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Complexity of Linear and General Cyclic Snake Networks

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Abstract: In this paper we prove that the number of spanning trees of the linear and general cyclic snake networks is the same using the combinatorial approach. We derive the explicit formulas for the subdivided fan network S(F_n) and the subdivided ladder graph S(L_n) . Finally, we calculate their spanning trees entropy and compare it between them.

Key Words: Number of spanning trees, Cyclic snakes networks, Entropy

AMS(2010): 05C05, 05C30

§1. Introduction

The complexity (the number of spanning trees) $\tau(G)$ of a finite connected undirected graph G is defined as the total number of distinct connected acyclic spanning subgraphs. There are many techniques to compute this number. Kirchhoff [1] gave the famous matrix tree theorem. In which $\tau(G)$ = any cofactor of $L\ (G\ )$, where $L\ (G\ )$ is equal to the degree matrix $D\ (G\ )$ of G minus the adjacency matrix $A\ (G\ )$ of G .

There are other methods for calculating $t(G)$. Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_p$ denote the eigenvalues of H matrix of a p point graph. Then it is easily shown that$\mu_p = 0$. In 1974, Kelmans and Chelnokov [2] shown that, $\tau(G) = \frac{1}{p}\prod_{k=1}^{p-1}\mu_k$. The formula for the number of spanning trees in a *d-regular* graph G can be expressed as $t(G) = \frac{1}{p}\prod_{k=1}^{p-1}(d-\mu_k)$ where $\lambda_0 = \lambda_1, \lambda_2, \cdots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such results is due to Cayley [3] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2}$, $n \geqslant 2$. Clark [4] proved that $\tau(K_{p,q}) = p^{q-1}q^{p-1}$, $p, q \geqslant 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively.

[1]Received December 05, 2016, Accepted August 18, 2017.

Therefore, many works derive formulas to calculate the complexity for some classes of graphs. Bogdanowicz [5] derived the explicit formula for the fan network if $n \geqslant 1$,

$$\tau(F_n) = \frac{1}{\sqrt{5}}[(\frac{3+\sqrt{5}}{2})^n - (\frac{3-\sqrt{5}}{2})^n].$$

Sedlacek [6] proposed a formula for the number of spanning trees in a ladder graph. The ladder L_n is the Cartesian product of P_2 and P_n. The number of spanning trees in L_n is given by

$$\tau(L_n) = \frac{\sqrt{3}}{6}[(2+\sqrt{3})^n - (2-\sqrt{3})^n]$$

for $n \geqslant 1$. A. Modabish and M. El Marraki investigated the number of spanning trees in the star flower planar graph [7]. In [8], E.M. Badr and B.Mohamed derived the explicit formulas for triangular snake ($\Delta_k - snake$),double triangular snake ($2\Delta_k - snake$) and the total graph of path $P_n(T(P_n))$. Badr and Mohamed [9] derived the explicit formulas for the subdivision of ladder, fan, wheel, triangular snake (Δ_k-*snake*), double triangular snake ($2\Delta_k$-*snake*) and the total graph of path $P_n(\,T(P_n)\,)$.

In this paper we prove that the number of spanning trees of the linear and general cyclic snake networks is the same using the combinatorial approach. We derive the explicit formulas for the subdivided fan network S(F_n) and the subdivided ladder graph S(L_n) . Finally, we calculate their spanning trees entropy and compare it between them.

§2. Preliminary Notes

The combinatorial method involves the operation of contraction of an edge. An edge e of a graph G is said to be contracted if it is deleted and its ends are identified. The resulting graph is denoted by $G.e$. Also we denote by $G - e$ the graph obtained from G by deleting the edge e.

Theorem 2.1([10]) *Let G be a planar graph (multiple edges are allowed in here). Then for any edge e,*

$$\tau(G) \ = \ \tau(G \ - \ e) \ + \ \tau(G \cdot e).$$

Remark 2.2 If G' is obtained from G by removing all the pendant edges of G, then

$$\tau(G^{'}) = \tau(G).$$

Remark 2.3 If G' is obtained from G by removing all the loops of G, then $\tau(G^{'}) = \tau(G)$.

