Logarithmic Extension of Real Numbers and Hyperbolic Representation of Generalized Lorentz Transforms

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We construct the logarithmic extension for real numbers in which the numbers, less then $-\infty$ exist. Using this logarithmic extension we give the single formula for hyperbolic representation of generalized tachyon Lorentz transforms.

1 Introduction

Generalized Lorentz transforms, to be considered in this paper, are connected with the theory of tachyons, that is the hypothetical objects, moving with velocity, greater, than the velocity of light. It should be noted that ukrainian physicist, academician, Oleksa-Myron Bilanyuk stood near the beginning the tachyon theory [1, 2]. In early works in this direction the theory of tachyons was considered in the framework of classical Lorentz transformations, and superlight speed for the frame of reference was forbidden. Later in the works of E. Recami, V. Olkhovsky and R. Goldoni [3–5] extension of classical Lorentz transforms for superlight velocity of reference frames was proposed (see also [6]). Latter the above extended Lorentz transformations were rediscovered in [7, 8]. Interest to this subject had been increased in 2010-2012 due to the experiments conducted in the framework of collaboration OPERA (results, which were not confirmed later). In particular B. Cox and J. Hill in the paper [7] have rediscovered the formulas of Recami-Olkhovsky-Goldoni's extended Lorentz transformations by means of new and elegant way of deduction them (the fact, that extended Lorentz transforms, obtained in [7] are not new is noted in the comment [9]). Also Recami-Olkhovsky-Goldoni's extended Lorentz transformations were investigated in [10] and generalized in [11, 12].

The hyperbolic representation of classical Lorentz Transforms is well-known. The hyperbolic representation of generalized Lorentz transforms for superlight reference frames can be found in the papers [7, 10]. At the same time, formula that would give a single hyperbolic representation for extended superluminal together with classical Lorentz transforms now is unknown. In the paper [10] author tries to give such formula, using so-called extended hyperbolic functions. But, in fact, such "extended hyperbolic functions" are not conventional hyperbolic functions. In our opinion, the main cause of this situation is the fact that classical hyperbolic functions are defined on all real axis $\mathbb{R}$, and substitution any real value as hyperbolic argument into the formulas of hyperbolic representation of Lorentz transforms does not lead to superluminal velocity of reference frame. In the present paper we are aiming to show, that going beyond of the real axis for hyperbolic argument in the formulas of hyperbolic representation of Lorentz transforms leads to reference frames with superluminal velocity.

In the next section we construct the ***logarithmic extension of real numbers***[1]. Next, using this logarithmic extension we present the simple formulas, which give the single hyperbolic representation for classical Lorentz transforms as well as generalized Lorentz transforms (in the sense of Recami-Olkhovsky-Goldoni).

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[1] Despite despite the fact that this extension is elementary, I have not found such extension in the scientific literature yet. Dear readers! If you know that this extension is already known, please share the link of corresponding publication to e-mail of the author of this work.

2 Logarithmic Extension of Real Numbers

2.1 Motivation

We start our considerations with one simple example.

Example 1. Consider the algebraic system of kind $(\mathbb{R}_+, +, \times)$, where $\mathbb{R}_+$ is the set of positive real numbers, $+$ and $\times$ are the operations of addition and multiplication of real numbers (correspondingly). This algebraic system has many beautiful properties. But it is not "completed" because the operation of subtraction (which is inverse to addition) is not defined on whole $\mathbb{R}_+$. Indeed, if $a, b \in \mathbb{R}_+$ and $a \le b$ then $a - b \notin \mathbb{R}_+$. So, in this case the equation $b + x = a$ has not any solution in $\mathbb{R}_+$. This means that algebraic system $(\mathbb{R}_+, +, \times)$ is not field. Apparently, we may "correct" this situation by means of "completion" $\mathbb{R}_+$ by zero and negative real numbers. By this way we obtain the algebraic system $(\mathbb{R}, +, \times)$, that is the usual field of real numbers.

