Three Dimensions from Motivic Quantum Gravity

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Abstract

Important polytope sequences, like the associahedra and permutohedra, contain one object in each dimension. The more particles labelling the leaves of a tree, the higher the dimension required to compute physical amplitudes, where we speak of an abstract categorical dimension. Yet for most purposes, we care only about low dimensional arrows, particularly associators and braiding arrows. In the emerging theory of motivic quantum gravity, the structure of three dimensions explains why we perceive three dimensions. Some higher dimensional data, such as the $\mathbf{e_8}$ lattice, is encoded in two or three dimensions. Here we give a very elementary overview of the key quantum data from an axiomatic perspective, focusing on the permutoassociahedra of Kapranov.

These notes originated in a series of lectures I gave at Quantum Gravity Research in February 2018. There I met Michael Rios, whom I originally taught about the associahedra and other operads many, many years ago. He discussed the magic star and exceptional periodicity. In March I met Tony Smith, the founder of $\mathbf{e_8}$ approaches to quantum gravity. In June I attended the Advances in Quantum Gravity workshop at UCLA, where I met the M theorist Alessio Marrani, and in July participated in Group32 in Prague, where I met Piero Truini, who I later talked to about operad polytopes in motivic quantum gravity. All these activities led to a greatly improved understanding of the emerging theory. I was then forced to return to a state of severe poverty, isolation and abuse, ostracised effectively from the professional community, from local communities, from family, and many online websites. I am working now on an ipad in my tent.

Although I hope the discussion here is elementary, the motivation for it comes from deep mathematical conundrums, as any good question in quantum gravity should. As a category theorist, the slow development of topological field theories was endlessly frustrating, with its separation of spatial and algebraic data. Like in gauge theories, one was supposed to mark spatial diagrams with objects from an algebraic category, such as a traditional category of Hilbert spaces. But the most interesting functors are monadic, and it seemed that gravity should canonically interpret its geometry in an algebraic way. Moreover, the common habit of grabbing $\mathbb{R}$ or $\mathbb{C}$ as a God given set of numbers was in conflict with both the axioms of a topos [1] and the need to construct classical manifolds from discrete quantum data.

We begin with the *power set monad* [1], which simply maps any set to the set of all subsets of the set. In the case of a three element set, there are eight possible subsets including the empty set, listed by the vertices of figure 1. A set with n elements gives a cube in dimension n. An edge on a cube, directed away from the empty set 1, is an inclusion of sets. The concatenation of elements, like $\alpha_i \alpha_j$, denotes the *union*. Conversely, reverse arrows restrict to a subset of a set, and the source of a square is the *intersection* of sets. A category theorist talks about the *duality* of reversing arrows, or the *opposite category* of the category of all sets.

But quantum mechanics is not about sets. Axioms based on set theoretic mathematics must be replaced by axioms for quantum logic. In the next section we see that n-cubes also play an important role in the quantum case. The signed vertices on a cube denote a state of n qubits in the theory of quantum computation [2][3][4], and we use the universality of special categories like the Fibonacci anyons [5][6] to argue that gauge groups emerge from braid group representations.

