specification was not satisfied for S (Equation (1)), since $S = \varnothing$, the axiom of empty set would still give S the status of set under ZFC. Therefore, the axioms of ZFC, while avoiding Russell's paradox, can't avoid Curry's paradox because S, as defined in the set theory version of Curry's paradox, is a perfectly valid set under ZFC.

Identifying the Problem and the Solution

The main statement of Curry's paradox is in the first step above:

$$Q \longleftrightarrow (Q \longrightarrow Y)\,, \tag{2}$$

where Y is false. This is generally taken by assumption or definition. When, for example, the paradox is applied to set theory, a set S is defined as follows: $S = \{x \mid x \in S \longrightarrow Y\}$ which is equivalent to $x \in S \longleftrightarrow (x \in S \longrightarrow Y)$. Taking $Q = x \in S$ we derive Equation (2) above.

However, let us actually plug in the truth values of Q and Y (T for *true* and F for /false) in Equation (2)) in order to. Obviously, if Y is false then there are two possibilities: either Q is true or Q is false. If Q is true we get:

$$(T \longleftrightarrow (T \longrightarrow F))\,. \tag{3}$$

This amounts to $T \longleftrightarrow F$ which is obviously false! If we take Q as false we get:

$$(F \longleftrightarrow (F \longrightarrow F)) \tag{4}$$

This amounts to $F \longleftrightarrow T$ which again is obviously false! This means that if Y is false, Equation (2) and consequently the definition it is derived from are necessarily false and therefore invalid! That's also the case when Q is false. The only case where the definition is true is when both Q and Y are true. Therefore, a simple checking of the definition for its truth value avoids all the versions of Curry's paradox. If the starting point (be it definition or assertion), while false, is assumed to be true then a contradiction will arise. But if it is recognized that the starting point is false then it is no contradiction and should be no surprise that it leads to a false statement. The contradiction and the surprise would be if it didn't! For example, in case of Q being true, step #3 presented earlier $(Q \longrightarrow Y)$ becomes $T \longrightarrow F$. Tis is false. But if it was true, as assumed, then Y must be true and the contradiction ensues.

Therefore, the solution merely amounts to recognizing there is a problem with the starting point of the paradox which turns out to be a false definition or assumption. This realization resolves all versions of the paradox, including the powerset version of the paradox.[3]

Analysis of the Problem-Solution

Curry started from the Kleene–Rosser paradox[6] and realized that the paradox can be simplified into what is now known as Curry's paradox. But the paradox can be simplified even further and arrived at by mere definition or assertion: $Q := false$! It is often that we complicate things and those complications are in the way of seeing the obvious.

Mathematical entities exist by virtue of being able to construct them. A mathematical definition, in effect, constructs the entity being defined, specifying the conditions under which it exists. The contradiction arises when the entity being asserted to exist is impossible to exist. A definition, by its very nature, is positive. It defines what something is not what something isn't. It defines when somethings exists or how something can be constructed (which means that it mathematically exists) not when something doesn't exist. Thus, a definition is assumed to be true. Defining something that is or should be known to be false makes no mathematical sense. Neither does making an assumption that is or should be known to be false. Just like a definition, an assumption is assumed to be true. But the definition or assertion behind Curry's paradox stipulates that an entity (or relation between entities) exists precisely when it doesn't (when conditions of its existence forbid its existence). In other words, a false statement is assumed or defined which then leads to a false conclusion.

The lack of prior test for the truth value of Equation (2) ($Q \longleftrightarrow (Q \longrightarrow Y)$) is perplexing given the fact that it's about as difficult as calculating "$1 \times (1 \times -1)$," (where 1 is equivalent to *true* and -1 to *false*), that three out of the possible four cases are false and that, ever since Curry, many students have studied the paradox.

Although "it is generally agreed that one of the hardest among the paradoxes is Curry's paradox,"[2] the solution turned out to be trivial and the paradox turned out to be no paradox at all but rather a trick played by mathematicians' assumptions. The result therefore shows the importance of verifying one's assumptions and how long time and how much effort goes wasted on a non-problem when this check is overlooked.

References

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[4] Haskell B. Curry, *The paradox of Kleene and Rosser*, Transactions of the American Mathematical Society **50** (1941), no. 3, 454–516.

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[6] S. C. Kleene and J. B. Rosser, *The inconsistency of certain formal logics*, Annals of Mathematics **36** (1935), no. 3, 630–636.

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