Now we consider the other algebraic system. We introduce the following new operations of "multiplication" and "addition" on the set $\mathbb{R}$ of real numbers:

$$x \,\widehat{\times}\, y := x + y;$$
$$x \,\widehat{+}\, y := \ln\left(e^x + e^y\right) \quad (\forall\, x, y \in \mathbb{R}).$$

It is easy to see, that the mapping

$$\mathbb{R} \ni x \longmapsto \exp(x) \in \mathbb{R}_+ \tag{1}$$

is bijection (one-to-one correspondence) between $\mathbb{R}$ and $\mathbb{R}_+$. Moreover, for any $x, y \in \mathbb{R}$ we have:

$$\exp\left(x \,\widehat{\times}\, y\right) = \exp(x) \exp(y); \tag{2}$$
$$\exp\left(x \,\widehat{+}\, y\right) = \exp(x) + \exp(y). \tag{3}$$

Thus, the algebraic systems $\left(\mathbb{R}, \widehat{+}, \widehat{\times}\right)$ and $(\mathbb{R}_+, +, \times)$ are isomorphic and the mapping (1) provides isomorphism between them. Hence, the algebraic system $\left(\mathbb{R}, \widehat{+}, \widehat{\times}\right)$ is not "completed", similarly to $(\mathbb{R}_+, +, \times)$. And (similarly to $(\mathbb{R}_+, +, \times)$), all real numbers are "positive" in respect of the algebraic system $\left(\mathbb{R}, \widehat{+}, \widehat{\times}\right)$, so "zero" and "negative" numbers are missing in $\left(\mathbb{R}, \widehat{+}, \widehat{\times}\right)$. Therefore the problem of constructing the natural expansion of algebraic system $\left(\mathbb{R}, \widehat{+}, \widehat{\times}\right)$ to field, including "zero" and "negative" elements arises. In the next subsection we solve the above problem.

2.2 Construction of Logarithmic Extension

Definition 1. *By **logarithmic extension of real numbers** we name the set:*

$$\mathbb{R}^{\widehat{+}} := \mathbb{R} \cup \{-\infty\} \cup \{a + \pi i \mid a \in \mathbb{R}\} \quad (\textit{where } i = \sqrt{-1}).$$

Further we will consider that

$$\exp(-\infty) := e^{-\infty} := 0. \tag{4}$$

Function $\exp(x)$, determined by formula (4) for $x = -\infty$, is defined on whole set $\mathbb{R}^{\widehat{+}}$. Moreover, this function is a bijection from $\mathbb{R}^{\widehat{+}}$ to $\mathbb{R}$. The following function is inverse to the function $\exp(\cdot) : \mathbb{R}^{\widehat{+}} \mapsto \mathbb{R}$:

$$\ln^{\widehat{+}}(x) = \begin{cases} \ln(x), & x > 0 \\ -\infty, & x = 0 \\ \ln|x| + \pi i, & x < 0 \end{cases} \quad (x \in \mathbb{R}).$$

For any $x, y \in \mathbb{R}^{\widehat{+}}$ we denote:

$$x \,\widehat{+}\, y := \ln^{\widehat{+}}\left(\exp(x) + \exp(y)\right); \tag{5}$$
$$x \,\widehat{\times}\, y := \ln^{\widehat{+}}\left(\exp(x) \exp(y)\right). \tag{6}$$

It is easy to see that for $x, y \in \mathbb{R}$ it is true the equality $x \,\widehat{\times}\, y = x + y$. So, the operation "$\widehat{\times}$" ("multiplication" for $\mathbb{R}^{\widehat{+}}$) is the extension of the operation "$+$" of usial addition from the set $\mathbb{R}$ to the set $\mathbb{R}^{\widehat{+}}$.

Now we extend the order relation from the set of real numbers $\mathbb{R}$ to $\mathbb{R}^{\widehat{+}}$.