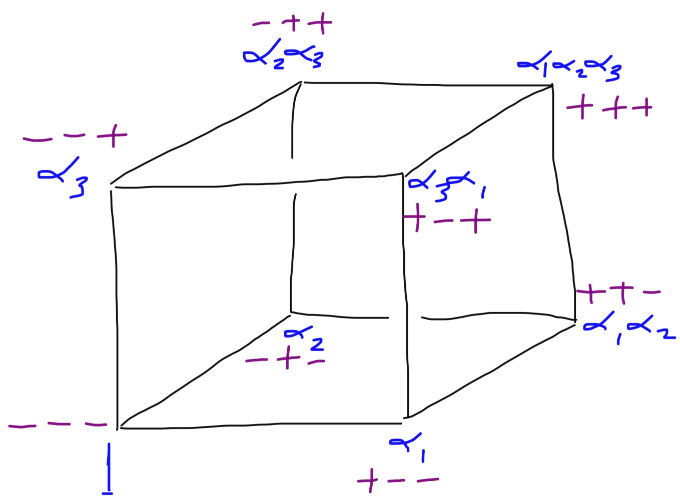

Figure 1: The parity cube in dimension 3

From cubes we move on to the associahedra [7] and permutohedra [8], the polytopes that are specified by rooted binary trees or permutations. The combination of these polytopes, including both Mac Lane pentagons [9] and braiding hexagons, are the *permutoassociahedra* of [10]. In particular, the 120 vertex permutoassociahedron in dimension 3 stands in for half the 240 roots of the 8 dimensional $\mathbf{e_8}$ lattice.

So what is a motive? Certainly, it does not begin with a variety based on

$\mathbb{C}$, or any other random field. A whole, infinite dimensional category of motives should be linked to our quantum axioms, taking motivic diagrams directly to physical results, following the original vision of Grothendieck [11]. A classical spacetime is a mere afterthought, a three time Minkowski space perhaps compactified to $SU(2) \times SU(2)$, constructed using the anyon braid group B_3 [12]. Ribbon diagrams are spatial, and integral octonions, for instance, are algebraic, but the category of motives thinks they are one and the same, thinks that a fundamental group drawn in knots is its own algebraic structure. And the Standard Model of particle physics lives here.

1 A motivic perspective

Geometric categorical axioms encode algebraic information. In the theory of quantum gravity, we require a canonical symbiosis of discrete geometry and algebraic rules. Motivated by the power set monad [1], we begin with the humble cube.

The vertices of figure 1 carry two sets of labels. Physically, the signs encode the fractional charges of the anyons [13] underlying the leptons and quarks, so that $---$ denotes 3 zero charges on a neutrino and $+++$ denotes the charge on a positron. There is a second cube for the negative leptons and quarks [14][15]. In quantum computation, three signs stand for a set of 3 qubit states. At the same time, the edges of a 3-cube move in the 3 directions of a *qutrit* space. A qutrit has three possible measurement outcomes, while a qubit has two. A path from the source 1 to the target $\alpha_1\alpha_2\alpha_3$ has three edges. These six paths clearly form a copy of the permutation group S_3, if we label the directions 1, 2 and 3.

Seven of the vertices give an unconventional Fano plane, as shown in figure 2. This gives us a basis for the integral octonions [16][17], which are appearing here automatically, once we decide to study sets carefully.

In quantum mechanics, an element of a set usually becomes a basis vector in a state space. Instead of three points and their unions, consider three generic lines in a Euclidean plane with respect to intersection in the plane. If no lines are selected, we have the whole plane. If one line is selected, we get a line. If two lines are selected, we obtain the point of intersection of the lines. And finally, if all three lines are selected, we have the empty set. Observe how these possibilities reverse the position of empty set, moving it from 1 to the target of the cube. The source 1 is now the entire plane, an axiomatic object of dimension 2. The singleton α_i are now dimension 1 objects and the $\alpha_i\alpha_j$ of dimension zero. Whereas sets are always zero dimensional, whatever the cardinality, quantum geometry interprets a whole number n as a dimension.

In this example, there are spaces of dimension 0, 1 and 2. A classical topologist would take these numbers and put them on the vertices of a triangle, calling the triangle a 2-*simplex*. We have already drawn the 2-simplex as the interior of three lines in a plane. It is also the diagonal slice of the 3-cube at the α_i positions, or rather, the points 001, 010 and 100. Now consider qutrit words which allow more than one step in each direction. As well as $231 \in S_3$ we have 2231,