Remark 2.4 If G' is obtained from G by removing one or more than one multiple edges of G, then $\tau(G^{'}) < \tau(G)$.

Definition 2.5([11]) *A triangular snake (Δ_k-snake) is a connected graph in which all blocks are triangles and the block-cut-point graph is a path.*

Definition 2.6 *C_n-cyclic snake is a connected graph in which all blocks are C_n and the block-cut-point graph is a path. Furthermore, if the length of its path is exactly k, we call it a kC_n-cyclic snake.*

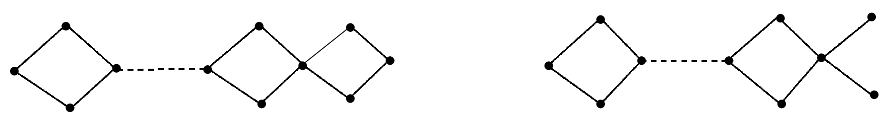
Definition 2.7 *A kC_n -snake is called linear if its block-cut-vertex graph of kC_n -snake has the property that the distance between any two consecutive cut-vertices is $\lfloor \frac{n}{2} \rfloor$.*

§3. Main Results

Theorem 3.1 *The number of spanning trees of the linear kC_4 -snake satisfies the following recursive relation:*

$$\tau(kC_4 - snake) = 4^k$$

Proof Let us consider a graph $kC_4^/ - snake$ constructed from $kC_4 - snake$ by deleting two edges. See Figure 1



KC_4-Snake $\qquad$ $kC_4^/$-Snake

Figure 1 Linear kC_4-Snake

We put $kC_4 - snake = \tau(kC_4 - snake)$ and $kC_4^/ - snake = \tau(kC_4^/ - snake)$.

It is clear that

$$kC_4 - snake = 3(k-1)C_4 - snake + 4(k-1)C_4^/ - snake$$

and

$$kC_4 - snake = 2(k-1)C_4 - snake - 4(k-1)C_4^/ - snake$$

with initial conditions $C_4 - snake = 4$, $C_4^/ - snake = 1$. Thus, we have

$$\begin{pmatrix} kC_4 - snake \\ kC_4^/ - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_4 - snake \\ (k-1)C_4^/ - snake \end{pmatrix},$$

where, $A = \begin{pmatrix} 3 & 4 \\ 2 & -4 \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_4 - snake \\ kC_4^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_4 - snake \\ (k-1)C_4^{/} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_4 - snake \\ C_4^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 20 = 0, \qquad \lambda_1 = -5 \text{ and } \lambda_2 = 4, \ \lambda_1 \neq \lambda_2.$$

Therefore, there is a matrix M invertible such that $A = MBM^{-1}$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{4} \end{pmatrix} \quad \Rightarrow \quad M^{-1} = \frac{1}{9/4} \begin{pmatrix} \frac{1}{4} & -1 \\ 2 & 1 \end{pmatrix}.$$

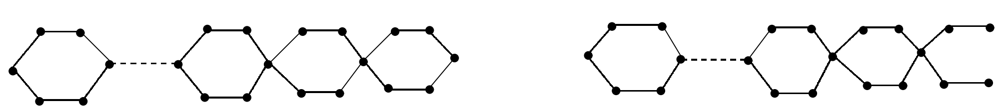
Notice that $A^{n-1} = MB^{n-1}M^{-1}$, where $B^{n-1} = \begin{pmatrix} (-5)^{n-1} & 0 \\ 0 & 4^{n-1} \end{pmatrix}$. We therefore obtain

$$A^{n-1} = \begin{pmatrix} \frac{(-5)^{n-1}}{9} + \frac{2*4^n}{9} & \frac{-4*(-5)^{n-1}}{9} + \frac{4^n}{9} \\ \frac{-2*(-5)^{n-1}}{9} + \frac{2*4^{n-1}}{9} & \frac{8*(-5)^{n-1}}{9} + \frac{4^{n-1}}{9} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.2 *The number of spanning trees of the linear $kc_6 - snake$satisfies the following recursive relation $\tau(\ kc_6 - snake\ ) = 6^k$*

