# Classification of an $AdS$ solution

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Dr. Thomas Günther[1]

*Hugo-Küching-Straße 1, D-59067 Hamm*

Abstract: McVittie's model [10], which interpolates between a Schwarzschild Black Hole and an expanding global (FLRW) spacetime, can be constructed by a simple coordinate replacement in Schwarzschild's isotropic intervall. Analogously, one gets a similarly generated exact solution of Einstein's equations based on a static transformation of de Sitter's metric, cf. [6]. The present article is concerned with the application of this method on the $AdS$ (Anti de Sitter) spacetime. Multiplying the radial coordinate and it's differential by a function $a(t)$ gives the basic line element. Einstein's equations for the modified interval reduce to a system of two differential equations, which are solved in the article. The resulting solution is classified depending on the value of the cosmological constant. Several promising theories like String theorie and $AdS/CFT$ correspondence [11] include spacetimes with higher dimensions. Thereby motivated, the previous results of this article are generalized to $D$ dimensions in the last section.

## 1. Introduction

The algebraic sign convention of Ricci tensor $R^i_k = \sum_n g^{in}\left\{\sum_a \left(\partial_a\Gamma^a_{kn} - \partial_k\Gamma^a_{an} + \sum_b\left(\Gamma^a_{ab}\Gamma^b_{kn} - \Gamma^a_{kb}\Gamma^b_{an}\right)\right)\right\}$, scalar curvature $R = \sum_k R^k_k$, metric signature or $\Lambda$ in the field equations is not unique in literature. The calculations in this article are based on Einstein's field equations

$$R^i_k - \frac{1}{2}R\delta^i_k + \Lambda\delta^i_k = \frac{8\pi\gamma}{c^4}T^i_k \tag{0.1}$$

where $\gamma$ is the gravitational constant, $\Lambda$ the cosmological constant and $\delta^i_k$ is the Kronecker delta: It is $\delta^i_k = 1$ for $i = k$ and $\delta^i_k = 0$ for $i \neq k$. The signature of the metric tensor is choosen to be $(-,+,+,+)$. With respect to this convention, de Sitter's metric

$$ds^2 = -\left[1 - \frac{\Lambda}{3}r^2\right]c^2dt^2 + \frac{1}{1-\frac{\Lambda}{3}r^2}dr^2 + r^2d\Omega^2_2\,, \qquad d\Omega^2_2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

is a solution of Einstein's empty space equations $R^i_k - \frac{1}{2}R\delta^i_k + \Lambda\delta^i_k = 0$ with positive cosmological constant $\Lambda > 0$. De Sitter's space has a positive Ricci scalar curvature $R = 4\Lambda$. On the other hand, there is the Anti de Sitter spacetime which has negative scalar curvature and solves Einstein's equations for negative cosmological constant $\Lambda < 0$. In this case it is $\Lambda = -|\Lambda|$ and the field equations (0.1) become $R^i_k - \frac{1}{2}R\delta^i_k = |\Lambda|\,\delta^i_k$. The four dimensional Anti de Sitter spacetime ($AdS_4$) can be described in spherical coordinates $\{t, r, \theta, \phi\}$ by the line element

$$ds^2 = -\left[1 + \frac{|\Lambda|}{3}r^2\right]c^2dt^2 + \frac{1}{1+\frac{|\Lambda|}{3}r^2}dr^2 + +r^2d\Omega^2_2. \tag{0.2}$$

It is similar to de Sitter's metric exept for the positive prefactor in front of $r^2$ in the metric components $g_{tt}$ and $g_{rr}$. The Ricci scalar curvature of the $AdS_4$ metric (0.2) is given by $R = -4|\Lambda|$. As well as de Sitter's metric, the $AdS_4$ line element can be transformed into a static and isotropic form. Let $q$ be the radial coordinate in an isotropic frame $\{t, q, \theta, \phi\}$ with time $t$, inclination $\theta$ and azimuth $\phi$. The isotropic $AdS_4$ It is obtained by using the coordinate transformation

$$r = \frac{q}{1-\frac{|\Lambda|}{12}q^2} \tag{0.3}$$

in (0.2). A possible way to obtain the coordinate transformation (0.3) is described in the last section. The isotropic $AdS_4$ metric reads:

$$ds^2 = -\left(\frac{1+\frac{|\Lambda|}{12}q^2}{1-\frac{|\Lambda|}{12}q^2}\right)^2 c^2dt^2 + \left(\frac{1}{1-\frac{|\Lambda|}{12}q^2}\right)^2\left(dq^2 + q^2d\Omega^2_2\right) \tag{0.4}$$