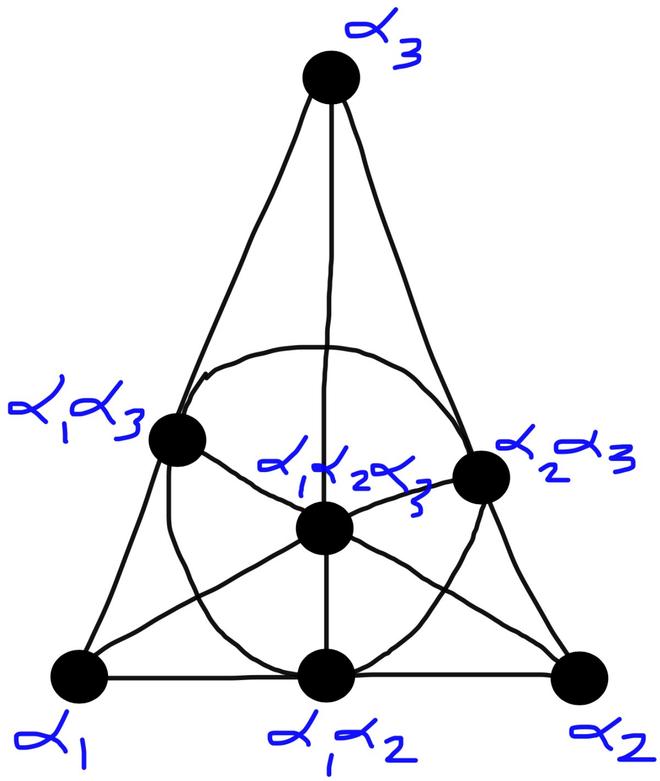

Figure 2: The Fano plane

taking two steps in one direction. Allowing arbitrary noncommutative words, we see discrete paths on rectangular blocks in any dimension. On a cubic array of paths, the diagonal slice fixes the length of the word, so that length 3 qutrit words collapse to the discrete simplex of figure 3. Simplices carry commutative words, while cubes carry the full set of noncommutative words.

Given any noncommutative word, a rectangular subspace of subwords sits below it. For example, 2231 sits on a 3-cube with three points along each edge, while its subpath 231 lies in the space of paths closer to the source vertex. If the stepping represents powers of a set of primes, as in $p_2^2 p_3 p_1$, then the subspace vertices list the divisors, and each dimension out to ∞ lists powers of an integral prime. Then the positive integers in $\mathbb{Z}$ use up every cubical point in every dimension.

A qubit corresponds to the prime 2 and a qutrit to the prime 3, because state spaces are characterised by Schwinger's mutually unbiased bases [18][19][20]. For qubits, the three Pauli matrices are replaced by the three unitary bases of

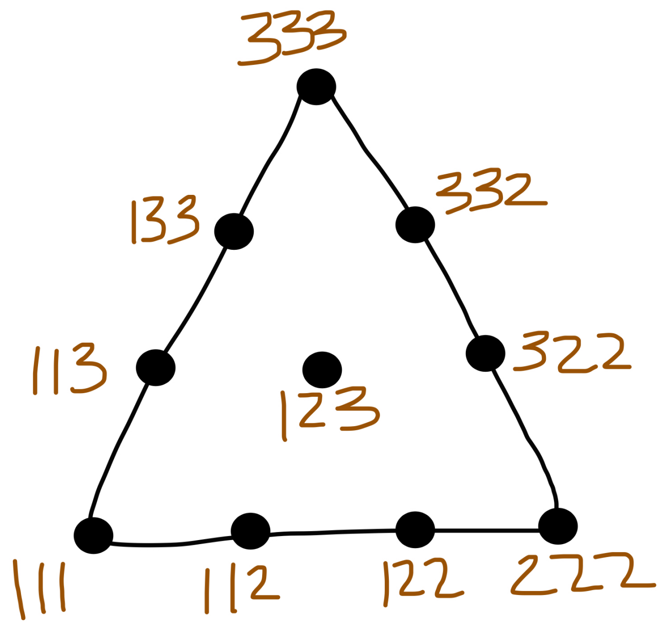

Figure 3: The commutative tetractys for three qutrits

eigenvectors,

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

For qutrits, the bases are

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix}, \quad R_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{pmatrix}, \quad R_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & \overline{\omega} \\ \overline{\omega} & 1 & 1 \\ 1 & \overline{\omega} & 1 \end{pmatrix}, \quad I_3, \tag{2}$$

