The quantum theory of a closed string.

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Abstract

The Virasoro problem of string theory is traced back to the non-intrinsic character of the dynamics of string theory, meaning that the dynamics depends too much upon the normal directions to the string. This is the disadvantage of the worldsheet formulation of Polyakov as well as Nambu and Goto and bcomes particularly clear in the context of covariant quantum theory.

1 Introduction.

The Virasoro problem in string theory arises most clearly in the covariant quantization where one has hermitian generators L_n with $n \in \mathbb{Z}$ which have to be regarded as constraints; that is physical states have to satisfy $L_n|\Psi\rangle = 0$ for $n \neq 0$ and $L_0|\Psi\rangle = a|\Psi\rangle$ with $a \neq 0$. The Virasoro algebra without central anomalies $c(n)$,

$$[L_n, L_m] = i(n-m)L_{n+m} + c(n-m)1$$

makes this impossible given that

$$0 = [L_n, L_{-n}]\,|\Psi\rangle = 2inL_0|\Psi\rangle = 2ina|\Psi\rangle$$

which contradicts $a \neq 0$. The "fix" of the problem is to keep the constraints $L_n|\psi\rangle = 0$ for $n > 0$ while dropping the others. This leads to physical operators changing particle species, spin and angular momentum causing all known conservation laws of particle physics to fail (but not largely in practice). The downside is that the geometrical description of the theory is totally lost at the quantum level even in a Minkowski background and that everything becomes therefore gauge dependent. This is not expected given that quantum theory works perfectly fine for flat geometries and we shall trace back the problem to the non-geometric character of quantum theory itself. In that context, the worldsheet formulation evaporates and only reparametrisations of the type $t'(t)$ and $\theta'(\theta)$ can be made such that the Virasoro problem dissapears giving rise to two mutually commuting symmetry algebra's as the *full* symmetry algebra.

2 Strings from the viewpoint of covariant quantum theory.

Given a closed string worldsheet $\gamma(t, \theta)$, we define two vectorfields $\mathbf{V} = \partial_t \gamma(t, s)$ and $\mathbf{Z} = \partial_s \gamma(t, s)$ where $t \in [0, T]$ and $\theta \in [0, 2\pi]$ with periodic boundary conditions; obviously $[\mathbf{V}, \mathbf{Z}] = 0$.

The law one is looking for clearly is of the kind

$$\nabla_{\mathbf{V}}\mathbf{V} = \mathbf{F}(\mathbf{V}, \mathbf{Z}, \nabla_{\mathbf{Z}}\mathbf{Z}, \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{R}(\mathbf{Z}, \mathbf{V})\mathbf{Z}, \mathbf{h})$$

where $\mathbf{A} = \nabla_{\mathbf{Z}}\mathbf{Z}$ is a kind of acceleration, $\mathbf{h}$ the, possibly degenerate, metric on the string and all Riemann curvature terms involve the intrinsic geometry of the string. The problem so far is that the velocity field $\mathbf{V}$ is randomly chosen and that therefore it is desirable to impose constraints on $\nabla_{\mathbf{V}}\mathbf{Z}$. We have basically two types: (a) one involving the extrinsic geometry and the latter only the intrinsic geometry. In other words, we have (I)

$$\nabla_{\mathbf{V}}\mathbf{Z} = \mathbf{G}(\mathbf{V}, \mathbf{Z}, \nabla_{\mathbf{Z}}\mathbf{Z}, \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{R}(\mathbf{Z}, \mathbf{V})\mathbf{Z}, \mathbf{h})$$

or (II)

$$\mathbf{g}(\nabla_{\mathbf{V}}\mathbf{Z}, \mathbf{V}) = P(\mathbf{V}, \mathbf{Z}, \nabla_{\mathbf{Z}}\mathbf{Z}, \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{R}(\mathbf{Z}, \mathbf{V})\mathbf{Z}, \mathbf{h})$$

whereas a condition of the kind

$$\mathbf{g}(\nabla_{\mathbf{V}}\mathbf{Z}, \mathbf{Z}) = Q(\mathbf{V}, \mathbf{Z}, \nabla_{\mathbf{Z}}\mathbf{Z}, \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{R}(\mathbf{Z}, \mathbf{V})\mathbf{Z}, \mathbf{h})$$

is meaningless given that consistency would bring it down to an algebraic condition on $\mathbf{g}(\mathbf{Z}, \mathbf{Z})$. Such theories are usually empty and therefore not interesting at all.