[1] *dr.thomas.guenther@gmail.com*

## 2. Basic line element and Einstein's equations

Let $l$ be an arbitary constant with physical unit $\mathrm{m}^{-1}$. Based on the latter metric (0.4) we get a suitable ansatz for the modified line element. Instead of the factor $|\Lambda|/12$ we use a positive constant $l^2$. According to the method desribed in [6], the radial coordinate $q$ is replaced by $a(t)\,q$ and the differential $dq^2$ by $a(t)\,dq$. It should be mentioned that this replacement is different from a coordinate transformation, e.g. $q = a\bar{q}$, since the $\dot{a}\bar{q}dt$ term in $dq = \dot{a}\bar{q}dt + ad\bar{q}$ is ignored. We receive the isotropic line element:

$$ds^2 = -\left(\frac{1 + l^2\,a^2(t)\,q^2}{1 - l^2\,a^2(t)\,q^2}\right)^2 c^2dt^2 + \frac{a^2(t)}{\left(1 - l^2\,a^2(t)\,q^2\right)^2}\left(dq^2 + q^2 d\Omega_2^2\right) \tag{0.5}$$

In order to work out Einstein's equations for this metric, we use the abbreviation

$$\nu = \nu(t,q) := l^2\,a^2(t)\,q^2 \tag{0.6}$$

in the following. Metric (0.5) then takes the simple form

$$ds^2 = -\left(\frac{1+\nu}{1-\nu}\right)^2 c^2dt^2 + \frac{a^2(t)}{(1-\nu)^2}\left(dq^2 + q^2 d\Omega_2^2\right). \tag{0.7}$$

Obviously, the metric is a special case of a general isotropic line element, which is given by:

$$ds^2 = -c^2 e^{\xi(q,t)}dt^2 + e^{\mu(q,t)}\left(dq^2 + q^2 d\Omega_2^2\right) \tag{0.8}$$

Einstein's tensor for general isotropic spacetimes can be found in various papers, cf. for example [2] or [5]. A detailed calculation for Einstein's isotropic tensor is given in [5]. Alternatively, the correspondig calculations can been verified e.g. with the computer algebra system MAXIMA[2]. Einstein's tensor $G^i_k = R^i_k - \frac{1}{2}R\delta^i_k + \Lambda\delta^i_k$ for the metric (0.8) reads:

$$\begin{aligned}
G^t_t &= -\frac{3\dot{\mu}^2 e^{-\xi}}{4c^2} + e^{-\mu}\left(\mu'' + \frac{1}{4}\mu'^2 + \frac{2}{q}\mu'\right) + \Lambda \\
G^q_t &= e^{-\mu}\left(\frac{1}{2}\dot{\mu}\xi' - \dot{\mu}'\right), \qquad G^t_q = \frac{e^{-\xi}}{c^2}\left(\dot{\mu}' - \frac{1}{2}\dot{\mu}\xi'\right) \\
G^q_q &= \frac{e^{-\xi}}{c^2}\left(\frac{1}{2}\dot{\mu}\dot{\xi} - \ddot{\mu} - \frac{3}{4}\dot{\mu}^2\right) + e^{-\mu}\left(\frac{\mu'\xi'}{2} + \frac{\mu'^2}{4} + \frac{\mu' + \xi'}{q}\right) + \Lambda \\
G^\theta_\theta &= \frac{e^{-\xi}}{c^2}\left(\frac{1}{2}\dot{\mu}\dot{\xi} - \ddot{\mu} - \frac{3}{4}\dot{\mu}^2\right) + \frac{e^{-\mu}}{2}\left(\mu'' + \xi'' + \frac{\xi'^2}{2} + \frac{\mu' + \xi'}{q}\right) + \Lambda \\
G^\phi_\phi &= G^\theta_\theta
\end{aligned} \tag{0.9}$$

With the set of equations (0.9), Einstein's tensor for (0.7) can be obtained as a special case. Metric (0.8) takes the form (0.7) if

$$\xi = 2\left\{\ln(1+\nu) - \ln(1-\nu)\right\}, \qquad \mu = 2\left\{\ln(a) - \ln(1-\nu)\right\}.$$