where ω is the cubed root of unity. In every prime power dimension d, there is a (one step) circulant generator R_d whose powers give d bases that are mutually unbiased. Thus each cubic point is associated to a set of $d = p^r$ matrices. Dimension 3 is special because circulants are Hermitian, like the elements of the exceptional Jordan algebra over the octonions [16][17]. The physical charges attached to figure 1 by the $\mathbb{C} \otimes \mathbb{O}$ algebra [14][15] correspond to ribbon twists in the particle spectrum of Bilson-Thompson [13][21], where three anyon strands braid in B_3 to give leptons and quarks.

Sometimes we view the α_i as differential forms, but we want to avoid any reference to manifolds over bothersome fields. Most complex numbers are obtained from the (half) integers in dimension 8 using $\mathbb{Q}(\sqrt{5})$ with the golden ratio. Let $\phi = (1 + \sqrt{5})/2$ and $\rho = \sqrt{\phi + 2}$. Then the Gaussian integer combinations $a + b\phi + c\rho + d\phi\rho$ fill the plane. Everything begins with discrete data.

2 Associahedra and Permutohedra

The permutation group S_n has a polytope in dimension $n-1$. The three dimensional *permutohedron* is shown in figure 4 with its S_4 labels. Think of the

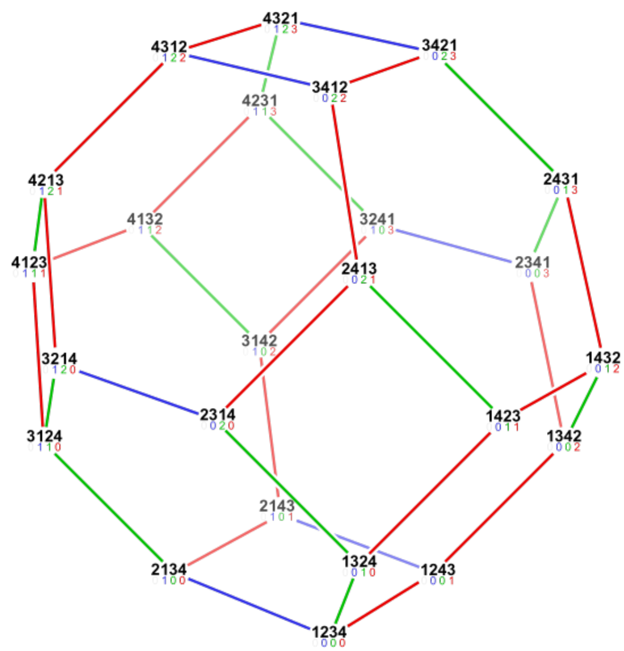

Figure 4: Three dimensional permutohedron

elements of S_4 as integral coordinates in dimension 4 and restrict to the three dimensional plane given by the fixed sum $1+2+3+4=10$. The infinite set $\coprod S_n$ of all permutohedra form an *operad*, meaning that natural compositions of permutations give larger permutations.

Another important operad is the set of *associahedra* polytopes [7], whose vertices are labeled by binary rooted planar trees. Figure 5 gives the pentagon associahedron for trees with 4 leaves. As an axiom for categories of Hilbert spaces (symmetric monoidal) the pentagon edges are directed associator maps $(12)3 \to 1(23)$ between bracketing options on nonassociative words. In every dimension n, the associahedron is embedded in a discrete simplex by the coordinates of [8]. The pentagon lies in the bottom left corner of the tetractys of figure 3, as follows. Take vertices such that each place i in the word has a value that is less than or equal to i. This is the set 111, 112, 122, 113 and 123. These words may be viewed (homework problem) geometrically as arbitrary forests of rooted trees on three nodes.

