THERE EXISTS A PRIME IN ANY INTERVAL FROM A^2 TO A(A±1), FOR ALL A>1

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The following is an outline demonstrating a method of proof for the title claim, which proves the Legendre conjecture and perhaps other conjectures.

For any odd integer a, the four intervals:

1 a(a+1) … (a+1)(a+1)
2 (a+1)(a+1) … (a+1)(a+2)
3 (a+1)(a+2) … (a+2)(a+2)
4 (a+2)(a+2) … (a+2)(a+3)

taken as non-inclusive of the first and last integers, have the following properties:

- each interval contains **(a+1)/2 odd integers**.
- each interval contains **(a-1)/2 potential odd composites**. (1)

[Example: a=11]

1 (132, 144): {133, 135, 137, 139, 141, 143}
2 (144, 156): {145, 147, 149, 151, 153, 155}
3 (156, 169): {157, 159, 161, 163, 165, 167}
4 (169, 182): {171, 173, 175, 177, 179, 181}

Each interval contains (a+1)/2 odd integers.

We will see by demonstration that each such interval contains (a-1)/2 potential odd composites. Thus, each interval must contain at least one fewer potential odd composite than odd integers, and there must always be at least one prime in any given interval.

Remarks on potential odd composites and the dynamic:

- Potential odd composites may overlap, leading to more primes in a given interval.
- Some potential odd composites may not appear in a given interval, leading to more primes.
- Potential odd composites are limited by the maximum possible configurations of composites in an interval.

Demonstration of the configurations of potential odd composites

Odd composites are formed as a product (2m+1)(2n+1) where m and n are integers, m > 0, n > 0.

For each of the four intervals with (a+1)/2 odd integers, the largest odd integer value could be formed by 4mn + 2n + 2m + 1. Given that each odd integer composite is of the form 2x+1 where x=2mn+m+n, and each odd integer prime is of the form 2x+1 where x≠2mn+m+n, we have {x | x ∈ N} and for any (m+n) the value of x is greatest where the difference between m and n is least.

We can show:
(1) the odd integers in each interval are represented by one and only one x value (in the form 2x+1).
(2) each x that represents a composite must have the form 2mn+m+n.
(3) given no square values are within the intervals, we can establish m>n for the odd composites in question.
(4) we regard each m,n combination as unique, i.e. m=2, n=1 is a unique configuration (though it may not give a unique value for 2mn+m+n).
(5) the greatest possible n value for the four given intervals is (a-1)/2.

(6) the odd composites in any interval are given by the unique configurations given by n=1, 2, 3… (a-1)/2 and a paired m value.

Demonstration: a=11

1 (132, 144): x = {66, 67, 68, 69, 70, 71}
2 (144, 156): x = {72, 73, 74, 75, 76, 77}
3 (156, 169): x = {78, 79, 80, 81, 82, 83}
4 (169, 182): x = {85, 86, 87, 88, 89, 90}

Remarks on (5), and proof

For any odd a, a^2 would be given as (2m+1)(2n+1) where m=n and a=(2n+1). For any such a, m, and n, (a+2)^2 would be given as (2(m+1) +1)(2(n+1) +1).

The intervals are defined in each case by either (a+1) or (a+2), and are non-inclusive of these values. The maximum factor of a non-square odd composite in the given four intervals, that is less than the square root of said odd composite, is therefore (2n+1).

In the example above where a=11, and a=2n+1, the maximum n value is 5.

Remarks on (6), and proof of the sufficiency of (a-1)/2 configurations for m and n:

Continuing from the remarks on (5), we can thus count all potential composites of each interval as being of the form 2x+1 where :

x=2m(5) + m + (5)
x=2m(4) + m + (4)
x=2m(3) + m + (3)
x=2m(2) + m + (2)
x=2m(1) + m + (1)

Where it is sufficient to count one unique m configuration for each interval, in the above order and by elimination, to cover all composites in the interval.

To understand why it is sufficient to count only one (m,n) configuration for each value of n from 1 to (a-1)/2, we must begin from the largest value of n and, by elimination, work down to the least factors. Since any odd composite can be factored as (2m+1) and (2n+1), and some such factors may be further subdivided into factors of like form, it will generally be possible to factor more odd integers in an interval with 2n+1 where n=1 than where n is greater.

An odd composite with a factor of 9 – (2n+1 where n=4) – will also appear as having a factor of 3 – (2n+1 where n=1). So rather than counting multiple instances of n=1 as a factor in the odd integers of an interval, we may count first the n=4 instance, then by elimination turn to any remaining odd integers to find n=1. Thus in this manner we can show that each interval contains, at most, one and only one unique configuration for each n value. It may be that a particular n value isn't contained in the interval, but there can be no others but the potential values.

Thus, there must always be fewer odd composites than odd integers in every interval from a^2 to a(a±1), for all a>1. Thus, there must exist at least one prime in each such interval.