With $H := \dot{a}/a$ it is $\dot{\nu} = 2H\nu$ and $\nu' = 2\nu/q$ and the required derivatives of the functions $\xi$ and $\mu$ are given by

$$\dot{\xi} = 8H\frac{\nu}{1-\nu^2}, \quad \xi' = \frac{8}{q}\frac{\nu}{1-\nu^2}, \quad \dot{\mu} = 2H\frac{1+\nu}{1-\nu}, \quad \mu' = \frac{4}{q}\frac{\nu}{1-\nu}, \quad \xi'' = \frac{8}{q^2}\frac{\nu\left(1+4\nu^2\right)}{\left(1-\nu^2\right)^2},$$

$$\ddot{\mu} = \frac{2}{(1-\nu)^2}\left\{\frac{\ddot{a}}{a}\left(1-\nu^2\right) + H^2\left(\nu^2 + 4\nu - 1\right)\right\}, \quad \dot{\mu}' = \frac{8H}{q}\frac{\nu}{(1-\nu)^2}, \quad \mu'' = \frac{4}{q^2}\frac{\nu(1+\nu)}{(1-\nu)^2}.$$

By using the above derivatives in (0.9) it is now relatively simple to compute Einstein's tensor for the metric (0.7). The only nonzero components are $G^t_t$ and $G^q_q = G^\phi_\phi = G^\theta_\theta$. Correspondingly, Einstein's equations lead to the following

[2] http://maxima-online.org/. A source code that computes $R^i_k - \frac{1}{2}R\delta^i_k$ for the general isotropic line element (0.8) is:

```
load(ctensor); dim: 4; ct_coords: [t,q,theta,phi];
lg: matrix( [-c^2*exp(xi(t,q)),0,0,0], [0,exp(mu(t,q)),0,0], [0,0,q^2*exp(mu(t,q)),0], [0,0,0,q^2*(sin(theta)^2)*exp(mu(t,q))] );
cmetric(); einstein(dis); expand(ein[1,1]); expand(ein[1,2]); expand(ein[2,1]); expand(ein[2,2]); expand(ein[3,3]);
```

two differential eqations for the function $a(t)$:

$$\frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2 = 12l^2+\Lambda \tag{0.10}$$

$$\frac{2\frac{\ddot{a}}{a}(1-\nu)+\left(\frac{\dot{a}}{a}\right)^2(1+5\nu)}{c^2(1+\nu)} = 12l^2+\Lambda \tag{0.11}$$

**3. Solution of the Field equations**

Equation (0.10) can be solved easily. From $\frac{da}{dt}=\pm a\cdot c\sqrt{4l^2+\frac{\Lambda}{3}}$ one gets $\int_{a_1}^{a}\frac{dx}{x}=\pm c\int_{t_1}^{t}\sqrt{4l^2+\frac{\Lambda}{3}}\,dx$ and thus

$$a(t)=a_0\exp\left(\pm\sqrt{4l^2+\frac{\Lambda}{3}}\,c\,t\right) \tag{0.12}$$

where $a_0$ contains all constants[3] of integration. Now it has to be shown, that the function (0.12) also solves the differential equation (0.11). Equation (0.10) directly leads to

$$\left(\frac{\dot{a}}{a}\right)^2=\left(4l^2+\frac{\Lambda}{3}\right)c^2 \tag{0.13}$$

The following lemma shows that if $(\dot{a}/a)^2$ is constant, the term is equal to $\ddot{a}/a$.

**Lemma 1.** *Let $k$ be an arbitrary nonzero constant and $a(t)$ a function with $(\dot{a}/a)^2=k$, then it applies*

$$\frac{\ddot{a}}{a}=\left(\frac{\dot{a}}{a}\right)^2 \tag{0.14}$$

*Proof.* Derivation of the constant term $(\dot{a}/a)^2$ with respect to the coordinate $t$ leads to:

$$0=\frac{d}{dt}\left[\left(\frac{\dot{a}}{a}\right)^2\right]=2\frac{\dot{a}}{a}\cdot\frac{\ddot{a}a-\dot{a}^2}{a^2}=2\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)=2\sqrt{k}\left(\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}\right)\quad\Rightarrow\quad\frac{\ddot{a}}{a}=\frac{\dot{a}^2}{a^2}$$