The edges of an associahedron are marked by a tree with one ternary node, which collapses the binary nodes on the end vertices. Similarly, each face of an

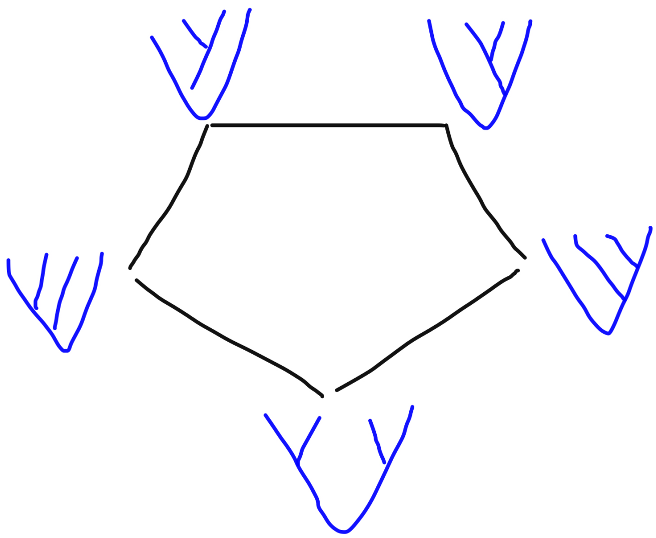

Figure 5: Mac Lane's pentagon axiom

associahedron is a tree with two collapsed nodes, and each cell in any dimension is labeled by a tree. The associahedra operad underlies scattering amplitude calculations in particle physics [22][23]. Particle number determines the required dimension of the calculation, which is therefore arbitrary.

Observe that labels on the leaves of a tree on the pentagon give an element of S_4. Allowing all possible labels on all trees, we obtain the three dimensional *permutoassociahedron* [10] of figure 7. We suspect now that cubes, simplices, associahedra and permutohedra are all closely related. In fact, they carry a rich algebraic structure, crucial to the observational arguments of the neutrino CMB correspondence [12][24]. We discuss now Solomon's algebra of descent, which maps the permutohedra to the sign labels on a cube of the same dimension.

3 Cubes and Solomon's Algebra

Finite sets are acted on by permutations. Taking an element of the hexagon S_3, we permute the α_i vertices on the cube, or reorder the word $\alpha_1\alpha_2\alpha_3$. Distinct actions slice a cube into its diagonal subsets. The cardinality of a diagonal is a binomial coefficient, and these sum to the total 2^n.

There are 12 ways to braid three objects into a permutation in S_3, either by flipping only two objects or by flipping one object around the other pair. These are the 12 edges of the octahedron in figure 6. The bracket on a word indicates the face on the cube dual to the octahedron, where a permutation is a three step path along edges on the cube. Accounting for all possible nonassociative words in S_3, there are two copies of the octahedron, giving all vertices of the 12-gon

