ABSTRACT: In this paper we define a new Mellin discrete convolution, which is related to Perron’s formula. Also we introduce new explicit formulae for arithmetic function which generalize the explicit formulae of Weil.

MELLIN DISCRETE CONVOLUTION:

We define the Mellin discrete convolution in the form

$$\sum_{n=1}^{\infty} a(n) f \left( \frac{n}{x} \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)G(s)x^s \quad (1)$$

Where $$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = G(s)$$ is the Dirichlet generating functio of the coefficients $$a(n)$$ and $$F(s) = \int_{0}^{\infty} dx f(x)x^{-s-1}$$

The proof is quite easy, first we apply the integral operator $$\int_{0}^{\infty} dx f(x)$$ to the left of (1) so if the series involving $$a(n)$$ is completely convergent, so we can switch between the series and the integral then, we have

$$\int_{0}^{\infty} \frac{dx}{x^{s+1}} \left( \sum_{n=1}^{\infty} a(n) f \left( \frac{n}{x} \right) \right) = \sum_{n=1}^{\infty} a(n) \int_{0}^{\infty} \frac{t^{s-1}f(nt)dt}{n^s} = \sum_{n=1}^{\infty} a(n) \int_{0}^{\infty} u^{s-1}f(u)du = G(s)F(s) \quad (2)$$

If we apply the inverse operator of $$\int_{0}^{\infty} \frac{dx}{x^{s+1}} f(x)$$ which is to both sides

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \int_{0}^{\infty} \frac{dx}{x^{s+1}} f(x) \right)x^s = f(x)$$ then we have proved (1).

this kind of discrete transform is a discrete analogue to the Mellin Convolution theorem defined for Mellin transforms.
\[
\int_0^\infty \frac{dt}{t} f\left(\frac{x}{t}\right) g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)G(s)x^{-s} \quad F(s) = \int_0^\infty dx f(x)x^{-1} \quad G(s) = \int_0^\infty dx g(x)x^{-1} \quad (3)
\]

Now, if we set \( f\left(\frac{1}{t}\right) = H(t-1) = \begin{cases} 1 & t > 1 \\ 0 & t < 1 \end{cases} \) we recover Perron's formula [5] for the Coefficients of the Dirichlet series

\[
\sum_{n=1}^\infty a(n)H\left(\frac{x}{n} - 1\right) = \sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s)\frac{x^s}{s} \quad \text{since} \quad F(s) = \frac{1}{s} = \int_1^\infty \frac{dx}{x^{s+1}} \quad (4)
\]

But one of the best applications of our Mellin convolution is related to several Dirichlet series(see [4] ) in the form \( \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = G(s) \), Where \( G(s) \) includes powers or quotients of the Riemann zeta function for example

\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \quad (5)
\]

\[
\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} \quad \frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} \quad (6)
\]

The definition of the functions inside () and () is as follows

- The Möbius function, \( \mu(n) = 1 \) if the number ‘n’ is square-free (not divisible by an square) with an even number of prime factors , \( \mu(n) = 0 \) if \( n \) is not squarefree and \( \mu(n) = (-1)^{\Omega(n)} \) if the number ‘n’ is square-free with an odd number of prime factors.
- The Von Mangoldt function \( \Lambda(n) = \log p \), in case ‘n’ is a prime or a prime power and takes the value 0 otherwise
- The Liouville function \( \lambda(n) = (-1)^{\Omega(n)} \Omega(n) \) is the number of prime factors of the number ‘n’
- \( |\mu(n)| \) is 1 if the number is square-free and 0 otherwise
- \( \varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \), the meaning of \( p \parallel n \) is that the product is taken only over the primes \( p \) that divide ‘n’.

To obtain the coefficients of the Dirichlet series we can use the Perron formula

\[
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = G(s) = \int_1^{\infty} \frac{A(x)}{x^s} \quad A(x) = \sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} G(s) ds \quad (7)
\]

If the function \( G(s) \) includes powers and quotients of the Riemann zeta function we can use Cauchy’s theorem to obtain the explicit formulae for example

