Generalized Lorentz Transformations

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This article presents the generalized Lorentz transformations of time, space, velocity and acceleration which can be applied in any inertial or non-inertial (non-rotating) frame.

Introduction

If we consider a (non-rotating) frame S relative to another inertial frame Σ then the time (t), the position ($\mathbf{r}$), the velocity ($\mathbf{v}$) and the acceleration ($\mathbf{a}$) of a (massive or non-massive) particle relative to the frame Σ are given by:

$$t \,=\, \int_0^{\mathtt{t}} \gamma \, \mathtt{dt} \,-\, \gamma \, \frac{\vec{r} \cdot \mathbf{V}}{c^2} \,+\, \mathtt{k}$$

$$\mathbf{r} \,=\, \vec{r} \,+\, \frac{\gamma^2}{\gamma+1} \, \frac{(\, \vec{r} \cdot \mathbf{V} \,) \, \mathbf{V}}{c^2} \,-\, \mathbf{R} \,-\, \frac{\gamma^2}{\gamma+1} \, \frac{(\, \mathbf{R} \cdot \mathbf{V} \,) \, \mathbf{V}}{c^2}$$

$$\mathbf{v} \,\doteq\, \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a} \,\doteq\, \frac{d\mathbf{v}}{dt}$$

where ($\mathtt{t}$, $\vec{r}$) are the time and the position of the particle relative to the frame S ($\mathbf{R}, \mathbf{V}, \mathbf{A}$) are the position, the velocity and the acceleration of the origin of the frame Σ relative to the frame S, ($\mathtt{k}$) is a particular constant between the frames Σ & S, (c) is the speed of light in vacuum, and $\gamma \,\doteq\, (1 - \mathbf{V} \cdot \mathbf{V} / c^2)^{-1/2}$

- $\frac{\gamma^2}{\gamma + 1} \frac{1}{c^2} = \frac{\gamma - 1}{\mathbf{V}^2} \qquad (\, \mathbf{V}^2 = \mathbf{V} \cdot \mathbf{V} \,)$

- $\vec{r} + \frac{\gamma^2}{\gamma + 1} \frac{(\, \vec{r} \cdot \mathbf{V} \,)\, \mathbf{V}}{c^2} = \gamma\, \vec{r} + \frac{\gamma^2}{\gamma + 1} \frac{(\, \vec{r} \times \mathbf{V} \,) \times \mathbf{V}}{c^2}$

- $\mathbf{R} + \frac{\gamma^2}{\gamma + 1} \frac{(\, \mathbf{R} \cdot \mathbf{V} \,)\, \mathbf{V}}{c^2} = \gamma\, \mathbf{R} + \frac{\gamma^2}{\gamma + 1} \frac{(\, \mathbf{R} \times \mathbf{V} \,) \times \mathbf{V}}{c^2}$

The frame S is inertial when ($\mathbf{A} = 0$)

The frame S is non-inertial (rectilinear accelerated motion) when ($\mathbf{A} \neq 0$) and ($\mathbf{A} \times \mathbf{V} = 0$)

The frame S is non-inertial (uniform circular motion) when ($\mathbf{A} \neq 0$) and ($\mathbf{A} \cdot \mathbf{V} = 0$)

If the frame S is inertial then the observer S must use a fixed origin O such that ($\mathbf{R} \times \mathbf{V} = 0$)

If the frame S is non-inertial (rectilinear accelerated motion) then the observer S must use a fixed origin O such that ($\mathbf{R} \times \mathbf{V} = 0$)

If the frame S is non-inertial (uniform circular motion) then the observer S must use a fixed origin O such that ($\mathbf{R} \cdot \mathbf{V} = 0$)

If the frame S is inertial then ($\mathbf{A} = 0$) , ($\mathbf{V} = \text{constant}$) , ($\gamma = \text{constant}$) ($\int_0^{\mathtt{t}} \gamma\, \mathtt{dt} = \gamma\, \mathtt{t}$) , ($\mathbf{R} = \mathbf{V}\, \mathtt{t} + \text{constant}$) and ($\mathbf{R} \times \mathbf{V} = 0$)

If the frame S is non-inertial (rectilinear accelerated motion) then ($\mathbf{A} \neq 0$) ($\mathbf{A} \times \mathbf{V} = 0$) and ($\mathbf{R} \times \mathbf{V} = 0$)

If the frame S is non-inertial (uniform circular motion) then ($\mathbf{A} \neq 0$) ($\mathbf{A} \cdot \mathbf{V} = 0$) , ($\gamma = \text{constant}$) , ($\int_0^{\mathtt{t}} \gamma\, \mathtt{dt} = \gamma\, \mathtt{t}$) and ($\mathbf{R} \cdot \mathbf{V} = 0$)

If the frame S is inertial then the observer S can use test particles such that ($\vec{r} \times \mathbf{V} = 0$)

If the frame S is non-inertial (rectilinear accelerated motion) then the observer S can use test particles such that ($\vec{r} \times \mathbf{V} = 0$)

If the frame S is non-inertial (uniform circular motion) then the observer S can use test particles such that ($\vec{r} \cdot \mathbf{V} = 0$)

General Observations

It is known that in inertial frames the local geometry is Euclidean and that in non-inertial frames the local geometry is in general non-Euclidean.

According to this article, the local line element in the frame S must be obtained from the local line element of the frame Σ.

The local line element in the frame Σ (in rectilinear coordinates) is given by:

- $ds^2 \;=\; c^2 dt^2 - d\mathbf{r}^2$

Additionally, the kinematic quantities ($t, \mathbf{r}, \mathbf{v}, \mathbf{a}$) are the proper kinematic quantities of the frame Σ. Therefore, the kinematic quantity (t) is a tensor of rank 0 and the kinematic quantities ($\mathbf{r}, \mathbf{v}, \mathbf{a}$) are tensors of rank 1.

According to this article, if the frame S is inertial or non-inertial (uniform circular motion) then:

- $\dfrac{d\mathbf{r}}{dt} \;=\; \left(\dfrac{d\mathbf{r}}{\mathtt{dt}} + \Omega \times \mathbf{r} \right) \left(\dfrac{1}{dt \,/\, \mathtt{dt}} \right)$

- $\dfrac{d\mathbf{v}}{dt} \;=\; \left(\dfrac{d\mathbf{v}}{\mathtt{dt}} + \Omega \times \mathbf{v} \right) \left(\dfrac{1}{dt \,/\, \mathtt{dt}} \right)$

- $\Omega \;\doteq\; \dfrac{\gamma^2}{\gamma + 1} \, \dfrac{(\, \mathbf{A} \times \mathbf{V} \,)}{c^2}$

Finally, the velocity of light in vacuum is ($\mathbf{c}$) in the frame Σ and ($\vec{c}$) in the frame S and ($\mathbf{c} \cdot \mathbf{c}$) & ($\vec{c} \cdot \vec{c}$) are constant in the frames Σ & S, respectively.

Bibliography

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