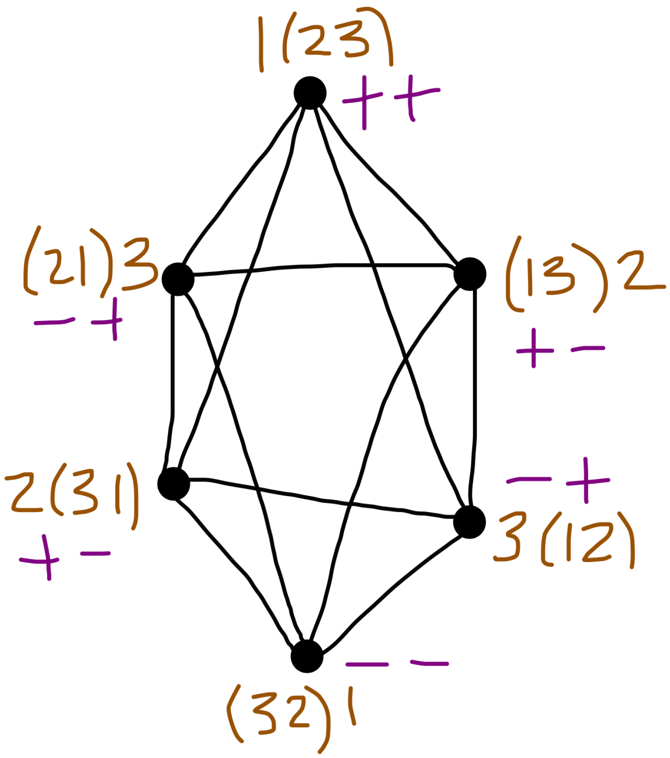

Figure 6: Octahedron for permutations

permutoassociahedron [10], using both swap maps and associators between the two octahedra. That is, we use three vertices in one octahedron, three in the other, and alternate these with the six associators in between.

Why such an abtuse drawing of a 12-gon? Because what we aim for, in the next section, is a special set of three dimensional pictures that capture the data we need to describe gravity. The 12-gon carries all possible terms in weak Jacobi rules, such as those required for the nonassociative algebras of exceptional periodicity based on the magic star [25][26].

Return to the permutohedron of figure 4. This polytope collapses to a 3-cube with signed vertices. The signs at each vertex of S_n are the *signature* of the permutation, given by following the increases and decreases in the numerical entries from left to right. Signs for S_3 appear in figure 6. For example, the signature of 4312 is $--+$. Signature sets occupy neighbouring sets in the S_n polytope.

Solomon's (Hopf) algebra [27] on signature classes uses integral elements of the group algebra product for S_n. An element of this group algebra, over $\mathbb{Z}$, is an arbitrary sum of permutations, written out like a polynomial. The product is like an ordered polynomial product. For example,

$$(231 + 312)(231) = 231 \cdot 231 + 312 \cdot 231 = (312 + 123). \tag{3}$$

For each signature of S_n write the sum of permutations with that signature. For example,

$$+- = (132 + 231). \tag{4}$$

Solomon's theorem shows that the polynomial product on the signatures always results in a signature, closing an algebra on the signed cube.