One has to demand now that the dynamics preserves the constraint; that is

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{Z} = \nabla_{\mathbf{V}}\mathbf{G} = \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V} + \nabla_{\mathbf{Z}}\mathbf{F}$$

a consistency condition. Note that

$$\nabla_{\mathbf{Z}}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}) = (\nabla_{\mathbf{Z}}\mathbf{R})(\mathbf{V}, \mathbf{Z})\mathbf{V} - \mathbf{R}(\mathbf{G}, \mathbf{Z})\mathbf{V} + \mathbf{R}(\mathbf{V}, \mathbf{A})\mathbf{V} - \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{G}$$

which can be reduced to, by means of the second Bianchi identity to

$$\nabla_{\mathbf{Z}}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}) = (\nabla_{\mathbf{V}}\mathbf{R})(\mathbf{V}, \mathbf{Z})\mathbf{Z} + \mathbf{R}(\mathbf{Z}, \mathbf{G})\mathbf{V} + \mathbf{R}(\mathbf{V}, \mathbf{A})\mathbf{V} - \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{G}.$$

On the other hand, a similar compuation gives that

$$\nabla_{\mathbf{V}}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{Z}) = (\nabla_{\mathbf{V}}\mathbf{R})(\mathbf{V}, \mathbf{Z})\mathbf{Z} + \mathbf{R}(\mathbf{F}, \mathbf{Z})\mathbf{Z} + \mathbf{R}(\mathbf{V}, \mathbf{G})\mathbf{Z} + \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{G}$$

where no second Bianchi identity has been used and the other terms do not allow for comparison between $\mathbf{F}$ and $\mathbf{G}$ by means of the latter identity. Contractions with the spacetime metric do allow for further use of the first Bianchi identity and gives rise to a larger margin to construct stringy laws. Hence, in light of the conservation law for the constraint,

$$\mathbf{F}(\mathbf{V}, \mathbf{Z}, \nabla_{\mathbf{Z}}\mathbf{Z}, \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{h})$$

and

$$\mathbf{G}(\mathbf{V}, \mathbf{Z}, \nabla_{\mathbf{Z}}\mathbf{Z}, \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{Z}, \mathbf{h})$$

with

$$\frac{\delta \mathbf{F}}{\delta \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}} = \frac{\delta \mathbf{G}}{\delta \mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{Z}}.$$

We also have that

$$\frac{\delta \mathbf{G}}{\delta \mathbf{Z}} \triangle \mathbf{G} + \frac{\delta \mathbf{G}}{\delta \mathbf{V}} \triangle \mathbf{F} + \frac{\delta \mathbf{G}}{\delta \mathbf{A}} \triangle \nabla_{\mathbf{V}} \mathbf{A}$$

$$= \frac{\delta \mathbf{F}}{\delta \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}} \triangle (-\mathbf{R}(\mathbf{G},\mathbf{Z})\mathbf{V} + \mathbf{R}(\mathbf{V},\mathbf{A})\mathbf{V} - \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{G} - \mathbf{R}(\mathbf{F},\mathbf{Z})\mathbf{Z} - \mathbf{R}(\mathbf{V},\mathbf{G})\mathbf{Z} - \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{G})$$

$$+ \frac{\delta \mathbf{F}}{\delta \mathbf{Z}} \triangle \mathbf{A} - \frac{\delta \mathbf{F}}{\delta \mathbf{V}} \triangle \mathbf{G} + \frac{\delta \mathbf{F}}{\delta \mathbf{A}} \triangle \nabla_{\mathbf{Z}} \mathbf{A} + \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}.$$