Definition 2. *Let $x, y \in \mathbb{R}^{\widehat{+}}$ be such, that $x \notin \mathbb{R}$ or $y \notin \mathbb{R}$. We say, that $x \le y$ ($x < y$) if and only if $\exp(x) \le \exp(y)$ ($\exp(x) < \exp(y)$) (correspondingly).*

Assertion 1. *For any $x, y \in \mathbb{R}^{\widehat{+}}$ the following equivalences are true:*
$x \le y \Longleftrightarrow \exp(x) \le \exp(y)$;
$x < y \Longleftrightarrow \exp(x) < \exp(y)$.

Proof. Indeed, for $x, y \in \mathbb{R}$ the both equivalences are trivial, whereas in the case $x \notin \mathbb{R}$ or $y \notin \mathbb{R}$ these equivalences follow from Definition 2. □

From Assertion 1 we get immediately the following Assertion.

Assertion 2. *1. The relation $\le$ is linear order relation on $\mathbb{R}^{\widehat{+}}$, that is the following propositions are true:*

(a) for any $x \in \mathbb{R}^{\widehat{+}}$ $x \le x$;

(b) if $x, y \in \mathbb{R}^{\widehat{+}}$, $x \le y$ and $y \le x$ then $x = y$;

(c) if $x, y, z \in \mathbb{R}^{\widehat{+}}$, $x \le y$ and $y \le z$ then $x \le z$;

(d) for any $x, y \in \mathbb{R}^{\widehat{+}}$ one of inequalities $x \le y$ or $y \le x$ is true.

2. The relation $<$ is strict linear order relation on $\mathbb{R}^{\widehat{+}}$, generated by the non-strict order $\le$, that is for any $x, y \in \mathbb{R}^{\widehat{+}}$ the correlation $x < y$ is true if and only if $x \le y$ and $x \ne y$.

Theorem 1. *1. The algebraic system $\left(\mathbb{R}^{\widehat{+}}, \widehat{+}, \widehat{\times}, <\right)$ is an ordered field.*

2. The element $0^{\widehat{+}} = -\infty$ is zero element of the field $\mathbb{R}^{\widehat{+}}$.

3. The number $1^{\widehat{+}} = 0$ is identity element of the field $\mathbb{R}^{\widehat{+}}$.

4. *The field $\mathbb{R}^{\widehat{+}}$ is isomorphic to the field $\mathbb{R}$ and the mapping $\mathbb{R}^{\widehat{+}} \ni x \mapsto \exp(x) \in \mathbb{R}$ provides isomorphism between these fields.*

Proof. **1,2,3:** To prove the first three items of Theorem, according to definitions of field and ordered field (see [13,14]), we need to verify the following properties:

(a) $a \widehat{+} b = b \widehat{+} a$ (for any $a, b \in \mathbb{R}^{\widehat{+}}$);

(b) $\left(a \widehat{+} b\right) \widehat{+} c = a \widehat{+} \left(b \widehat{+} c\right)$ (for any $a, b, c \in \mathbb{R}^{\widehat{+}}$);

(c) $a \widehat{\times} b = b \widehat{\times} a$ (for any $a, b \in \mathbb{R}^{\widehat{+}}$);

(d) $\left(a \widehat{\times} b\right) \widehat{\times} c = a \widehat{\times} \left(b \widehat{\times} c\right)$ (for any $a, b, c \in \mathbb{R}^{\widehat{+}}$);

(e) $a \widehat{+} 0^{\widehat{+}} = a \widehat{\times} 1^{\widehat{+}} = a$ (for any $a \in \mathbb{R}^{\widehat{+}}$);

(f) for every element $a \in \mathbb{R}^{\widehat{+}}$ there exist the element $\widehat{-}\, a \in \mathbb{R}^{\widehat{+}}$ such, that $a \widehat{+} \left(\widehat{-}\, a\right) = 0^{\widehat{+}}$;

(g) for every element $a \in \mathbb{R}^{\widehat{+}}$ such, that $a \neq 0^{\widehat{+}}$ there exist the element $a^{\widehat{-}1} \in \mathbb{R}^{\widehat{+}}$ such, that $a \widehat{\times} a^{\widehat{-}1} = 1^{\widehat{+}}$.