Since $k\neq 0$ one gets (0.14). □

Now we apply (0.13) on the left side of equation (0.11). Due to lemma 1 the terms $(\dot{a}/a)^2$ and $\ddot{a}/a$ are identically and can be placed outside the brackets, one gets:

$$\frac{2\frac{\ddot{a}}{a}(1-\nu)+\left(\frac{\dot{a}}{a}\right)^2(1+5\nu)}{c^2(1+\nu)} = \left(4l^2+\frac{\Lambda}{3}\right)c^2\cdot\frac{2(1-\nu)+1+5\nu}{c^2(1+\nu)} = \left(4l^2+\frac{\Lambda}{3}\right)\cdot\frac{3+3\nu}{1+\nu} = 12l^2+\Lambda$$

All in all, the above considerations show that (0.5) with the function $a(t)$ given by (0.12) solves Einstein's equations (0.1) for the empty space, i.e. $T_k^i=0$:

$$ds^2=-\left[\frac{1+l^2q^2a_0^2\exp\left(\pm2\sqrt{4l^2+\frac{\Lambda}{3}}\,c\,t\right)}{1-l^2q^2a_0^2\exp\left(\pm2\sqrt{4l^2+\frac{\Lambda}{3}}\,c\,t\right)}\right]^2c^2dt^2+\frac{a_0^2\exp\left(\pm2\sqrt{4l^2+\frac{\Lambda}{3}}\,c\,t\right)}{\left[1-l^2q^2a_0^2\exp\left(\pm2\sqrt{4l^2+\frac{\Lambda}{3}}\,c\,t\right)\right]^2}\left(dq^2+q^2d\Omega_2^2\right) \tag{0.15}$$

The latter interval essentially belongs to the category of real-valued solutions with positive cosmological constant, however $\Lambda$ may be negative as long as $\Lambda>-12l^2$. This leads to the question if there is a solution for negative $\Lambda<-12l^2$. In this case we use again $\Lambda=-|\Lambda|$ and receive the field equations

$$\frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2=12l^2-|\Lambda|\ ,\quad\frac{2\frac{\ddot{a}}{a}(1-\nu)+\left(\frac{\dot{a}}{a}\right)^2(1+5\nu)}{c^2(1+\nu)}=12l^2-|\Lambda|$$

analogously to (0.10) and (0.11). If one assumes real-valued $g_{ik}$, the solution

$$a(t)=a_0\exp\left(\pm\sqrt{4l^2-\frac{|\Lambda|}{3}}\,c\,t\right) \tag{0.16}$$

[3] $a_0:=\exp\left(\ln a_1\pm\sqrt{4l^2+\frac{\Lambda}{3}}\,c\,t_1\right)$

is confined to $|\Lambda| < 12l^2$. Hence, there is no real-valued solution for $\Lambda < -12l^2$ in any case.

## 4. Classification of the solution

This section is concerned with the classification of metric (0.15) for different cosmological constant values.

### *4.1. The case $\Lambda > 0$*

At first we consider a positive cosmological constant $\Lambda > 0$. In this case, metric (0.15) solves Einstein's empty space equations $R^i_k - \frac{1}{2}R\delta^i_k + \Lambda\delta^i_k = 0$ with positive cosmological constant and has constant positive Ricci scalar curvature $R = 4\Lambda > 0$. One can easily confirm this with the MAXIMA source code in the footnote[4]. Replacing the function $a$ in the source code with it's negative branch, that is $a = a_0 \exp\left(-\sqrt{4l^2 + \frac{\Lambda}{3}}\, c\,t\right)$, does not change the value of the scalar curvature. Space-times with constant scalar curvature are classified as de-Sitter ($R > 0$), Minkowski ($R = 0$) or Anti-de-Sitter ($R < 0$). Cf. [7]: "*The space of constant curvature with $R = 0$ is Minkowski space-time. The space for $R > 0$ is de Sitter space-time* [...]" Accordingly, the metric (0.15) with $\Lambda > 0$ shares these properties with de-Sitter's space-time.

### *4.2. The case $\Lambda < 0$*

The case of a negative cosmological constant is restricted to $-12l^2 < \Lambda < 0$, since there is no real-valued solution (0.15) for $\Lambda < -12l^2$. Here one receives a constant negative curvature $R = -4\,|\Lambda| < 0$ and the metric solves $R^i_k - \frac{1}{2}R\delta^i_k - |\Lambda|\,\delta^i_k = 0$. This features are associated with the $AdS_4$ space-time. Again, the results are the same for both branches ($\pm$) in (0.16).