From a generalist point of view, this would suggest

$$-\mathbf{R}(\mathbf{G},\mathbf{Z})\mathbf{V} + \mathbf{R}(\mathbf{V},\mathbf{A})\mathbf{V} - \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{G} - \mathbf{R}(\mathbf{F},\mathbf{Z})\mathbf{Z} - \mathbf{R}(\mathbf{V},\mathbf{G})\mathbf{Z} - \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{G} = 0$$

as well as

$$\frac{\delta \mathbf{G}}{\delta \mathbf{Z}} \triangle \mathbf{G} + \frac{\delta \mathbf{G}}{\delta \mathbf{V}} \triangle \mathbf{F} + \frac{\delta \mathbf{G}}{\delta \mathbf{A}} \triangle \nabla_{\mathbf{V}} \mathbf{A} = \frac{\delta \mathbf{F}}{\delta \mathbf{Z}} \triangle \mathbf{A} - \frac{\delta \mathbf{F}}{\delta \mathbf{V}} \triangle \mathbf{G} + \frac{\delta \mathbf{F}}{\delta \mathbf{A}} \triangle \nabla_{\mathbf{Z}} \mathbf{A} + \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}.$$

It is immediately seen that, in general and independent of this ansatz,

$$\frac{\delta \mathbf{F}}{\delta \mathbf{A}} = 0$$

given that higer spatial derivatives do not occur elsewhere in the formula and therefore

$$\mathbf{F}(\mathbf{V},\mathbf{Z},\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V})$$

given that we have already neglected $\mathbf{h}$. On the other hand

$$\nabla_{\mathbf{V}} \mathbf{A} = \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{Z} + \nabla_{\mathbf{Z}} \mathbf{G}$$

which implies that

$$\frac{\delta \mathbf{G}}{\delta \mathbf{A}} = 0$$

due to consistency given that no algebraic relations are allowed for between higher spatial derivatives. Hence,

$$\mathbf{G}(\mathbf{V},\mathbf{Z},\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{Z})$$

and we conclude from the remaining master equation that only intrinsic contractions of the Riemann tensor with $\mathbf{V},\mathbf{Z}$ are allowed for to eliminate the nasty

$$\mathbf{R}(\mathbf{V},\mathbf{A})\mathbf{V}$$

term. This however happens in two different ways $\mathbf{g}(\mathbf{R}(\mathbf{V},\mathbf{A})\mathbf{V},\mathbf{V}) = 0$ identically whereas contractions of the kind

$$\mathbf{g}(\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V},\mathbf{Z})$$

require a balancing between

$$\frac{\delta \mathbf{F}}{\delta \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}} \triangle \mathbf{R}(\mathbf{V},\mathbf{A})\mathbf{V}$$

and

$$\frac{\delta \mathbf{F}}{\delta \mathbf{Z}} \triangle \mathbf{A}$$

in the sense that they have to be equal to one and another due to the first Bianchi identity. As a conclusion, we further specify that

$$\mathbf{F} = X(\mathbf{V}, \mathbf{Z}, \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{Z}))\mathbf{V} + Y(\mathbf{V}, \mathbf{Z}, \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{Z}))\mathbf{Z}$$

which automatically satifies this requirement by means of symmetries of the Riemann tensor. This further limits

$$\mathbf{G} = R(\mathbf{V}, \mathbf{Z}, \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{Z}, \mathbf{Z}))\mathbf{V} + S(\mathbf{V}, \mathbf{Z}, \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{Z}, \mathbf{V}))\mathbf{Z}$$

with

$$\frac{\delta X}{\delta \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{Z})} = -\frac{\delta R}{\delta \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{Z}, \mathbf{Z})}$$

and

$$\frac{\delta Y}{\delta \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{Z})} = -\frac{\delta S}{\delta \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{Z}, \mathbf{Z})}.$$

This shows that ∇ cannot be the Christoffel connection of a Riemannian metric and $\mathbf{R}$ its associated Riemann tensor. Although the Riemann tensor of any connection satisfies the second Bianchi identies, the first Bianchi identies and the associated symmetries of the Riemann tensor follow from the metric and torsionless character. Therefore, the connection needs torsion for the subsequent analysis to hold.