(h) if $a, b, c \in \mathbb{R}^{\widehat{+}}$, $a < b$ and $b < c$ then $a < c$;

(i) for any $a, b \in \mathbb{R}^{\widehat{+}}$ exactly one of the correlations $a = b$, $a < b$ or $b < a$ is true;

(j) if $a, b, c \in \mathbb{R}^{\widehat{+}}$ and $a < b$ then $a \widehat{+} c < b \widehat{+} c$;

(k) if $a, b, c \in \mathbb{R}^{\widehat{+}}$, $a < b$ and $c > 0^{\widehat{+}}$ then $a \widehat{\times} c < b \widehat{\times} c$.

Applying the formulas (5), (6), for any any $a, b, c \in \mathbb{R}^{\widehat{+}}$ we obtain:

$$
\begin{aligned}
a \widehat{+} b &= \ln^{\widehat{+}}(\exp(a) + \exp(b)) = \\
&= \ln^{\widehat{+}}(\exp(b) + \exp(a)) = b \widehat{+} a; \\
\left(a \widehat{+} b\right) \widehat{+} c &= a \widehat{+} \left(\ln^{\widehat{+}}(\exp(b) + \exp(c))\right) = \\
&= \ln^{\widehat{+}}(\exp(a) + (\exp(b) + \exp(c))) = \\
&= \ln^{\widehat{+}}((\exp(a) + \exp(b)) + \exp(c)) = \\
&= a \widehat{+} \left(b \widehat{+} c\right).
\end{aligned}
$$

So, properties (a),(b) have been verified. The verification of properties (c),(d),(e) is conducted similarly. To verify properties (f),(g) we put:

$$
\widehat{-}\, a := \begin{cases} a + \pi i, & a \in \mathbb{R} \\ 0^{\widehat{+}}, & a = 0^{\widehat{+}} \\ a - \pi i, & a \in \mathbb{R}^{\widehat{+}} \setminus \left(\mathbb{R} \cup \left(0^{\widehat{+}}\right)\right) \end{cases} \quad \left(a \in \mathbb{R}^{\widehat{+}}\right);
$$

$$
a^{\widehat{-}1} := \begin{cases} -a, & a \in \mathbb{R} \\ -x + \pi i, & a = x + \pi i, \\ & \text{where } x \in \mathbb{R} \end{cases} \quad \left(\begin{array}{c} a \in \mathbb{R}^{\widehat{+}}, \\ a \neq 0^{\widehat{+}} \end{array} \right),
$$

and then apply the formulas (5), (6), taking into account, that $\exp\left(0^{\widehat{+}}\right) = 1^{\widehat{+}}$ and $\ln^{\widehat{+}}\left(1^{\widehat{+}}\right) = 0^{\widehat{+}}$.

To verify Property (h), let us suppose that $a, b, c \in \mathbb{R}^{\widehat{+}}$, $a < b$ and $b < c$. Then, according to Assertion 2 (item 2), we have $a \leq b$ and $b \leq c$. So, by Assertion 2 (item 1), we obtain, $a \leq c$. Assume, that $a = c$. Then (since $a \leq b$ and $b \leq c$) we have $a \leq b$ and $b \leq a$. So, by Assertion 2 (item 1), $a = b$, which contradicts to correlation $a < b$ and Assertion 2 (item 2). Therefore, assumption $a = c$ is false. Thus, we have proved, that $a \leq c$ and $a \neq c$. And, by Assertion 2 (item 2), we have $a < c$.

