A class of integrable mixed Liénard-type differential equations

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Abstract

This letter is devoted to show the existence of a general class of integrable mixed Liénard-type equations that includes some physically important nonlinear differential equations like the generalized modified Emden-type equation (MEE) through the first integral under differentiation approach.

1. Theory

Let us consider that the first integral of the desired generalized nonlinear differential equation of interest may be written in the form

$$a(x,\dot{x}) = g(x)\dot{x} + xf(x) \qquad (1)$$

where $g(x) \neq 0$, and $f(x)$ are arbitrary functions of x, and the dot over a symbol means differentiation with respect to time. Suppose, now, that the function $a(x,\dot{x})$ satisfies

$$\frac{da(x,\dot{x})}{dt} = 0 \qquad (2)$$

Substitution of the equation (1) into (2), after a few mathematical rearrangements, yields immediately the desired class of solvable mixed Liénard-type differential equations

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 + \frac{f'(x)}{g(x)}x\dot{x} + a\frac{f(x)}{g(x)^2} - \frac{f(x)^2}{g(x)^2}x = 0 \qquad (3)$$

where prime designates differentiation with respect to x.

2. Application

Let $g(x) = a_1 x^m$, and $f(x) = a_1^2 x^{2m+1}$, where the exponent m is a real number. So, the equation (3) reduces to

$$\ddot{x} + m\frac{\dot{x}^2}{x} + (2m+1)a_1 x^{m+1}\dot{x} + ax - a_1^2 x^{2m+3} = 0 \qquad (4)$$

The equation (4) consists of a generalized mixed Liénard-type equation [1].

Now, substitution of $m = 0$, into the equation (4), leads immediately to the generalized modified Emden type equation [1]

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$$\ddot{x}+a_1x\dot{x}+ax-a_1^2x^3=0 \qquad (5)$$

Also, $m=-\frac{1}{2}$, gives, taking into account the equation (4)

$$\ddot{x}-\frac{1}{2}\frac{\dot{x}^2}{x}+ax-a_1^2x^2=0 \qquad (6)$$

The equation (6) is known as a quadratic Liénard-type differential equation [2,3]. It is interesting to note that a more generalization of equation (1) may be written in the form

$$a(x,\dot{x})=g(x)\dot{x}+x^l f(x) \qquad (7)$$

where the exponent l is a real number.

References

[1] M. Lakshmanan, V. K. Chandrasekar, Generating finite dimensional integrable nonlinear dynamical systems, arXiv:1307.0273v1.(2013).

[2] M. D. Monsia, J. Akande, K.K. Adjaï, L.H. Koudahoun, Y.J.F. Kpomahou, A theory of exactly integrable quadratic Liénard type equations, viXra:1607.0199v1.(2016).

[3] J. Akande, D. K. K. Adjaï, L. H. Koudahoun, Y. J. F. Kpomahou, M. D. Monsia, A general class of exactly solvable inverted quadratic Liénard type equations, viXra :1608.0124v1.(2016).