Given that one would expect only curvature to occur in the acceleration law of the string and moreover that the acceleration is of the geodesic type so that the sring t coordinate is nothing but a rescaling of the geodesic time, reparamentrization invariance has to be given up in the light of the fact that no $g(\mathbf{Z}, \mathbf{Z})$ or $g(\mathbf{Z}, \mathbf{V})$ terms may occur due to an inappropriate appearance of $\mathbf{A}$ in the

$$\frac{\delta \mathbf{F}}{\delta \mathbf{Z}} \triangle \mathbf{A}$$

term. Therefore,

$$\nabla_{\mathbf{V}} \mathbf{V} = \frac{c}{L^3} \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{V}, \mathbf{Z})\mathbf{V}$$

where c is the speed of light and L has units of meters. This is the correct way of looking at it given that the curves are ordinary geodesics again but then reparametrized in a way as to balance the tidal forces; t can be reparametrized but generally speaking only *one* worldline of a point of the circle can have unit time parametrization. This is a salient feature given that strings will not induce superluminal effects in this way by means of its nonlocal character. In particular, we have that if x is a point past to the string and $\mathbf{V}$ is a future pointing timelike vectorfield, then the entire string will remain within $I^+(x)$. Finally,

$$\nabla_{\mathbf{V}} \mathbf{Z} = K(\mathbf{g}(\mathbf{V}, \mathbf{Z}), \mathbf{g}(\mathbf{V}, \mathbf{V}), \mathbf{g}(\mathbf{Z}, \mathbf{Z}))\mathbf{V} +$$
$$-\frac{c}{L^3} \mathbf{g}(\mathbf{R}(\mathbf{V}, \mathbf{Z})\mathbf{Z}, \mathbf{Z}))\mathbf{V} + L(\mathbf{g}(\mathbf{V}, \mathbf{Z}), \mathbf{g}(\mathbf{V}, \mathbf{V}), \mathbf{g}(\mathbf{Z}, \mathbf{Z}))\mathbf{Z}.$$

The consistency equation has now been reduced to

$$\frac{\delta \mathbf{G}}{\delta \mathbf{Z}} \triangle \mathbf{G} + \frac{\delta \mathbf{G}}{\delta \mathbf{V}} \triangle \mathbf{F} =$$

$$-\frac{\delta \mathbf{F}}{\delta \mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}}(\mathbf{R}(\mathbf{G},\mathbf{Z})\mathbf{V}+\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{G}+\mathbf{R}(\mathbf{F},\mathbf{Z})\mathbf{Z}+\mathbf{R}(\mathbf{V},\mathbf{G})\mathbf{Z}+\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{G})$$

$$-\frac{\delta \mathbf{F}}{\delta \mathbf{V}}\triangle \mathbf{G}+\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}.$$

As expected one page ago, this equation can only have solution in case $\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}$ equals its projection on the string worldsheet determined by the $\mathbf{V},\mathbf{Z}$ plane which is in general impossible except for Einstein spaces. Therefore, it might be possible to develop a type I string theory for Einstein spaces with torsion but given such restriction it is utterly clear that type II is the only physical case.

Here, we might try to arrive at a theory with equation of motion

$$\nabla_{\mathbf{V}}\mathbf{V}=\mathbf{F}$$

and constraint equations

$$\begin{aligned}\mathbf{g}(\mathbf{V},\nabla_{\mathbf{V}}\mathbf{Z}) &= \alpha\mathbf{g}(\mathbf{V},\mathbf{G})\\ \mathbf{g}(\mathbf{E}_i,\nabla_{\mathbf{V}}\mathbf{Z}) &= \alpha\mathbf{g}(\mathbf{E}_i,\mathbf{G})\\ \frac{1}{2}\mathbf{g}(\nabla_{\mathbf{V}}\mathbf{Z},\nabla_{\mathbf{V}}\mathbf{Z}) &= \alpha\mathbf{g}(\nabla_{\mathbf{V}}\mathbf{Z},\mathbf{G})\end{aligned}$$