Proof Consider a graph $kC_6^{/} - snake$ constructed from $kC_6 - snake$ by deleting two edges. See Figure 2 following.



kC_6-Snake $\qquad$ $kC_6^{/}$-Snake

Figure 2 Linear kC_6-Snake

We put $kC_6 - snake = \tau(kC_6 - snake)$ and $kC_6^{/} - snake = \tau(kC_6^{/} - snake)$. It is clear that

$$kC_6 - snake = 5((k-1)C_6 - snake) + 6((k-1)C_6^{/} - snake)$$

and

$$kC_6^{/} - snake = 2((k-1)C_6 - snake) - 6((k-1)C_6^{/} - snake)$$

with initial conditions $(C_1 - snake) = 6,\ (C_1^{/} - snake) = 1$. Thus, we have

$$\begin{pmatrix} kC_6 - snake \\ kC_6^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_6 - snake \\ (k-1)C_6^{/} - snake \end{pmatrix},$$

where, $A = \begin{pmatrix} 5 & 6 \\ 2 & -6 \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_6 - snake \\ kC_6^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_6 - snake \\ (k-1)C_6^{/} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_6 - snake \\ C_6^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 42 = 0, \qquad \lambda_1 = -7 \text{ and } \lambda_2 = 6,\ \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MDM^{-1}$, where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{6} \end{pmatrix} \quad \Rightarrow \quad M^{-1} = \begin{pmatrix} \frac{1}{13} & \frac{-6}{13} \\ \frac{12}{13} & \frac{6}{13} \end{pmatrix} \quad \Rightarrow \quad A^{n-1} = MB^{n-1}M^{-1},$$

where $B^{n-1} = \begin{pmatrix} (6)^{n-1} & 0 \\ 0 & (-7)^{n-1} \end{pmatrix}$. We therefore obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} \frac{(6)^{n-1}}{13} + \frac{12*(-7)^{n-1}}{13} & \frac{-(6)^n}{13} + \frac{6*(-7)^{n-1}}{13} \\ \frac{-2(6)^{n-1}}{13} + \frac{2*(-7)^{n-1}}{13} & \frac{2*(6)^n}{13} + \frac{(-7)^{n-1}}{13} \end{pmatrix} \frac{1}{2}$$

and hence the result follows. □

Theorem 3.3 *The number of spanning trees of the linear $(kC_n - snake)$ satisfies the following recursive relation $\tau(kC_n - snake) = n^k$.*

Proof Consider a graph $kC_n^{/} - snake$ constructed from $kc_n - snake$ by deleting two edges. See Figure 3 following.

kC_n-Snake $\qquad\qquad$ $kC_n^{/}$-Snake

Figure 3 Linear kC_n-Snake

We put $kC_n - snake = \tau(kC_n - snake)$ and $kC_n^{/} - snake = \tau(kC_n^{/} - snake)$. It is clear that

$$kC_n - snake = 5((k-1)C_n - snake) + 6((k-1)C_n^{/} - snake)$$

and

$$kC_n^{/} - snake = 2((k-1)C_n - snake) - 6((k-1)C_n^{/} - snake)$$

with initial conditions $(C_n - snake) = n$, $C_n^{/} - snake) = 1$. Thus, we have

$$\begin{pmatrix} kC_n - snake \\ kC_n^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_n - snake \\ (k-1)C_n^{/} - snake \end{pmatrix},$$

where, $A = \begin{pmatrix} n-1 & n \\ 2 & -n \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_n - snake \\ kC_n^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_n - snake \\ (k-1)C_n^{/} - snake \end{pmatrix}, \quad = \cdots = A^{n-1} \begin{pmatrix} C_n - snake \\ C_n^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 42 = 0, \qquad \lambda_1 = -(n+1) \text{ and } \lambda_2 = n,\ \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MDM^{-1}$, where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{n} \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} \frac{1}{(2n+1)} & \frac{-n}{(2n+1)} \\ \frac{2n}{(2n+1)} & \frac{n}{(2n+1)} \end{pmatrix} \Rightarrow A^{n-1} = MB^{n-1}M^{-1},$$

where $B^{n-1} = \begin{pmatrix} (n)^{n-1} & 0 \\ 0 & (-(n+1))^{n-1} \end{pmatrix}$. We therefore obtain

