Lagrangian Analysis of a Class of Quadratic Liénard-Type Oscillator Equations with Exponential-Type Restoring Force function

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Abstract

This research work proposes a Lagrangian and Hamiltonian analysis for a class of exactly integrable quadratic Liénard-type harmonic nonlinear oscillator equations and its inverted version admitting a position-dependent mass dynamics.

1. Analysis of the class of quadratic Liénard-type harmonic nonlinear oscillator equations

This section is devoted to the analysis of a class of quadratic Liénard-type nonlinear dissipative oscillator equations that admits exact analytical harmonic periodic solutions. Consider the equation [1, 2]

\[ \ddot{x} - \gamma \varphi(x)x^2 + \omega^2 x e^{2\varphi(x)} = 0 \]  

(1)

that represents the class of equations under analysis. \( \gamma \) and \( \omega \) are arbitrary parameters, and \( \varphi(x) \) is an arbitrary function of \( x \). The dot over a symbol means differentiation with respect to time, and prime holds for differentiation with respect to \( x \). The equation (1) is of the general form

\[ \ddot{x} + f(x)x^2 + g(x) = 0 \]  

(2)

for which the first integral is given by [3]

\[ I(\dot{x}, x) = \dot{x}^2 e^{2 \int f(x) dx} + 2 \int g(x) e^{2 \int f(x) dx} \]  

(3)

So, a first integral of (1) may be written as

\[ I(\dot{x}, x) = \dot{x}^2 e^{-2\varphi(x)} + \omega^2 x^2 \]  

(4)

By application of the formula [4]

\[ L(\dot{x}, x) = \dot{x} \int \frac{I(\dot{x}, x)}{x^2} d\dot{x} \]  

(5)

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the Lagrangian of the equation (1) becomes

\[ L(\dot{x}, x) = x^2 e^{-2y(x)} - \omega^2 x^2 \]  

(6)

Applying the Euler-Lagrange equation

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \]

(7)
to the equation (6), gives the equation (1). Now, using [3]

\[ H(p, x) = p\dot{x} - L(x, \dot{x}) \]

(8)
one may deduce from (6) the Hamiltonian

\[ H(p, x) = \frac{p^2}{4} e^{2y(x)} + \omega^2 x^2 \]

(9)

Let us now consider, as illustration, some specific examples of (1). Let \( \varphi(x) = x \). Then (1) becomes

\[ \ddot{x} - 2p^2 x + \omega^2 x e^{2y(x)} = 0 \]

(10)
The equation (10) admits the first integral

\[ I(\dot{x}, x) = x^2 e^{-2y(x)} + \omega^2 x^2 \]

(11)
which provides the Lagrangian function

\[ L(\dot{x}, x) = \dot{x}^2 e^{-2y(x)} - \omega^2 x^2 \]

(12)
The application of the Euler-Lagrange equation (7) to (12) gives, as expected, (10). In this regard the Hamiltonian associated to (10) takes the form

\[ H(p, x) = \frac{p^2}{4} e^{2y(x)} + \omega^2 x^2 \]

(13)
So, the Hamilton equations

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p} \\
\dot{p} &= -\frac{\partial H}{\partial x}
\end{align*}
\]

(14)
yield for (13)
\[
\begin{align*}
\dot{x} &= \frac{p}{2} e^{2yx} \\
\dot{p} &= -\frac{p^2}{2} y e^{2yx} - 2\omega^2 x
\end{align*}
\] (15)

The explicit expression for the canonically conjugate momentum \( p \), as a function of \( x \) and \( \dot{x} \) takes then the form
\[
\dot{p} = -2e^{-2yx} \left( \dot{\gamma} x^2 + \omega^2 x e^{2yx} \right)
\] (16)

Putting now \( \varphi(x) = \frac{1}{2} x^2 \), into (1), one may obtain as equation
\[
\ddot{x} - \dot{\gamma} x^2 + \omega^2 x e^{2yx} = 0
\] (17)

A first integral of (17) takes then the form
\[
I(\dot{x}, x) = \dot{x}^2 e^{-\gamma x^2} + \omega^2 x^2
\] (18)