where E_i is a $n-2$ bein orhogonal to $\mathbf{V},\mathbf{Z}$. In vector language, this gives

$$\nabla_{\mathbf{V}}\mathbf{Z}-\alpha\mathbf{G}=\mathbf{W}$$

with $\mathbf{W}$ perpendicular to the $n-1$ plane defined by $\mathbf{V},\mathbf{E}_i$. Moreover,

$$\mathbf{g}(\mathbf{W}-\alpha\mathbf{G},\nabla_{\mathbf{V}}\mathbf{Z})=0.$$

Hence,

$$\mathbf{g}(\mathbf{W},\mathbf{W})=\alpha^2\mathbf{g}(\mathbf{G},\mathbf{G}).$$

The structure of these equations is as such that they are preserved during time evolution. Time evolution of the first gives

$$\mathbf{g}(\mathbf{F},\nabla_{\mathbf{V}}\mathbf{Z})+\mathbf{g}(\mathbf{V},\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V})+\mathbf{g}(\mathbf{V},\nabla_{\mathbf{Z}}\mathbf{F})=$$

$$\alpha\mathbf{g}(\mathbf{F},\mathbf{G})+\alpha\mathbf{g}(\mathbf{V},\nabla_{\mathbf{V}}\mathbf{G})$$

which generically leads to

$$\mathbf{g}(\mathbf{F},\nabla_{\mathbf{V}}\mathbf{Z}-\alpha\mathbf{G})=0,\ \ \mathbf{g}(\mathbf{V},\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}+\nabla_{\mathbf{Z}}\mathbf{F}-\alpha\nabla_{\mathbf{V}}\mathbf{G})=0.$$

The other equations are

$$\mathbf{g}(\mathbf{E}_i,\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}+\nabla_{\mathbf{Z}}\mathbf{F}-\alpha\nabla_{\mathbf{V}}\mathbf{G})=0$$

and

$$\mathbf{g}(\nabla_{\mathbf{V}}\mathbf{Z},\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V}+\nabla_{\mathbf{Z}}\mathbf{F}-\alpha\nabla_{\mathbf{V}}\mathbf{G})=0$$

supplemented with

$$\mathbf{g}(\nabla_{\mathbf{Z}}\mathbf{F}+\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V},\mathbf{G})=0$$

which obviously gives the same problems as before. It appears some more delicate analysis is necessary: clearly, one would like

$$\nabla_{\mathbf{V}}\mathbf{V} = \mathbf{F}(\mathbf{V},\mathbf{Z},\mathbf{A},\mathbf{g}(\mathbf{R}(\mathbf{V},\mathbf{Z})\mathbf{V},\mathbf{Z}))$$

given that $\mathbf{Z}$ is chosen according to arc length and evolution should only depend upon the intrinsic geometry and only as far on the directions perpendicular to the infinitesimal string surface as the acceleration goes. That is

$$\mathbf{g}(\mathbf{Z},\mathbf{A}) = 0$$

and one would like to preserve this property under evolution, in either keep it as a constraint. Time evolution gives

$$\mathbf{g}(\nabla_{\mathbf{V}}\mathbf{Z},\mathbf{A}) + \mathbf{g}(\mathbf{Z},\nabla_{\mathbf{Z}}\nabla_{\mathbf{V}}\mathbf{Z}) = 0 = \nabla_{\mathbf{Z}}(\mathbf{g}(\mathbf{Z},\nabla_{\mathbf{V}}\mathbf{Z})).$$

Therefore, we should add as constraint

$$\mathbf{g}(\mathbf{Z},\nabla_{\mathbf{V}}\mathbf{Z}) = \nabla_{\mathbf{Z}}\mathbf{g}(\mathbf{Z},\mathbf{V}) - \mathbf{g}(\mathbf{A},\nabla_{\mathbf{V}}\mathbf{V}) = 0$$

which follows from

$$\mathbf{g}(\mathbf{Z},\mathbf{V}) = \mathbf{g}(\mathbf{A},\mathbf{V}) = 0.$$

In ordinary string theory in flat Minkowski $\mathbf{F} = \mathbf{A}$ and the latter two conditions give by means of the equation of motion

$$\mathbf{g}(\mathbf{Z},\mathbf{V}) = \nabla_{\mathbf{V}}\mathbf{g}(\mathbf{V},\mathbf{V}) = 0.$$