---

2
\[ M(x) = \sum_{n \leq x} \mu(n) = -2 + \sum_{\rho} \frac{x^\rho}{\rho \zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{\zeta'(-2n)(-2n)} \quad (8) \]

\[ \Psi(x) = \sum_{n \leq x} \Lambda(n) = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho} \frac{x^\rho}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{(-2n)} \quad (9) \]

\[ L(x) = \sum_{n \leq x} \lambda(n) = 1 + \frac{\sqrt{x}}{\zeta(1/2)} + \sum_{\rho} \frac{x^\rho \zeta(2\rho)}{\rho \zeta'(\rho)} \quad (10) \]

\[ Q(x) = \sum_{n \leq x} |\mu(n)| = 1 + \frac{6x}{\pi^2} + \sum_{\rho} \frac{x^\rho \zeta\left(\frac{\rho}{2}\right)}{\rho \zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{x^{-n} \zeta(-n)}{(-2n)\zeta'(-2n)} \quad (11) \]

\[ \Phi(x) = \sum_{n \leq x} \varphi(n) = 1 + \frac{3x^2}{\pi^2} + \sum_{\rho} \frac{x^\rho \zeta(\rho-1)}{\rho \zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{x^{-2n} \zeta(-2n-1)}{(-2n)\zeta'(-2n)} \quad (12) \]

Under the assumption that all the Riemann Non-trivial zeros are simple.

Also we have for the Riemann zeta function and its derivatives

\[ \zeta'(-2n) = \frac{(-1)^n \zeta(2n+1)(2n)!}{2^{2n+1} \pi^{2n}} \quad \zeta'(0) = -\frac{1}{2} \log(2\pi) \quad \zeta(0) = -\frac{1}{2} \quad (13) \]

The reader will remember the relation between Perron’s formula and our discrete convolution, using the work of Baillie [ ] we will give different explicit formulae, to do so we need to use Cauchy’s theorem on complex integration and evaluate the closed mellin inverse transform by using the residue theorem

\[ \frac{1}{2\pi i} \oint_{C} F(s)G(s)x^s \quad \text{where 'C' is a closed circuit including all the poles of the} \]

Dirichlet series G(s), we can do this assuming all the Riemann zeros are simple and that the Mellin transform F(s) has no poles inside 'C', in this case we have the 'explicit formulae'

\[ \sum_{n=1}^{\infty} \Lambda(n)f\left(\frac{n}{x}\right) = xF(1) - \sum_{\rho} x^\rho F(\rho) - \sum_{n=1}^{\infty} F(-2n) \frac{1}{x^{2n}} \quad (14) \]

\[ \sum_{n=1}^{\infty} \mu(n)f\left(\frac{n}{x}\right) = \sum_{\rho} x^\rho \frac{F(\rho)}{\zeta'(\rho)} + \sum_{n=1}^{\infty} F(-2n) \frac{1}{\zeta'(-2n) x^{2n}} \quad (15) \]

\[ \sum_{n=1}^{\infty} \lambda(n)f\left(\frac{n}{x}\right) = \frac{\sqrt{x}}{2\zeta\left(\frac{1}{2}\right)} F\left(\frac{1}{2}\right) + \sum_{\rho} x^\rho \frac{\zeta(2\rho)F(\rho)}{\zeta'(\rho)} \quad (16) \]
\[
\sum_{n=1}^{\infty} \varphi(n)f\left(\frac{n}{x}\right) = \frac{6}{\pi^2} F(2)x^2 + \sum_{\rho} \int_{-\infty}^{\infty} \frac{\zeta'(\rho) F(\rho)}{\zeta(\rho)} + \sum_{n=1}^{\infty} \frac{F(-2n)}{x^{2n}} \zeta(-2n) - \frac{\zeta(2n)}{2\zeta(-2n)} \tag{17}
\]

\[
\sum_{n=1}^{\infty} \mu(n) f\left(\frac{n}{x}\right) = \frac{6}{\pi^2} F(1)x + \sum_{\rho} \int_{-\infty}^{\infty} \frac{\frac{\zeta(\rho)}{2} F\left(\frac{\rho}{2}\right)}{2\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{F(-n)}{x^{2n}} \zeta(-n) - \frac{\zeta(n)}{2\zeta(-2n)} \tag{18}
\]

If the Mellin transform has poles inside the closed circuit \( 'C' \) \( \oint F(s)G(s)x^{s} \), then this poles will contribute with a remainder term due to the Residue theorem \([1]\) in this case we have the extra term

\[
r(x) = \sum_{k} \text{Re}\left\{ F(s)G(s)x^{s}\right\}_{s=k} \quad \text{with} \quad F(k) = \int_{0}^{\infty} dx f(x)x^{k-1} = \infty \quad (19)
\]

this is what happens in Perron formula, due to the step function \( H(x-1) \) in this case its Mellin transform has a pole at \( s = 0 \) since \( F(s) = \frac{1}{s} \) this is why in formulae (8-12) there is a constant term.