Consider any elements $a, b \in \mathbb{R}^{\widehat{+}}$. If we suppose, that $a \neq b$ then, according to Assertion 2 (items 1 and 2), one of correlations $a < b$ or $b < a$ must be fulfilled. The correlations $a < b$ and $a = b$, according to Assertion 2 (item 2), can not be fulfilled simultaneously. Similarly, correlations $b < a$ and $a = b$ are incompatible. Correlations $a < b$ and $b < a$ also are incompatible, becuse otherwise, according to property (h) we obtain $a < a$, and so, by Assertion 2 (item 2), $a \neq a$, which is impossible. Hence, Property (i) also has been verified.

From equalities (5), (6) it follows, that equalities (2), (3) can be extended for any $x, y \in \mathbb{R}^{\widehat{+}}$. Properties (j),(k) may be easy verified, using the analogical properties of real numbers, Assertion 1 and equalities (2), (3), extended to any $x, y \in \mathbb{R}^{\widehat{+}}$.

4: Function $\exp(x)$ is a bijection from $\mathbb{R}^{\widehat{+}}$ to $\mathbb{R}$. Using equalities (2), (3), extended to any $x, y \in \mathbb{R}^{\widehat{+}}$ as well as Assertion 1 it is easy to verify, that this function is isomorphism between the fields $\mathbb{R}^{\widehat{+}}$ and $\mathbb{R}$. □

Further we will use the term "field $\mathbb{R}^{\widehat{+}}$" meaning by this term all the algebraic structure $\left(\mathbb{R}^{\widehat{+}}, \widehat{+}, \widehat{\times}, \leq\right)$.

Remark 1. Using Assertion 1 it can be easy proved, that for $a \in \mathbb{R}^{\widehat{+}}$ the condition $a < 0^{\widehat{+}}$ is satisfied if and only if the number a can be represented in the form $a = x + \pi i$, where $x \in \mathbb{R}$. Taking into account, that $0^{\widehat{+}} = -\infty$, we have seen, that in the field $\mathbb{R}^{\widehat{+}}$ the numbers, less then $-\infty$ exist (namely, numbers of kind $a = x + \pi i$, where $x \in \mathbb{R}$).

3 Hyperbolic Representation of Generalized Lorentz Transforms

3.1 Case of One Space Dimension

According to results of the papers [3–8,10], for the case of one space dimension, any generalized Lorentz transform for finite velocity of reference frame (in the sense of Recami-Olkhovsky-Goldoni) may be represented in the form:

$$
ct' = s\frac{ct - \frac{Vx}{c}}{\sqrt{\left|1 - \frac{V^2}{c^2}\right|}}; \quad x' = s\frac{x - Vt}{\sqrt{\left|1 - \frac{V^2}{c^2}\right|}}, \tag{7}
$$

where:

- (t, x) are coordinates of some point in a fixed reference frame $\mathfrak{l}$.
- (t', x') are the coordinates of the point (t, x) in the reference frame $\mathfrak{l}'$, moving relatively the frame $\mathfrak{l}$ with the constant velocity V ($|V| \neq c$).
- c is a positive real constant, which has the physical content of the speed of light in vacuum.
- $s \in \{-1, 1\}$ is constant, that can take only two values (1 and -1). This constant is responsible for the direction of time in moving reference frame $\mathfrak{l}'$ relatively the fixed frame $\mathfrak{l}$.

Note, that in the case $|V| < c$ and $s = 1$ formula (7) gives the classical Lorentz transforms.