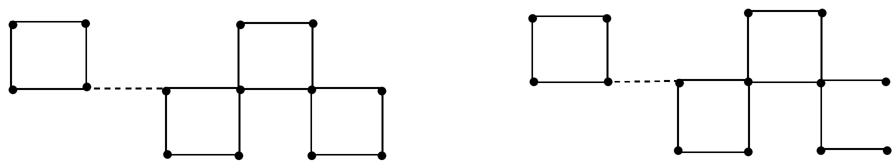
$$\mathrm{A}^{n-1} = \begin{pmatrix} \frac{(n)^{n-1}}{(2n+1)} + \frac{2n*(-(n+1))^{n-1}}{(2n+1)} & \frac{-(n)^{n}}{(2n+1)} + \frac{n*(-(n+1))^{n-1}}{(2n+1)} \\ \frac{-2(n)^{n-1}}{(2n+1)} + \frac{2*(-(n+1))^{n-1}}{(2n+1)} & \frac{2*(n)^{n}}{(2n+1)} + \frac{(-(n+1))^{n-1}}{(2n+1)} \end{pmatrix}$$

and hence the result follows. □

Remark 3.4 The number of spanning trees of the subdivision of linear $(kC_n - snake)$satisfies the following recursive relation $\tau(S(kC_n - snake)) = 2n\,\tau((k-1))C_n - snake$, where k is the number of blocks and n is the number of vertices for each block.

Theorem 3.5 *The number of spanning trees of the general $kC_4 - snake$ satisfies the following recursive relation $\tau(\ kC_4 - snake\ ) = 4^k$, where k is the number of blocks.*

Proof Consider a graph $kC_4^{/} - snake$ constructed from $kC_4 - snake$ by deleting two edges. See Figure 4 following.



kC_4-Snake $\qquad$ $kC_4^{/}$-Snake

Figure 4 General kC_4-Snake

We put $kC_4 - snake = \tau(kC_4 - snake)$ and $kC_4^{/} - snake = \tau(kC_4^{/} - snake)$. It is clear that

$$kC_4 - snake = 3(k-1)C_4 - snake + 4(k-1)C_4^{/} - snake$$

and

$$kC_4 - snake = 2(k-1)C_4 - snake - 4(k-1)C_4^{/} - snake$$

with initial conditions $C_4 - snake = 4$, $C_4^{/} - snake = 1$. Thus, we have

$$\begin{pmatrix} kC_4\text{−snake} \\ kC_4^{/}\text{−snake} \end{pmatrix} = A \begin{pmatrix} (k-1)C_4\text{−snake} \\ (k-1)C_4^{/}\text{−snake} \end{pmatrix},$$

where $A = \begin{pmatrix} 3 & 4 \\ 2 & -4 \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_4\text{−snake} \\ kC_4^{/}\text{−snake} \end{pmatrix} = A \begin{pmatrix} (k-1)C_4\text{−snake} \\ (k-1)C_4^{/}\text{−snake} \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_4\text{−snake} \\ C_4^{/}\text{−snake} \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A-\lambda I_2)=\lambda^2+\lambda-20=0, \qquad \lambda_1=-5 \text{ and } \lambda_2=4,\ \lambda_1\neq\lambda_2$$

Then, there is a matrix M invertible such that $A=MBM^{-1}$, where $B=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M=\begin{pmatrix}1 & 1\\ -2 & \frac{1}{4}\end{pmatrix} \quad\Rightarrow\quad M^{-1}=\frac{1}{\frac{9}{4}}\begin{pmatrix}\frac{1}{4} & -1\\ 2 & 1\end{pmatrix} \quad\Rightarrow\quad A^{n-1}=MB^{n-1}M^{-1}\ ,$$