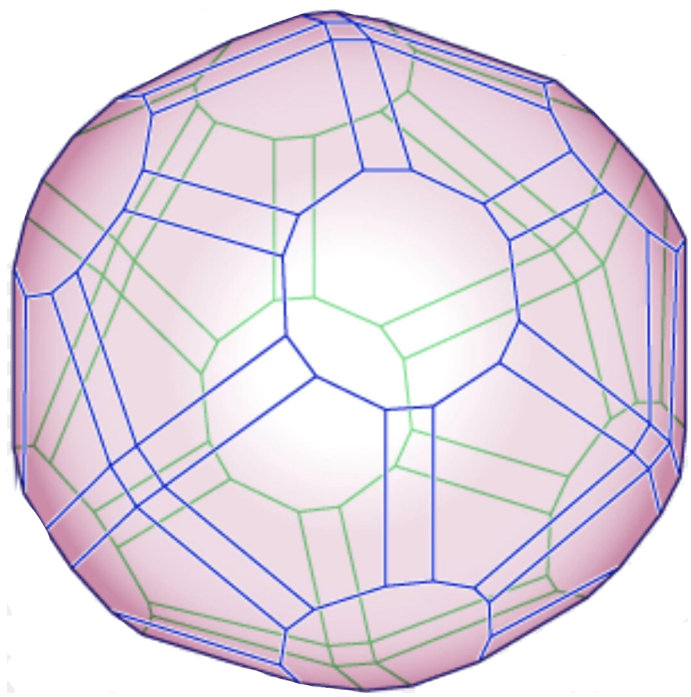

Figure 7: The permutoassociahedron

The permutoassociahedra vertices in dimension n list all associahedra trees with permutation labels, characterising the essential data for the scattering of $n+1$ particles. However, the essential axioms belong to categories in three dimensions: the braided monoidal categories of quantum computation. We are interested in finding the three dimensional essence of these polytopes, for application to quantum gravity. This means defining a sequence of trivalent graphs, picking out three special neighbouring permutations to any permutation in S_n. In the case of S_3, there are three other permutations resulting from a flip S_2. Restricting to a subset of 3 particles in a set of $n+1$, we can identify three outgoing edges.

A double copy of figure 7 has 240 vertices, the number of roots in the $\mathbf{e_8}$ Lie algebra lattice. The connection between octonions and this lattice was often used in Lie group compactification approaches to gravity, originating decades ago in the work of Tony Smith [28]. We study the relations between 24 dimensional lattices (for three qubits and one qutrit) and three copies of $\mathbf{e_8}$. To obtain the 196560 vertices of the Leech lattice [29], multiply the 240 by 3 times 273, where $273 = 1 + 16 + 256$. That is, double a 3-cube to 16 vertices (particles and antiparticles), double a 7-cube to 256 vertices (magnetic information), and add half a 1-cube. The cubes of dimension 1, 3 and 7 represent the spheres with parallelisable vector fields. From moonshine mathematics, we remember that the j-invariant Fourier transform for three dimensional gravity [30] is closely related to the modular form for the Leech lattice.

4 Three Dimensions in Quantum Gravity

The number of vertices in S_n is $n!$. Starting with the octahedron for S_3, we might blow up each vertex by a factor of 4 to find the 24 vertices of figure 4. Indeed, the truncated octahedron is the permutohedron polytope. For each $\sigma \in S_3$ there are 4 positions to place the letter 4 to make an element of S_4. For how long can we keep going this way? Blowing up each vertex of S_4 by a pentagon gives the permutoassociahedron, now a representation of S_5. Many higher dimensional edges are lost as blowups continue in dimension 3, but the set S_n is recovered.

Let us look more closely now at the trivalent nature of restricted S_n polytopes. For each blowup by an n-gon, the n edges that are not used in drawing the n-gon must join a neighbouring n-gon vertex. So we must partition these n edges into three sets, to follow the original three edges of the initial graph. For $n \mod 3 \equiv 0$, an equal partition is possible and the new graph is clearly well defined. For $n \mod 3 \in 1, 2$, one out of three outgoing edges is the odd man out. How do we know we can define a new polytope with the odd man edges connecting vertices consistently? If there was a Hamiltonian circuit on the initial S_n polytope, then each trivalent vertex automatically has an odd man out missing from the circuit.

Observe that S_4 and figure 7 both have Hamiltonian circuits. Since S_4 has a Hamiltonian circuit, and $5 \equiv 2$ is an odd man case, the pentagon blowup exists as figure 7 (ignoring for now the 4-valent part). Does a Hamiltonian circuit always exist on the new graph? In general, finding circuits is an NP-complete problem, but we have a specific sequence of graphs.

Let us first build a set of loops in the blowup using the original circuit as a template. The set covers every vertex once. Since an edge set from one polygon to another creates a bunch of square faces, one can only cover all vertices on an n-gon if one goes in and/or out across a leg using neighbouring edges. We stick to two pairs of edges. On each edge bunch, use either the first two or last two edges. These pairs can always link edges around the polygon in an obvious way, so that every vertex is covered.

Now there are two circuits around the original graph, one for each of the two edges along a bunch. For a Hamiltonian circuit, we need to join these two.

This is easily done by flipping the edge pair around one single square, since the used two edges on the bunch belong one to each circuit. We have established the following.