Now, we introduce the new variable ψ ($\psi \in \widehat{\mathbb{R}}^{+}$, $\psi \neq -\infty$) such, that:

$$\exp\left(-\frac{1}{2}\mathsf{sign}^{\widehat{+}}\psi\right)\cosh\frac{\psi}{2} = \frac{s_c(V)}{\sqrt{\left|1-\frac{V^2}{c^2}\right|}}; \qquad (8)$$

$$\exp\left(-\frac{1}{2}\mathsf{sign}^{\widehat{+}}\psi\right)\sinh\frac{\psi}{2} = \frac{s_c(V)\,\frac{V}{c}}{\sqrt{\left|1-\frac{V^2}{c^2}\right|}}, \qquad (9)$$

where

$$s_c(V) = \begin{cases} 1, & |V| < c \text{ or } V > 0 \\ -1, & |V| > c \text{ and } V < 0 \end{cases};$$

$$\mathsf{sign}^{\widehat{+}} x = \ln^{\widehat{+}}\left(\mathsf{sign}\left(\exp(x)\right)\right) =$$

$$= \begin{cases} 1^{\widehat{+}} & x \in \mathbb{R} \\ 0^{\widehat{+}} & x = 0^{\widehat{+}} \\ \widehat{-}1^{\widehat{+}} & x \in \mathbb{R}^{\widehat{+}} \setminus \left(\mathbb{R} \cup \left\{0^{\widehat{+}}\right\}\right) \end{cases} =$$

$$= \begin{cases} 0 & x \in \mathbb{R} \\ -\infty, & x = -\infty \\ \pi i & x < -\infty \end{cases} \quad \left(x \in \mathbb{R}^{\widehat{+}}\right)$$

($\mathsf{sign}^{\widehat{+}} x$ is the function, which represents the analogue of the real function $\mathsf{sign}\,(x)$ in the field $\mathbb{R}^{\widehat{+}}$).

Parameter ψ in (8), (9) is uniquely determined by the parameter V. Indeed, from (8), (9), taking into account equality $\cosh^2\frac{\psi}{2} - \sinh^2\frac{\psi}{2} = 1$ we obtain:

$$\exp\left(-\mathsf{sign}^{\widehat{+}}\psi\right) = \frac{1-\frac{V^2}{c^2}}{\left|1-\frac{V^2}{c^2}\right|} = \mathsf{sign}\left(c - |V|\right). \qquad (10)$$

Hence,
$\exp\left(\mathsf{sign}^{\widehat{+}}\psi\right) = \left(\mathsf{sign}\left(c-|V|\right)\right)^{-1} = \mathsf{sign}\left(c-|V|\right)$,
and so:

$$\exp\left(-\frac{1}{2}\mathsf{sign}^{\widehat{+}}\psi\right) =$$

$$= \exp\left(-\frac{\ln^{\widehat{+}}\left(\mathsf{sign}\left(c-|V|\right)\right)}{2}\right). \qquad (11)$$

Thus, in the case $|V| < c$, according to (10), (11) and (9), we have, $\psi \in \mathbb{R}$ ($\psi > -\infty$) and $\sinh\frac{\psi}{2} = \frac{V/c}{\sqrt{\left|1-\frac{V^2}{c^2}\right|}}$.

Therefore in this case we deliver:

$$\psi = 2\ln\left(\frac{V/c+1}{\sqrt{1-\frac{V^2}{c^2}}}\right).$$

Similarly in the case $|V| > c$ we obtain $\psi < -\infty$ (so ψ can be represented in the form $\psi = \alpha + \pi i$, where $\alpha \in \mathbb{R}$) and $-i\cosh\frac{\psi}{2} = \frac{s_c(V)}{\sqrt{\frac{V^2}{c^2}-1}}$. The last equality together with $\psi = \alpha + \pi i$ leads to $\sinh\left(\frac{\alpha}{2}\right) = \frac{\mathsf{sign}(V)}{\sqrt{\frac{V^2}{c^2}-1}}$.

Hence $\alpha = 2\ln\left(\frac{\frac{|V|}{c}+\mathsf{sign}(V)}{\sqrt{\frac{V^2}{c^2}-1}}\right)$, and:

$$\psi = 2\ln\left(\frac{\frac{|V|}{c}+\mathsf{sign}(V)}{\sqrt{\frac{V^2}{c^2}-1}}\right) + \pi i.$$

So, we have seen that in the both cases the parameter ψ is uniquely determined by the parameter V.