### *4.3. The case $\Lambda = 0$*

Finally, in case of zero cosmological constant, we should get back the properties of Minkowski spacetime. Indeed, for $a = a_0 \exp(\pm 2lct)$ metric (0.5) is a solution of $R^i_k - \frac{1}{2}R\delta^i_k = 0$ with zero scalar curvature $R = 0$. This could be easily confirmed by adjusting the definition of $a$ in the MAXIMA source code. Additionally, one can compute the Kretschmann scalar[5] for (0.15) with $\Lambda = 0$:

$$\mathcal{K} = \sum_{i,j,k,l} R_{ijkl}R^{ijkl} \; = 0$$

It also coincides with that of Minkowski's spacetime and shows that the spacetime is flat.

## 5. Solution based on the de Sitter line element

Alternatively, the above method can be applied on the de Sitter intervall instead of the $AdS_4$ line element, cf. [5] and [6]. With the above algebraic sign convention of $\Lambda$ in the field equations and $(-,+,+,+)$ signature of the metric (which differs from my choice in [5] and [6]), the basic intervall is given by

$$ds^2 = -\left(\frac{1 - l^2\,a^2\,(t)\,q^2}{1 + l^2\,a^2\,(t)\,q^2}\right)^2 c^2dt^2 + \frac{a^2\,(t)}{\left(1 + l^2\,a^2\,(t)\,q^2\right)^2}\,\left(dq^2 + q^2 d\Omega_2^2\right) \tag{0.17}$$

Evaluating Einstein's equations (0.1) in case of positive cosmological constant $\Lambda > 0$ leads to a sightly different solution for $a\,(t)$:

$$a\,(t) = a_0 \exp\left(\pm\sqrt{\frac{\Lambda}{3} - 4l^2}\, c\,t\right) \tag{0.18}$$

The spacetime given by (0.17) with (0.18) has a positive scalar curvature $R = 4\Lambda$. Analogously, one gets back a spacetime with $R = -4\,|\Lambda|$ or $R = 0$ in case of $\Lambda < 0$ respectivly $\Lambda = 0$. But metric (0.17) with (0.18) remains a real valued solution only in case of $\Lambda > 12l^2$. Changing the signature of the metric into $(+,-,-,-)$, cf. [5, 6], just reverses the algebraic sign of the scalar curvature.

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[4]
```
load(ctensor); dim: 4; ct_coords: [t,q,theta,phi];
a:exp((sqrt(4*l^2+Lambda/3)*c*t)); v:(l*a*q)^2;
lg: matrix([-c^2*((1+v)/(1-v))^2,0,0,0],[0,(a/(1-v))^2,0,0],[0,0,q^2*(a/(1-v))^2,0],[0,0,0,q^2*(sin(theta)^2)*(a/(1-v))^2]);
cmetric(); einstein(dis); scurvature(); ratsimp(tracer);
```

[5]
```
/* calculate the Kretschmann scalar */ lriemann(false); uriemann(false); rinvariant(); ratsimp(%);
```

**6. De Sitter and Anti-de Sitter in $D$ dimensions**

Physical theories like String theorie and anti-de Sitter/conformal field theory correspondence include spacetimes with more than four dimensions: "*Recently, it has been proposed by Maldacena that large $N$ limits of certain conformal field theories in $d$ dimensions can be described in terms of supergravity (and string theory) on the product of $d+1$-dimensional AdS space with a compact manifold.*" cf. [13]. The rising interest in higherdimensional theories motivates to ask whether our model can be extended also. This section is concerned with the generalization of the previous results to $D \in \mathbb{N}$ dimensions. In the following $d\Omega^2_{D-2}$ denotes the line element on the unit $(D-2)$ −sphere:

$$d\Omega^2_{D-2} = d\theta_1^2 + \sin^2(\theta_1)\, d\theta_2^2 + \sin^2(\theta_1)\sin^2(\theta_2)\, d\theta_3^2 + \cdots + \prod_{k=1}^{D-3} \sin^2(\theta_k)\, d\theta^2_{D-2}$$