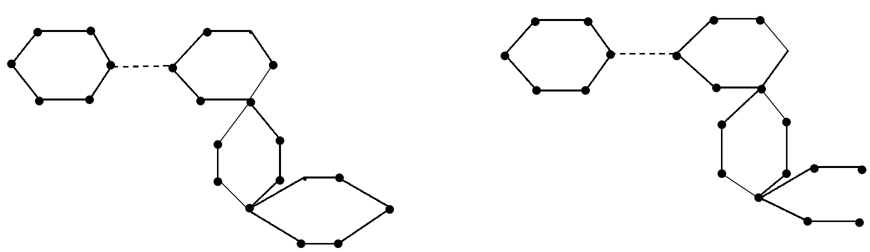
where, $B^{n-1}=\begin{pmatrix}(-5)^{n-1} & 0\\ 0 & (4)^{n-1}\end{pmatrix}$. We therefore obtain

$$\mathrm{A}^{n-1}=\begin{pmatrix}\frac{(-5)^{n-1}}{9}+\frac{2*(4)^{n}}{9} & \frac{-4*(-5)^{n-1}}{9}+\frac{(4)^{n}}{9}\\ \frac{-2*(-5)^{n-1}}{9}+\frac{2*(4)^{n-1}}{9} & \frac{8*(-5)^{n-1}}{9}+\frac{4^{n-1}}{9}\end{pmatrix}$$

and hence the result follows. □

Theorem 3.6 *The number of spanning trees of the general $kC_6-snake$ satisfies the following recursive relation $\tau(\ kC_6-snake\ )=6^k$.*

Proof Consider a graph $kC_6-snake$ constructed from $kC_6^{/}-snake$ by deleting two edges. See Figure 5.



kC_6-Snake $\qquad$ $kC_6^{/}$-Snake

Figure 5 General kC_6-Snake

We put $kC_6-snake=\tau(kC_6-snake)$ and $kC_6^{/}-snake=\tau(kC_6^{/}-snake)$. It is clear that

$$kC_6-snake=5((k-1)C_6-snake)+6((k-1)C_6^{/}-snake)$$

and

$$kC_6^{/}-snake=2((k-1)C_6-snake)-6((k-1)C_6^{/}-snake)$$

with initial conditions $(C_1 - snake) = 6(C_1^{/} - snake) = 1$. Thus we have

$$\begin{pmatrix} kC_6 - snake \\ kC_6^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_6 - snake \\ (k-1)C_6^{/} - snake \end{pmatrix},$$

where $A = \begin{pmatrix} 5 & 6 \\ 2 & -6 \end{pmatrix}$, which implies that

$$\begin{pmatrix} kC_6 - snake \\ kC_6^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_6 - snake \\ (k-1)C_6^{/} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_6 - snake \\ C_6^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 42 = 0, \qquad \lambda_1 = -7 \text{ and } \lambda_2 = 6,\ \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MDM^{-1}$, where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{6} \end{pmatrix} \quad \Rightarrow \quad M^{-1} = \begin{pmatrix} \frac{1}{13} & \frac{-6}{13} \\ \frac{12}{13} & \frac{6}{13} \end{pmatrix} \quad \Rightarrow \quad A^{n-1} = MB^{n-1}M^{-1},$$

where, $B^{n-1} = \begin{pmatrix} (6)^{n-1} & 0 \\ 0 & (-7)^{n-1} \end{pmatrix}$. From which, we obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} \frac{(6)^{n-1}}{13} + \frac{12*(-7)^{n-1}}{13} & \frac{-(6)^{n}}{13} + \frac{6*(-7)^{n-1}}{13} \\ \frac{-2(6)^{n-1}}{13} + \frac{2*(-7)^{n-1}}{13} & \frac{2*(6)^{n}}{13} + \frac{(-7)^{n-1}}{13} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.7 *The number of spanning trees of general $(kC_n - snake)$satisfies the following recursive relation $\tau(kC_n - snake) = n^k$.*

Proof Consider a graph $kC_n - snake$ constructed from $kC_n^{/} - snake$ by deleting two edges. See Figure 6 following.