Using the new parameter ψ, the formulas (7) may be rewritten as follows:

$$ct' = \widetilde{s}\exp\left(-\frac{1}{2}\mathsf{sign}^{\widehat{+}}\psi\right)\left(ct\cosh\frac{\psi}{2} - x\sinh\frac{\psi}{2}\right); \qquad (12)$$

$$x' = \widetilde{s}\exp\left(-\frac{1}{2}\mathsf{sign}^{\widehat{+}}\psi\right)\left(x\cosh\frac{\psi}{2} - ct\sinh\frac{\psi}{2}\right), \qquad (13)$$

$$\text{where}\quad \widetilde{s} = s s_c(V).$$

Since the parameter s takes the values from the set $\{-1, 1\}$, the parameter $\widetilde{s}$ also takes the values from $\{-1, 1\}$. Thus, ***any generalized Lorentz transform may be represented in the form (12)-(13), where*** $\widetilde{s} \in \{-1, 1\}$ ***and*** $\psi \in \mathbb{R}^{\widehat{+}} \setminus \{-\infty\}$. Besides, for $\psi \in \mathbb{R}$ we obtain the classical Lorentz transforms and for $\psi \in \mathbb{R}^{\widehat{+}}$, $\psi < -\infty$ we obtain the generalized Lorentz transforms for superluminal velocities of reference frame. In the case $\psi = \pi i$ we obtain the generalized Lorentz transforms for infinite velocities of reference frame (cf [7]).

3.2 Case of General Real Hilbert Space

In the papers [11, 12] the generalized Lorentz transforms (in the sense of Recami-Olkhovsky-Goldoni) had been introduced and investigated for the most general case, where the "geometric variable" runs over arbitrary real Hilbert space. Moreover, in these papers the generalized Lorentz transforms were introduced for arbitrary orientation of coordinate axes. The most general representation of these generalized Lorentz transforms gives [12, Theorem 4.2 and Corollary 4.2] with the help of functions:

$$\varphi_0(\theta) = \frac{1+\theta|\theta|}{2|\theta|}; \quad \varphi_1(\theta) = \frac{1-\theta|\theta|}{2|\theta|}, \tag{14}$$

where $\theta \in \mathbb{R} \setminus \{0\}$ (see also [15, Corollary II.17.1]). Functions $\varphi_0(\theta)$ and $\varphi_1(\theta)$ have some properties similar to the properties of hyperbolic functions (for example, $\varphi_0^2(\theta) - \varphi_1^2(\theta) = \operatorname{sign}\theta$ and so $\varphi_0^2(\theta) - \varphi_1^2(\theta) = 1$ for $\theta > 0$). However, actually, these functions are not hyperbolic. But now we are going to show that these functions can be reduced to hyperbolic functions, defined on $\widehat{\mathbb{R}^{+}}$. For this aim we introduce the new variable ψ such, that:

$$\frac{1}{\theta} = \exp\left(\frac{\psi + \operatorname{sign}^{\widehat{+}}\psi}{2}\right); \quad \psi \in \widehat{\mathbb{R}^{+}},\ \psi \neq -\infty. \tag{15}$$

The variable ψ in (15), is uniquely determined by the variable θ. Indeed, in the case $\theta > 0$, according to (15), we have $\psi \in \mathbb{R}$ (that is $\psi > -\infty$) , because for $\psi < -\infty$ we will have $\exp\left(\frac{\psi+\operatorname{sign}^{\widehat{+}}\psi}{2}\right) < 0$. So, in this case the equality (15) may be rewritten in the form, $\frac{1}{\theta} = \exp\left(\frac{\psi}{2}\right)$. Therefore $\psi = 2\ln\left(\frac{1}{\theta}\right)$ for $\theta > 0$. Similarly, in the case $\theta < 0$ we have $\psi \notin \mathbb{R}$, consequently $\psi < -\infty$ and $\psi = \alpha + \pi i$, where $\alpha \in \mathbb{R}$. Hence, in this case the equality (15) may be rewritten in the form, $\frac{1}{\theta} = \exp\left(\frac{\alpha+2\pi i}{2}\right)$. Thus, we get $\frac{1}{\theta} = -\exp\left(\frac{\alpha}{2}\right)$, $\alpha = 2\ln\left(-\frac{1}{\theta}\right)$ and $\psi = 2\ln\left(-\frac{1}{\theta}\right) + \pi i$.