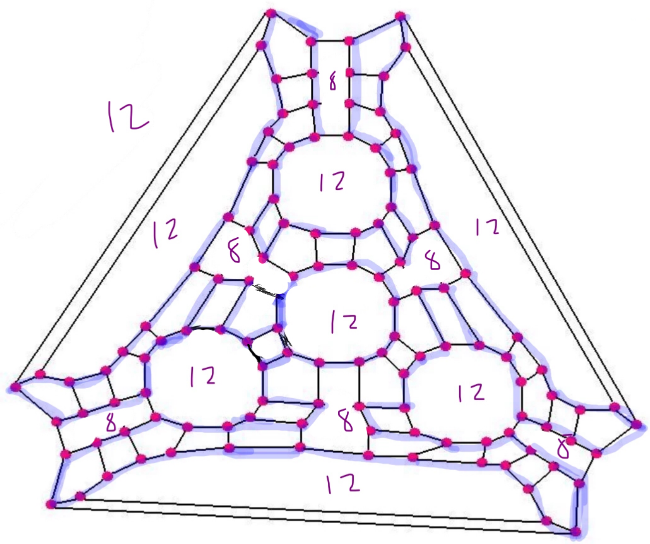

Figure 8: A Hamiltonian circuit on the permutoassociahedron

There exists a sequence of trivalent polytope graphs with Hamiltonian circuits on $n!$ vertices for any $n \geq 3$, each graph obtained from the previous one by an n-gon blowup at each vertex.

Figure 8 paints a Hamiltonian circuit on S_5. Its blowup is S_6, with 720 vertices. From these three dimensional reduced permutohedra, we obtain a set of reduced associahedra. Each $\sigma \in S_n$ is represented by a binary tree whose nodes have distinct levels, as opposed to the trees of figure 5, where the word (12)(34) does not distinguish the order of the bracketing. The hexagon S_3 collapses to the pentagon. The S_4 polytope collapses to the three dimensional associahedron with the contraction of three squares joined to two hexagons. Define the *reduced associahedron* for S_n in dimension 3 by collecting vertices according to the tree type labels. Finally, an associahedron permutohedron pair, each from dimension 3, defines a low dimensional reduced permutoassociahedron [10].

In three dimensions, we play with 3×3 matrices. The 27 dimensional exceptional Jordan algebra is defined using three qutrits, following [31]. These $1, 2, 3$ valued qutrits are the powers i, j and k of the matrices

$$\omega^i X^j P^k, \tag{5}$$

for X and P given by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}. \tag{6}$$

Of course X and P stand for space and momentum. Recall that these 27 words label the cubic paths for the tetractys of figure 3. This Jordan algebra J of Hermitian matrices over the octonions marks a vertex on the magic star [25] characterisation of the algebra $\mathbf{e_8}$ [26]. There is a triangle of J algebras and a triangle of $\overline{J}$. We can obtain the 13 points of a magic star from a three qutrit cube by projection along the diagonal. The source, target and centre point go to the centre of the star, while the first and last simplex points make the triangles of the star. Finally, the outer hexagon of the underlying $SU(3)$ plane come from the remaining edge midpoints on the cube. Observe that, geometrically, magic star planes are tiled by tetractys simplices.

With almost no input, we have uncovered some of the algebraic ingredients needed in quantum gravity. As is well known, physical rest masses for leptons and quarks are grouped as eigenvalues of 3×3 Hermitian matrices with special phases related to charge [12]. The 3-cube charges augment the basis neutrino state. The duality between the CMB neutrino cutoff (maximal cosmological wavelengths) and the Planck scale defines a Higgs scale via the inverse see-saw $m_H = \sqrt{m_\nu m_{\mathrm{Pl}}}$, and we see quantum masses growing in the braid soup from this pairing of (right handed) CMB neutrinos with local neutrino states, in a cosmological localisation a la Langlands.

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