We put $kC_n - snake = \tau(kC_n - snake)$ and $kC_n^{/} - snake = \tau(kC_n^{/} - snake)$. It is clear that

$$kC_n - snake = 5((k-1)C_n - snake) + 6((k-1)C_n^{/} - snake)$$

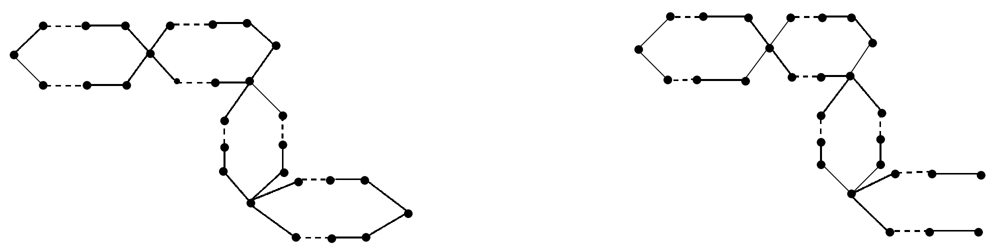
and

$$kC_n^{/} - snake = 2((k-1)C_n - snake) - 6((k-1)C_n^{/} - snake)$$

with initial conditions $(C_n - snake) = n$, $(C_n^{/} - snake) = 1$. Thus we have

$$\begin{pmatrix} kC_n - snake \\ kC_n^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_n - snake \\ (k-1)C_n^{/} - snake \end{pmatrix},$$

where $A = \begin{pmatrix} n-1 & n \\ 2 & -n \end{pmatrix}$,



kC_n-Snake $kC_n^{/}$-Snake

Figure 6 General kC_6-Snake

which implies that

$$\begin{pmatrix} kC_n - snake \\ kC_n^{/} - snake \end{pmatrix} = A \begin{pmatrix} (k-1)C_n - snake \\ (k-1)C_n^{/} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} C_n - snake \\ C_n^{/} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 42 = 0, \quad \lambda_1 = -(n+1) \text{ and } \lambda_2 = n, \ \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MDM^{-1}$, where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{n} \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} \frac{1}{(2n+1)} & \frac{-n}{(2n+1)} \\ \frac{2n}{(2n+1)} & \frac{n}{(2n+1)} \end{pmatrix} \Rightarrow A^{n-1} = MB^{n-1}M^{-1},$$

where $B^{n-1} = \begin{pmatrix} (n)^{n-1} & 0 \\ 0 & (-(n+1))^{n-1} \end{pmatrix}$. From which, we therefore obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} \frac{(n)^{n-1}}{(2n+1)} + \frac{2n*(-(n+1))^{n-1}}{(2n+1)} & \frac{-(n)^n}{(2n+1)} + \frac{n*(-(n+1))^{n-1}}{(2n+1)} \\ \frac{-2(n)^{n-1}}{(2n+1)} + \frac{2*(-(n+1))^{n-1}}{(2n+1)} & \frac{2*(n)^n}{(2n+1)} + \frac{(-(n+1))^{n-1}}{(2n+1)} \end{pmatrix}$$

and hence the result follows. $\square$

Remark 3.8 The number of spanning trees of the subdivision of general $S(kC_n - snake)$ satisfies the following recursive relation: $\tau(S(kC_n)) = 2n\tau(S(k-1)C_n - snake) = (2n)^k$ where k is the number of blocks.

Theorem 3.9 *The number of spanning trees of the subdivided fan graph satisfies the following recurrence relation*

$$\tau(S(F_n)) = \frac{1}{2\sqrt{5}}[(3+\sqrt{5})^n - (3-\sqrt{5})^n],$$

where $\tau($ S(F_1) $)=1$ *and* $\tau($ S(F_2) $)=6$.

Proof Consider a graph $S(F_n)$ constructed from $S(F_n^{/})$ by deleting two edges. See Figure 7 following.