The first of those is the usual Virasoro constraint

$$\partial_t X.\partial_x X = 0$$

whereas the second equals

$$\partial_t(\partial_t X.\partial_t X) = 0$$

which is the time derivative of one of the other constraints. Our original constraint was

$$\partial_x(\partial_x X.\partial_x X) = 0$$

which is the space derivative of the last Virasoro constraint. It is now possible to *impose* the constraints

$$\mathbf{g}(\mathbf{Z},\mathbf{V}) = 0 = \mathbf{g}(\mathbf{V},\mathbf{V})$$

where we have eliminated one integration function depending upon x only. Similarly, we could demand that

$$\mathbf{g}(\mathbf{Z},\mathbf{Z}) = 0$$

where we have eliminated a space integration constant β arising from

$$\nabla_{\mathbf{Z}}\mathbf{g}(\mathbf{Z},\mathbf{Z}) = 2\mathbf{g}(\mathbf{A},\mathbf{Z}) = 0.$$

We show now that the remaining two constraints close under time evolution

$$\nabla_{\mathbf{V}}\mathbf{g}(\mathbf{V},\mathbf{Z}) = \mathbf{g}(\mathbf{F},\mathbf{Z}) + \mathbf{g}(\mathbf{V},\nabla_{\mathbf{V}}\mathbf{Z}) = \mathbf{g}(\mathbf{F},\mathbf{Z})$$

where we have used the torsionless character of the Levi Civita connection and the commuting of the coordinate fields. This does not impose any constraints on the $\mathbf{F}$ field given the constraints. Finally

$$\nabla_{\mathbf{V}}\mathbf{g}(\mathbf{V},\mathbf{V}) = 2\mathbf{g}(\mathbf{V},\mathbf{F}) = 0$$

for similar reasons. Actually, in case $\beta \neq 0$, a restriction on $\mathbf{F}$ occurs in the sense that the $\mathbf{Z}$ vector dependency has to vanish. Hence, the most general case for $\mathbf{F}$ is the one with the ordinary Virasoro constraints

$$\mathbf{g}(\mathbf{Z},\mathbf{Z}) = \mathbf{g}(\mathbf{V},\mathbf{Z}) = \mathbf{g}(\mathbf{V},\mathbf{V}) = 0.$$

Classically, these equations cannot be solved for in a spacetime with a Lorentzian signature unless $\mathbf{V} \sim \mathbf{Z}$ which is rather boring and actually occurs in "string theory". It therefore appears clear that standard string theory would require at least two independent time directions which would endager the whole edifice of causality and make no sense at all unless those time directions are compactified of some sort and far beyond our scale of observation. Hence, a fibre structure is needed for the spacetime manifold with a four dimensional Lorentzian base manifold and Lorentzian fibre. In standard quantum theory of the string, one solves for right and left moving strings which should be kept strictly separate to impose the constraints. Alas such, line of reasoning inconsistent with the Heisenberg commutation relations, hence the Virasoro problem. In our setup, there are two possibilities, either one keeps $\beta < 0$ so that $\mathbf{g}(\mathbf{Z},\mathbf{Z}) = \beta$ and therefore $\mathbf{Z}$ is always spacelike involving $\mathbf{F}(\mathbf{V},\mathbf{A})$ *or* one goes over to the higher time formalism such that the projection of $\mathbf{Z}$ on the base manifold is spacelike and varying in case the fibre is one dimensional and the standard Virasoro picture with a more general force field may hold.

We now discuss these things in the next section.

3 Quantization of the string.

References

[1] J. Noldus, Foundations of a theory of quantum gravity, Vixra:1106.0028