The associated Lagrangian becomes
\[
L(\dot{x}, x) = \dot{x}^2 e^{-\gamma x^2} - \omega^2 x^2
\] (19)

The application of the Euler-Lagrange equation (7) to (19) gives with satisfaction (17). So, the associated Hamiltonian may be written as
\[
H(p, x) = \frac{p^2}{4} e^{\gamma x^2} + \omega^2 x^2
\] (20)

Such that the Hamilton equations take the form
\[
\begin{align*}
\dot{x} &= \frac{p}{2} e^{\gamma x^2} \\
\dot{p} &= -\frac{p^2}{2} x e^{\gamma x^2} - 2\omega^2 x
\end{align*}
\] (21)

The relation between \( \dot{x} \) and \( \dot{p} \) reads in this perspective
\[
\dot{p} = -2x e^{-\gamma x^2} \left( \gamma x^2 + \omega^2 e^{\gamma x^2} \right)
\] (22)

2. Analysis of inverted versions

Consider now the inverted version of (1)
\[
\ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 x e^{2\varphi(x)} = 0
\] (23)

which gives for \( \varphi(x) = x \), the following equation
\[ \ddot{x} + \rho x^2 + \omega^2 xe^{2\gamma x} = 0 \]  

(24)

The first integral of (24) may be then deduced from (3) as

\[ I(\dot{x}, x) = \dot{x}^2 e^{2\gamma x} + \frac{\omega^2}{2\gamma} xe^{4\gamma x} - \frac{\omega^2}{8\gamma^2} e^{4\gamma x} \]  

(25)

Therefore, the Lagrangian for (24) may be written in the form

\[ L(\dot{x}, x) = \dot{x}^2 e^{2\gamma x} + \frac{\omega^2}{2\gamma} xe^{4\gamma x} - \frac{\omega^2}{8\gamma^2} e^{4\gamma x} \]  

(26)

In this regard, it may be verified that the application of the Euler-Lagrange equation (7) to (26) yields, as expected, (24). The Hamiltonian for (24) may also be computed as

\[ H(p, x) = \frac{p^2}{4} e^{-2\gamma x} + \frac{\omega^2}{2\gamma} xe^{4\gamma x} - \frac{\omega^2}{8\gamma^2} e^{4\gamma x} \]  

(27)

which gives the Hamiltonian equations

\[
\begin{align*}
\dot{x} &= \frac{p}{2} e^{-2\gamma x} \\
\dot{p} &= \frac{p^2}{2} \gamma xe^{-2\gamma x} - 2\omega^2 xe^{4\gamma x}
\end{align*}
\]

(28)

from which the canonically conjugate momentum becomes

\[ \dot{p} = 2e^{2\gamma x} \left( \rho x^2 - \omega^2 xe^{2\gamma x} \right) \]  

(29)

By analysis, other forms of equations are also suggested by the previous studied equations. So, the following equations may also be considered in the perspective of this study, that is

\[ \ddot{x} + \rho \dot{x}^2 x + \omega^2 xe^{\alpha x^2} = 0 \]  

(30)

or in general

\[
\begin{align*}
\ddot{x} + \gamma \dot{\varphi}'(x) \dot{x}^2 + \omega^2 xe^{\gamma \varphi(x)} &= 0 \\
\ddot{x} - \gamma \dot{\varphi}'(x) \dot{x}^2 + \omega^2 xe^{\gamma \varphi(x)} &= 0
\end{align*}
\]

(31)

(32)

Finally one may consider the following more generalizations

\[
\begin{align*}
\ddot{x} + \gamma \dot{\varphi}'(x) \dot{x}^2 + \omega^2 h(x)e^{\gamma \varphi(x)} &= 0 \\
\ddot{x} - \gamma \dot{\varphi}'(x) \dot{x}^2 + \omega^2 h(x)e^{\gamma \varphi(x)} &= 0
\end{align*}
\]

(33)

(34)
\[ \ddot{x} + \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{2\gamma \varphi(x)} = 0 \quad \text{(35)} \]

\[ \ddot{x} - \gamma \varphi'(x) \dot{x}^2 + \omega^2 h(x) e^{2\gamma \varphi(x)} = 0 \quad \text{(36)} \]

These equations will be investigated in a subsequent work.

**References**