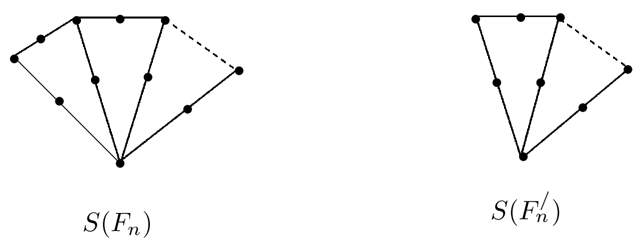


Figure 7 Subdivided Fan Graph

We put $S(F_n) = \tau(S(F_n))$ and $S(F_n^{/}) = \tau(S(F_n^{/}))$, It is clear that

$$S(F_n) = 32S(F_{n-2}) - 24S(F_{n-3}^{/}),$$

where $S(F_n^{/})$ is the number of odd block and

$$S(F_n^{/}) = 6S(F_{n-1}) - 4S(F_{n-2}^{/}),$$

where $S(F_n)$ is the number of even block with initial conditions $S(F_1) = 6$, $S(F_1^{/}) = 1$ and

$$\begin{pmatrix} S(F_n) \\ S(F_n^{/}) \end{pmatrix} = A \begin{pmatrix} S(F_{n-1}) \\ S(F_{n-1}^{/}) \end{pmatrix},$$

where, $A = \begin{pmatrix} 6 & -4 \\ 32 & -24 \end{pmatrix}$, which implies that

$$\begin{pmatrix} S(F_n) \\ S(F_n^{/}) \end{pmatrix} = A \begin{pmatrix} S(F_{n-1}) \\ S(F_{n-1}^{/}) \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} S(F_1) \\ S(F_1^{/}) \end{pmatrix},$$

$$\lambda_1 = \frac{1061}{1250} \quad \text{and} \quad \lambda_2 = \frac{23561}{1250}, \quad \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MBM^{-1}$ where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ 1.2878 & 6.2121 \end{pmatrix} \; ; \; \mathrm{M}^{-1} = \begin{pmatrix} 1.2615 & -0.2031 \\ -0.2615 & 0.2031 \end{pmatrix} \; ; \; \mathrm{A}^{n-1} = MB^{n-1}M^{-1} \, ,$$

$$B^{n-1} = \begin{pmatrix} (0.8488)^{n-1} & 0 \\ 0 & (-18.8488)^{n-1} \end{pmatrix} .$$

From which, we therefore obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} 1.2615(0.8488)^{n-1} - 0.2615(-18.8488)^{n-1} & -0.2031(0.8488)^{n-1} + 0.2031(-18.8488)^{n-1} \\ 1.6246(0.8488)^{n-1} - 1.6245(-18.8488)^{n-1} & -0.2616(0.8488)^{n-1} + 1.2617(-18.8488)^{n-1} \end{pmatrix}$$

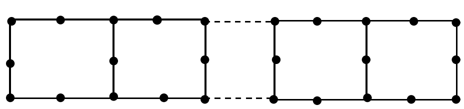
and hence the result follows. □

Theorem 3.10 *The number of spanning trees of the subdivided ladder graph satisfies the following recurrence relation*

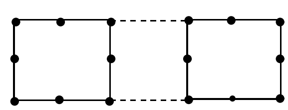
$$\tau(S(L_n)) = \frac{2^{n-2}}{\sqrt{3}}[(2+\sqrt{3})^n - (2-\sqrt{3})^n]$$

for any $n \geqslant 1$, where $\tau(\, S(L_1)\,) = 1$ and $\tau(\, S(L_2)\,) = 8$.

Proof Consider a graph $S(F_n)$ constructed from $S(F_n^{/})$ by deleting two edges. See Figure 8 following.