It is easy to verify, that equality (15) leads to the following equality:

$$\frac{1}{|\theta|} = \exp\left(\frac{\psi - \operatorname{sign}^{\widehat{+}}\psi}{2}\right). \tag{16}$$

Using the equalities (15) and (16), we may express the functions $\varphi_0(\theta)$, $\varphi_1(\theta)$ via new parameter ψ:

$$\begin{aligned}
\varphi_0(\theta) &= \frac{1}{2}\left(\frac{1}{|\theta|} + \theta\right) = \\
&= \tfrac{1}{2}\left(\exp\left(\tfrac{\psi-\operatorname{sign}^{\widehat{+}}\psi}{2}\right) + \exp\left(-\tfrac{\psi+\operatorname{sign}^{\widehat{+}}\psi}{2}\right)\right) = \\
&= \exp\left(-\frac{1}{2}\operatorname{sign}^{\widehat{+}}\psi\right)\cosh\left(\frac{\psi}{2}\right); \\
\varphi_1(\theta) &= \frac{1}{2}\left(\frac{1}{|\theta|} - \theta\right) = \\
&= \exp\left(-\frac{1}{2}\operatorname{sign}^{\widehat{+}}\psi\right)\sinh\left(\frac{\psi}{2}\right).
\end{aligned}$$

Applying the last equalities as well as [12, Corollary 4.2] (or [15, Corollary II.17.1]) we obtain the following Theorem 2 (in this Theorem we use the system of notions and denotations, accepted in [11, 12, 15]).

Theorem 2. *Operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{OT}(\mathfrak{H}, c)$ if and only if there exist numbers $s \in \{-1, 1\}$, $\psi \in \widehat{\mathbb{R}^{+}} \setminus \{-\infty\}$, vector $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$ and operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ such, that for any $\mathrm{w} \in \mathcal{M}(\mathfrak{H})$ vector $L\mathrm{w}$ can be represented by the formula:*

$$\begin{aligned}
L\mathrm{w} = \exp&\left(-\frac{1}{2}\operatorname{sign}^{\widehat{+}}\psi\right)\cdot \\
&\cdot\left(s\cosh\left(\frac{\psi}{2}\right)\mathcal{T}(\mathrm{w}) - \sinh\left(\frac{\psi}{2}\right)\frac{\langle \mathbf{n}, \mathrm{w}\rangle}{c}\right)\mathbf{e}_0 + \\
&+ J\Bigg(\exp\left(-\frac{1}{2}\operatorname{sign}^{\widehat{+}}\psi\right)\cdot \\
&\cdot\left(c\sinh\left(\frac{\psi}{2}\right)\mathcal{T}(\mathrm{w})\,\mathbf{n} - s\cosh\left(\frac{\psi}{2}\right)\mathbf{X}_1[\mathbf{n}]\,\mathrm{w}\right) + \\
&+ \mathbf{X}_1^{\perp}[\mathbf{n}]\,\mathrm{w}\Bigg).
\end{aligned}$$

Linear coordinate transform operator L is v-determined if and only if $\psi \neq \pi i$, and in this case:

$$\mathcal{V}(L) = cs\frac{\sinh\left(\frac{\psi}{2}\right)}{\cosh\left(\frac{\psi}{2}\right)}\mathbf{n} = cs\tanh\left(\frac{\psi}{2}\right)\mathbf{n}.$$

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