Anti-de Sitter space $AdS_D$ in $D$ dimensions with respect to the coordinates $\{t,\, r,\, \theta_1, \ldots, \theta_{D-2}\}$ is given by

$$ds^2 = -\alpha(r)\, c^2 dt^2 + \alpha^{-1}(r)\, dr^2 + r^2 d\Omega^2_{D-2}\,, \qquad \alpha(r) = 1 + u_D^2 r^2. \tag{0.19}$$

cf. for example [4], respectively for the de Sitter case [1]. The constant $u_D$ used here depends on the dimension $D$ and is given by the following series of numers:

$$u_D^2 = \frac{2\,|\Lambda|}{(D-2)(D-1)} \tag{0.20}$$

In case of four dimensions we get $u_4 = |\Lambda|/3$ and metric (0.19) reduces to (0.2). With respect to a system of isotropic coordinates $\{t,\, q,\, \theta_1, \ldots, \theta_{D-2}\}$ metric (0.19) takes the form

$$ds^2 = -A(q)\, c^2 dt^2 + B(q)\left(dq^2 + q^2 d\Omega^2_{D-2}\right) \tag{0.21}$$

with suitable functions $A$ and $B$. These functions are determined by comparison of (0.19) and (0.21):

$$\alpha(r) = A(q)\ , \quad \alpha^{-1}(r)\, dr^2 = B(q)\, dq^2\, , \quad r^2 = q^2 B(q) \tag{0.22}$$

Second and third condition can be combined to get $\alpha^{-1}(r)\, dr^2 = \left(r^2/q^2\right) dq^2$. From that we receive the differential equation $q^{-1}dq = \pm r^{-1}\alpha^{-\frac{1}{2}} dr$ and therewith

$$q = q_0 \exp\left(\pm \int_{r_0}^{r} \frac{dx}{x\sqrt{\alpha(x)}}\right) = q_0 \exp\left(\pm \int_{r_0}^{r} \frac{dx}{x\sqrt{1+u_D^2 x^2}}\right) \tag{0.23}$$

where $q_0$ contains the constants of integration.

**Lemma 2.** *Let $u \in \mathbb{R}$ be an arbitrary constant. It is*

$$\int \frac{dx}{x\sqrt{1+u^2x^2}} = -\mathit{Arsinh}\left(\frac{1}{ux}\right) + c_0 \tag{0.24}$$

*Proof.* The integral can be solved by substitution $x = \dfrac{1}{uz}$ from which we get $dx = -\dfrac{dz}{uz^2} = -\dfrac{1}{uz}\cdot\dfrac{dz}{z} = -x \cdot \dfrac{dz}{z}$ and

$$\int \frac{dx}{x\sqrt{1+u^2x^2}} = -\int \frac{dz}{z\sqrt{1+\frac{1}{z^2}}} = -\int \frac{dz}{\sqrt{z^2+1}} = -\mathrm{Arsinh}(z) + c_0$$

□

With Lemma 2 it is easy to get the $q$ coordinate from (0.23). At first we consider only the positive branch of "±" in (0.23). Together with the negative algebraic sign in (0.24) one gets $\frac{q}{q_0} = \exp\left[\text{-Arsinh}\left(\frac{1}{u_D r}\right)\right]$. The multiplicative inverse of this equation reads

$$\exp\left[\mathrm{Arsinh}\left(\frac{1}{u_D r}\right)\right] = \frac{q_0}{q}.$$

Taking logarithm and hyperbolic sine of this equation leads to:

$$\frac{1}{u_D r} = \sinh\left[\ln\left(\frac{q_0}{q}\right)\right] = \frac{1}{2}\left\{\exp\left[\ln\left(\frac{q_0}{q}\right)\right] - \exp\left[-\ln\left(\frac{q_0}{q}\right)\right]\right\} = \frac{1}{2}\left\{\frac{q_0}{q} - \frac{q}{q_0}\right\} = \frac{q_0}{2q}\left(1 - \frac{q^2}{q_0^2}\right)$$

Finally we get the transformation for the radial $r-$coordinate from that. With the choice of $q_0 = 2/u_D$ one receives

$$r = \frac{q}{1 - \frac{1}{4}u_D^2 q^2} = \frac{q}{1 - \frac{|\Lambda|}{2(D-2)(D-1)}q^2}.$$