$S(L_n)$



$S(L_n^{/})$

Figure 8 Subdivided Ladder Graphs $S(L_n)$ and $S(L_n^{/})$

We put $S(L_n) \;\; = \tau(S(L_n)\,)$ and $S(L_n^{/}) \;\; = \tau(S(L_n^{/})\,)$, It is clear that

$$S(L_n) = 8S(L_{n-1}^{/}) \; - \; 4S(L_{n-2}) \; ,$$

where $S(L_n)$ is the number of even block,

$$S(L_n^{/}) = 60S(L_{n-2}^{/}) \; - \; 32S(L_{n-3})$$

with $S(L_n^{/})$ the number of its odd block with initial conditions $S(L_1) = 8$, $S(L_1^{/}) = 1$. Thus, we have

$$\begin{pmatrix} S(L_n) \\ S(L_n^{/}) \end{pmatrix} = A \begin{pmatrix} S(L_{n-1}) \\ S(L_{n-1}^{/}) \end{pmatrix},$$

where $A = \begin{pmatrix} 8 & -4 \\ 60 & -32 \end{pmatrix}$, which implies that

$$\begin{pmatrix} S(L_n) \\ S(L_n^{/}) \end{pmatrix} = A \begin{pmatrix} S(L_{n-1}) \\ S(L_{n-1}^{/}) \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} S(L_1) \\ S(L_1^{/}) \end{pmatrix},$$

$$\lambda_1 = 0.49 \quad \text{and} \quad \lambda_2 = -24.49\ , \quad \lambda_1 \neq \lambda_2.$$

Then, there is a matrix M invertible such that $A = MBM^{-1}$ where $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ 1.8775 & 8.1225 \end{pmatrix} \ ; \ \mathrm{M}^{-1} = \begin{pmatrix} 1.3006 & -0.1601 \\ -0.3006 & 0.1601 \end{pmatrix} \ ; \ \mathrm{A}^{n-1} = MB^{n-1}M^{-1}\ ,$$

with $B^{n-1} = \begin{pmatrix} (0.49)^{n-1} & 0 \\ 0 & (-24.49)^{n-1} \end{pmatrix}$. From which, we therefore obtain

$$\mathrm{A}^{n-1} = \begin{pmatrix} 1.3006(0.49)^{n-1} - 0.3006(-24.49)^{n-1} & -0.1601(0.49)^{n-1} + 0.1601(-24.49)^{n-1} \\ 2.4419(0.49)^{n-1} - 2.4416(-24.49)^{n-1} & -0.3022(0.49)^{n-1} + 1.3004(-24.49)^{n-1} \end{pmatrix}$$

and hence the result follows. □

§4. **Spanning Tree Entropy**

The entropy of spanning trees of a network or the asymptotic complexity is a quantitative measure of the number of spanning trees and it characterizes the network structure. We use this entropy to quantify the robustness of networks. The most robust network is the network that has the highest entropy. We can calculate its spanning tree entropy which is a finite number and a very interesting quantity characterizing the network structure, defined as in [15, 16] as:

$$Z(G) = \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|},$$

$$Z(KC_4\text{−snake}) = \lim_{n\to\infty} \frac{\ln 4^k}{3k+1} = 0.4621,$$

$$Z(KC_6\text{−snake}) = \lim_{n\to\infty} \frac{\ln 6^k}{5k+1} = 0.3584$$

$$Z(KC_n - snake) = \lim_{k\to\infty} \frac{\ln n^k}{(n-1)k+1} = \frac{\ln(n)}{n-1},$$

$$Z(S(F_n)) = \lim_{n\to\infty} \frac{\ln(\frac{1}{2\sqrt{5}} * (3+\sqrt{5})^n - (3-\sqrt{5})^n)}{3n+1} = \ln(\sqrt[3]{3+\sqrt{5}}) = 0.5513$$

$$Z(S(L_n)) = \lim_{n\to\infty} \frac{\ln(\frac{2^{n-2}}{\sqrt{3}} * (2+\sqrt{3})^n - (2-\sqrt{3})^n)}{5n-2} = \ln(\sqrt[5]{2+\sqrt{3}}) + \frac{\ln(2)}{5} = 0.4020$$

§5. Conclusion

In this paper, we described how to propose the combinatorial approach to facilitate the calculation of the number of spanning trees in linear and general cyclic snake networks. In particular, we derived the explicit formulas for the linear $kc_4 - snake$, linear $kc_6 - snake$ and linear $kc_n - snake$. Finally, we derived explicit formulas for the general $kc_4 - snake$, general $kc_6 - snake$ and general $kc_n - snake$.

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