As a curious final example of our Mellin discrete convolution, if we use the Dirichlet generating function \( G(s) = \zeta(s-k) \) and the floor function as a test function so \( \int^\infty_0 \frac{dx}{x^{s+1}} \left\lfloor x \right\rfloor = \frac{\zeta(s)}{s} \), then our Mellin discrete convolution becomes the identity for the k-th order sum of the divisor function

\[
\sum_{n=1}^{\infty} \sigma_k(n) = \sum_{n=1}^{\infty} n^k \left\lfloor \frac{x}{n} \right\rfloor = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds x^s \zeta(s-k) \zeta(s) \tag{20}
\]

We have previously investigated this kind of explicit formula \([3]\) but instead of the Mellin transform we used the Fourier transform and Fourier convolution theorem for test functions \( g(x) \) and \( h(x) \) related by a dualFourier transform, so the integral \( h(c) = \int_{-\infty}^{\infty} dxg(x)e^{icx} \) exists and is finite for every real number (positive or negative) ‘c’, and \( g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dxh(x)e^{-i\alpha x} \) or \( g(\alpha) = \frac{1}{\pi} \int_{0}^{\infty} dxh(x)\cos(\alpha x) \)

depending on if the test function are even or not \( h(x) = h(-x) \).

For the case of the Liouville function, there is no contribution due to the nontrivial Riemann zeroes -2,-4,-6,... since the Dirichlet generating functions for this case \( \frac{\zeta(2s)}{\zeta(s)} \) is Holomorphic on the region of the complex plane \( \text{Re}(s) < 0 \)
In our previous work [3] we have established similar formulae to (14-18) but in terms only of the imaginary part of the Riemann zeros

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{\sqrt{n}} g(\log n) = \sum_{\rho} \frac{h(\gamma)}{\zeta'(-\rho)} + \sum_{n=1}^{\infty} \frac{1}{\zeta'(-2n)} \int_{-\infty}^{\infty} dx g(x) e^{-i(2n+\frac{1}{2})} \tag{21}
\]

\[
\sum_{n=1}^{\infty} \lambda(n) \frac{g(\log n)}{\sqrt{n}} = \frac{1}{2\zeta(1/2)} \int_{-\infty}^{\infty} dx g(x) + \sum_{\rho} \frac{\zeta(2\rho) h(\gamma)}{\zeta'(-\rho)} \tag{22}
\]

\[
\sum_{n=1}^{\infty} \varrho(n) \frac{g(\log n)}{\sqrt{n}} = \frac{6}{\pi^2} \int_{-\infty}^{\infty} dx g(x) e^{\gamma x} + \sum_{\rho} \frac{h(\gamma)}{2\zeta'(-\rho)} \zeta(\rho-1) + \sum_{n=1}^{\infty} \frac{\zeta(2n-1)}{\zeta'(-2n)} \int_{-\infty}^{\infty} dx g(x) e^{-i(2n+\frac{1}{2})} \tag{23}
\]

\[
\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{\gamma}} \frac{g(\log n)}{\sqrt{n}} = \frac{6}{\pi^2} \int_{-\infty}^{\infty} dx g(x) e^{\gamma x} + \sum_{\rho} \frac{h(\gamma)}{2\zeta'(-\rho)} \zeta(\rho-1) + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{\zeta'(-2n)} \int_{-\infty}^{\infty} dx g(x) e^{-i(2n+\frac{1}{2})} \tag{24}
\]

And finally the explicit formula for the divisor function \( \sigma(n) \) which is the sum of divisors of \( n \) \( \sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28 \), given by

\[
\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^{\gamma}} \frac{g(\log n)}{\sqrt{n}} = -\frac{1}{12} \int_{-\infty}^{\infty} g(x) e^{\gamma x} dx - \frac{1}{2} \int_{-\infty}^{\infty} g(x) e^{\gamma x} dx + \frac{15\zeta(3)}{\pi^2} \int_{-\infty}^{\infty} g(x) e^{\gamma x} + \sum_{\rho} \frac{h(\gamma)}{2\zeta'(-\rho)} \zeta(\rho) \zeta\left(\rho + 1\right) \tag{25}
\]

Where the sum inside (21-25) are over the imaginary part of the zeros of the Riemann zeta function on the critical line, and \( \rho = \frac{1}{2} + i\gamma \).