Together with the matching conditions (0.22) one gets

$$A(q) \;=\; 1 + u_D^2\left(\frac{q}{1 - \frac{1}{4}u_D^2 q^2}\right)^2 \;=\; \left(\frac{1 + \frac{1}{4}u_D^2 q^2}{1 - \frac{1}{4}u_D^2 q^2}\right)^2, \qquad B(q) \;=\; \frac{1}{\left(1 - \frac{1}{4}u_D^2 q^2\right)^2}$$

so that the $D-$dimensional $AdS$ metric finally takes the form

$$ds^2 = -\left(\frac{1 + \frac{|\Lambda|}{2(D-2)(D-1)}q^2}{1 - \frac{|\Lambda|}{2(D-2)(D-1)}q^2}\right)^2 c^2 dt^2 + \frac{1}{\left(1 - \frac{|\Lambda|}{2(D-2)(D-1)}q^2\right)^2}\left(dq^2 + q^2 d\Omega_{D-2}^2\right) \tag{0.25}$$

**7. Basic line element in higher dimensions**

Corresponding to (0.25), it would be interesting to examine whether metric (0.5) with $d\Omega_2^2$ replaced by $d\Omega_{D-2}^2$ provides a solution of Einstein's equations. The basic ansatz is given by

$$ds^2 = -\left(\frac{1 - l^2 a^2(t) q^2}{1 + l^2 a^2(t) q^2}\right)^2 c^2 dt^2 + \frac{a^2(t)}{\left(1 + l^2 a^2(t) q^2\right)^2}\left(dq^2 + q^2 d\Omega_{D-2}^2\right) \tag{0.26}$$

Here we consider the case $D = 5$ in detail. Einstein's empty space equations for the five-dimensional line element reduce to[6]

$$6\frac{\dot{a}^2}{c^2 a^2} \;=\; 24l^2 + \Lambda \tag{0.27}$$

$$\frac{\frac{\dot{a}^2}{a^2}(9v+3) - \frac{\ddot{a}}{a}(3v-3)}{c^2(v+1)} \;=\; 24l^2 + \Lambda \tag{0.28}$$

where $v$ is again given by (0.6). Since equation (0.27) reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(4l^2 + \frac{\Lambda}{6}\right)c^2 \tag{0.29}$$

one can use lemma 1 on the left side of (0.28):

$$\frac{\frac{\dot{a}^2}{a^2}(9v+3) - \frac{\ddot{a}}{a}(3v-3)}{c^2(v+1)} = \left(4l^2 + \frac{\Lambda}{6}\right)\frac{6v+6}{v+1} = 12l^2 + \Lambda$$

The $D = 5$ case of (0.26) is a real-valued solution to Einstein's empty space equations for

$$a(t) = a_0 \exp\left(\pm\sqrt{4l^2 + \frac{\Lambda}{6}}\, c\, t\right)$$

with $\Lambda > -24l^2$. It is an obvious guess that in general, the prefactors of $\Lambda$ constitute similar to those in (0.20) and the $D$ dimensional solution is given by (0.26) where

$$a(t) = a_0 \exp\left(\pm\sqrt{4l^2 + \frac{2\Lambda}{(D-2)(D-1)}}\, c\, t\right). \tag{0.30}$$

**Conclusion**

With the coordinate replacement method we obtained a basic line element, partly based on the $AdS$ intervall, which leads to new solutions of Einstein's field equations. Similar to the de Sitter, $AdS$ and Minkowski case, these solutions have constant positive, negative or zero scalar curvature. However, the coordinate replacement which is applied on the $AdS$ metric is basically different from a coordinate transformation. The results can be generalised to $D$ dimensions.

[6] load(ctensor); dim: 5; ct_coords: [t,q,theta,phi,alpha];
v:(l*a(t)*q)^2; lg: matrix([-c^2*((1+v)/(1-v))^2,0,0,0,0],[0,(a(t)/(1-v))^2,0,0,0],[0,0,q^2*(a(t)/(1-v))^2,0,0],
[0,0,0,q^2*(sin(theta)^2)*(a(t)/(1-v))^2,0], [0,0,0,0,q^2*(sin(theta)^2)*(sin(phi)^2)*(a(t)/(1-v))^2]);
cmetric(); einstein(dis); trigsimp(ratsimp(ein[1,1])); trigsimp(ratsimp(ein[2,2])); trigsimp(ratsimp(ein[3,3]));
trigsimp(ratsimp(ein[4,4])); trigsimp(ratsimp(ein[5,5]));

After a detailed consideration of the five dimensional case, a conjecture for a new $D-$ dimensional solution of Einstein's equations is proposed. This spacetime is given by metric (0.26) with (0.30).

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