Equations (14-18) are equivalent to the equations (21-25) but in one hand we use the Mellin transform and in the other hand we use the Fourier transform \( g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx h(x) e^{-i\alpha x} \), the use of the Fourier transform is in analogy to the Riemann-Weil explicit formula for the Von Mangoldt function

\[
\sum_{n=1}^{\infty} \Lambda(n) \frac{g(\log n)}{\sqrt{n}} = \frac{1}{4\pi} \int_{-\infty}^{\infty} dr \frac{\Gamma'(\frac{1}{4} + ir)}{\Gamma(\frac{1}{4} + ir)} + h\left(\frac{i}{2}\right) - \frac{h(0)}{2} \log \pi - \sum_{k=0}^{\infty} h(\gamma_k) \tag{26}
\]

In the formula (26) the sum is over the positive imaginary parts of the Riemann zeros. For the case of the explicit formulae which involve the test function \( g(x) \) the Laplace Bilateral transform of this function defined by
must be finite, or at least regularizable,

Formulae (14-18) and (21-24) are related if we use the test function $e^{au} \phi(xe^u)$ for some real positive number 'a' ($a = \frac{1}{2}, \frac{1}{4}, \frac{5}{4}, \ldots$) and a nonzero number 'x', in this case using 2 change of variables

$$\int_{-\infty}^{\infty} du e^{au} \phi(xe^u) e^{iuu} \rightarrow \int_{0}^{\infty} dt t^{iu+1} \phi(t) \rightarrow \frac{1}{x'} \int_{0}^{\infty} dt t^{iu+1} \phi(t)$$

With $s = a + i\omega$, then the Fourier transform turns into a Mellin transform

**An easier derivation of our explicit formulae**

There is an easier derivation for our explicit formulae, in general after Perron's formula is applied we find the following identity

$$\sum_{n < x} a_n = P + Qx^d + \sum_{\rho} h(\rho)x^\rho + \sum_{n=1}^{\infty} c_{2n}x^{-2nr}$$

For some real constants $P, Q, d, c_{2n}, r$ and a function $h(\rho)$ which includes the Riemann zeta function and its first derivative

Taking the distributional derivative for an step function of the form $\sum_{n < x} a_n$

$$\frac{d}{dx} \left( \sum_{n < x} a_n \right) = \sum_{n=1}^{x} a_n \delta(x-n) = dQx^{d-1} + \sum_{\rho} h(\rho)x^{\rho-1} + \sum_{n=1}^{\infty} c_{2n}(-2nr)x^{-2nr-1}$$

So if we apply a certain test function with a parameter 'x' $f\left(\frac{t}{x}\right)$ and its Mellin transform defined by

$$\int_{0}^{\infty} dt f\left(\frac{t}{x}\right) t^{s-1} \rightarrow x^s \int_{0}^{\infty} dt f(t) t^{s-1} = x^s F(s)$$

Then we find the desired explicit formula
\[
\sum_{n=1}^{\infty} a_n f\left(\frac{n}{x}\right) = dQF(d) + \sum_{\rho} h(\rho) \rho F(\rho) + \sum_{n=1}^{\infty} c_{2n}(-2nr)F(-2nr) \quad (32)
\]

In general for Dirichlet series generating functions \(\sum_{n=1}^{\infty} \frac{a_n}{n^s} = G(s)\) the zeroeth coefficient is \(a_0 = 0\) so the series inside (31) begins at \(n = 1\).

A similar method can be applied to derive the Poisson summation formula, let be the Floor function \([x]\), then we have a formula valid on the whole real line

\[
[x] = x - \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} \quad (33)
\]

Taking the distributional derivative of (33) and using the Euler's formula for the cosine function

\[
\frac{d}{dx} [x] = \sum_{n=-\infty}^{\infty} \delta(x - n) = 1 + 2 \sum_{n=1}^{\infty} \cos(2\pi nx) \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (34)
\]

Now if we use a test function inside (33) we have the Poisson summation formula

\[
\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x) \sum_{n=-\infty}^{\infty} e^{2\pi i nx} \quad (35)
\]

REFERENCES:


