Quantum Field Theory in fractal space-time with negative Hausdorff-Colombeau dimensions.The solution cosmological constant problem.

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Abstract. The cosmological constant problem arises because the magnitude of vacuum energy density predicted by quantum field theory is about 120 orders of magnitude larger than the value implied by cosmological observations of accelerating cosmic expansion. We pointed out that the fractal nature of the quantum space-time with negative Hausdorff- Colombeau dimensions can resolve this tension. The canonical Quantum Field Theory is widely believed to break down at some fundamental high-energy cutoff Λ_* and therefore the quantum fluctuations in the vacuum can be treated classically seriously only up to this high-energy cutoff. In this paper we argue that Quantum Field Theory in fractal space-time with negative Hausdorff-Colombeau dimensions gives high-energy cutoff on natural way. In order to obtain disered physical result we apply the canonical Pauli-Villars regularization up to Λ. It means that there exist the ghost-driven acceleration of the univers hidden in cosmological constant.

Content

I.Introduction

I.1.The formulation of the cosmological constant problem.

The cosmological constant problem arises at the intersection between general relativity and quantum field theory, and is regarded as a fundamental unsolved problem in modern physics. Remind that a peculiar and truly quantum mechanical feature of the quantum fields is that they exhibit zero-point fluctuations everywhere in space, even in regions which are otherwise 'empty' (i.e.devoid of matter and radiation).This vacuum energy density is believed to act as a contribution to the cosmological constant Λ appearing in Einstein's field equations from 1917,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T'_{\mu\nu} \tag{1.1.1}$$

where $R_{\mu\nu}$ and R refer to the curvature of space-time, $g_{\mu\nu}$ is the metric, $T'_{\mu\nu}$ the energy-momentum tensor,

$$T'_{\mu\nu} = T_{\mu\nu} + \frac{c^4\Lambda}{8\pi G}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.1.2}$$

where $T_{\mu\nu}$ is the energy- momentum tensor of matter. Thus $T'_{00} = T_{00} + \varepsilon_\Lambda$, $T'_{\alpha\beta} = T_{\alpha\beta} + \delta_{\alpha\beta} P_\Lambda$, where

$$\varepsilon_\Lambda = -P_\Lambda = c^4\Lambda/8\pi G. \tag{1.1.3}$$

Remind that under Lorentz transformations $(\varepsilon_\Lambda, P_\Lambda) \to \varepsilon'_\Lambda, (\varepsilon_\Lambda, P_\Lambda) \to P'_\Lambda$ the quantities ε_Λ and P_Λ are changes by law

$$\varepsilon'_\Lambda = \frac{\varepsilon_\Lambda + \beta^2 P_\Lambda}{1-\beta^2}, P'_\Lambda = \frac{P_\Lambda + \beta^2 \varepsilon_\Lambda}{1-\beta^2}. \tag{1.1.4}$$

Thus for the quantities ε_Λ and P_Λ Lorentz invariance holds by Eq(1.1.3) [1].

In modern cosmology it is assumed that the observable universe was initially vacuumlike, i.e., the cosmological medium was non-singular and Lorentz invariant. In the earlier, non-singular Friedmann cosmology the Friedmann universe comes into being during the phase transition of an initial vacuumlike state to the state of 'ordinary' matter [2],[3].

The Friedmann equations start with the simplifying assumption that the universe is spatially homogeneous and isotropic, i.e. the cosmological principle; empirically, this is justified on scales larger than ~100 Mpc. The cosmological principle implies that the metric of the universe must be of the form Robertson-Walker metric [2].

Robertson-Walker metric reads

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \quad (1.1.5)$$

For such a metric, the Ricci curvature scalar is $R = -6k$ and it is said that space has the curvature k.The scaling factor $a(t)$ rescales this curvature for a given time t, producing a curvature $k(t) = k/a(t)$.The scaling factor $a(t)$ is given by two independent Friedmann equations for modeling a homogeneous, isotropic universe reads

$$\dot{a}^2 = \frac{G}{3}\varepsilon a^2 - k, \ddot{a} = -\frac{G}{6}(\varepsilon + 3p) \quad (1.1.6)$$

and the equation of state

$$p = p(\varepsilon), \quad (1.1.7)$$

where p is pressure and ε is a density of the cosmological medium. For the case of the vacuumlike cosmological medium equation of state reads [2],[3],[4]:

$$p = -\varepsilon. \quad (1.1.8)$$

By virtue of Friedman's equations (1.1.6) in the universe filled with a vacuum-like medium, the density of the medium is preserved, i.e. $\varepsilon =$ const, but the scale factor $a(t)$ grows exponentially. By virtue of continuity, it can be assumed that the admixture of a substance does not change the nature of the growth of the latter, and the density of the medium hardly changes. This growth, interpreted by analogy with the Friedmann models as an expansion of the universe, but almost without changing the density of the medium! - was named inflation. The idea of inflation is the basis of inflation scenarios [2].

Non-singular cosmology [2],[4] suggests that the initial state of the observable universe was vacuum-like, but unstable with respect to the phase transition to the ordinary non-Lorentz-invariant medium. This, for example, takes place if, by virtue of the equations of state of the medium, a fluctuation decrease in its density d violates the condition of vacuum-like degeneration, $p = -\varepsilon$ or, which is the same, $3p + \varepsilon = -2\varepsilon < 0$, replacing it with

$$-2\varepsilon < 3p + \varepsilon < 0. \quad (1.1.9)$$

According to Friedman's equations, it corresponds to an accelerated expansion of the cosmological medium, accompanied by a drop in its density, which makes the process irreversible [2]. The impulse for expansion in this scenario, the vacuum-like environment, is not reported to itself (bloating), but to the emerging Friedmann environment.

In review [5], Weinberg indicates that the first published discussion of the contribution of quantum fluctuations to the cosmological constant was a 1967 paper by Zel'dovich [6].In his article [1] Zel'dovich emphasizes that zeropoint energies of particle physics theories cannot be ignored when gravitation is taken into account, and since he explicitly discusses the discrepancy between estimates of vacuum energy and observations, he is clearly pointing to a cosmological constant problem. As well known zeropoint energy density of scalar quantum field,etc.is divergent

$$\varepsilon(m) = \frac{2\pi c}{(2\pi\hbar)^3}\int_0^\infty \sqrt{p^2 + m^2c^2}\, p^2 dp = \infty. \quad (1.1.10)$$

In order avoid difficultnes mentioned above in article [3] Zel'dovich has applied Pauli-Villars regularization [7],[8] and obtain an finite result (his formulas (VIII.12-VIII.13) p.228)

$$\varepsilon_{vac} = -p_{vac} = \frac{1}{8}\int_0^\infty f(\mu)\mu^4(\ln\mu)d\mu = \frac{c^4\Lambda}{8\pi G}, \tag{1.1.11}$$

where

$$\int_0^\infty f(\mu)d\mu = \int_0^\infty f(\mu)\mu^2 d\mu = \int_0^\infty f(\mu)\mu^4 d\mu = 0. \tag{1.1.12}$$

Remark 1.1.1. Unfortunately the Eq(1.1.11)-Eq(1.1.12) gives nothing in order to obtain disered numerical values of the zero-point energy density ε.

In his paper [3], Zel'dovich arrives at a zero-point energy (his formula (IX.1))

$$\varepsilon_{\mathbf{vac}} = m\left(\frac{mc}{\hbar}\right)^3 \sim 10^{17}g/cm^3, \Lambda \sim 10^{-10}cm^{-2}, \tag{1.1.13}$$

where m (the ultra-violet cut-of) is taken equal to the proton mass. Zel'dovich notes that since this estimate exceeds observational bounds by 46 orders of magnitude it is clear that "...such an estimate has nothing in common with reality".

In his paper [3], Zel'dovich wroted:" Recently A.D. Sakharov proposed a theory of gravitation, or, more precisely, a justification GR equations based on consideration of vacuum fluctuations.In this theory, the essential assumption is that there is some elementary length L or the corresponding limiting momentum $p_0 = \hbar/L$. Shorter lengths or for large impulses theory is not applicable. Sakharov gets the expression of gravitational constant G through L or p_0 (his formula (IX.6))

$$G = \frac{c^3L^2}{\hbar} = \frac{\hbar c^3}{p_0^2}. \tag{1.1.14}$$

This expression has been known since the days of Planck, but it was read "from right to left": gravity determines the length L and the momentum p_0. According to Sakharov, L and p_0 are primary. Substitute (IX. 6) in the expression (IX. 4), we get

$$\rho_\Lambda = \frac{m^6c^5}{p_0^2\hbar^3}, \varepsilon_{vac} = \frac{m^6c^7}{p_0^2\hbar^3}. \tag{1.1.15}$$

That is expressions that the first members (in the formulas (VIII.10), (VIII. 11)) which are vanishes (with $p_0 \to \infty$).Thus, we can suggest the following interpretation of the cosmological constant: there is a theory of elementary particles, which would give (according to the mechanism that has not been revealed at the present time) identically zero vacuum energy, if this theory were applicable infinitely, up to arbitrarily large momentum; there is a momentum p_0, beyond which the theory is nont aplicable; along with other implications, modifying the theory gives different from zero vacuum energy; general considerations make it likely that the effect is portional p_0^{-2}.Clarification of the question of the existence and magnitude of the cosmological constant will also be of fundamental importance for the theory of elementary particles".

In contrast with Zel'dovich paper [3] we assume that Poincaré group is deformed at some fundamental high-energy cutoff Λ_* [9],[10],[11] in accordance on the basis of the following deformed Poisson brackets

$$\begin{gathered}\{x^\mu, x^\nu\} = \varkappa^{-1}(x^\mu\eta^{0\nu} - x^\nu\eta^{\mu 0}), \{p^\mu, p^\nu\} = 0,\\ \{x^\mu, p^\nu\} = -\eta^{\mu\nu} + \varkappa^{-1}\eta^{\mu 0}p^\nu\end{gathered} \tag{1.1.16}$$

where $\mu, \nu, = 0, 1, 2, 3,\ \eta^{\mu\nu} = (+1, -1, -1, -1)$ and is a parameter identified as the ratio between the high-energy cutoff Λ_* and the light speed. The corresponding to (1.1.16) momentum transformation reads [11]

$$\begin{aligned} p_0' &= \frac{\gamma(p_0 - up_x)}{1 + (c\varkappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, p_x' = \frac{\gamma(p_x - up_0/c^2)}{1 + (c\varkappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, \\ p_y' &= \frac{p_y}{1 + (c\varkappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, p_z' = \frac{p_z}{1 + (c\varkappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, \end{aligned} \tag{1.1.17}$$

and coordinate transformation reads [11]

$$\begin{aligned} t' &= \frac{\gamma(t - ux/c^2)}{1 + (c\varkappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, x' = \frac{\gamma(x - ut)}{1 + (c\varkappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, \\ y' &= \frac{y}{1 + (c\varkappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, z' = \frac{z}{1 + (c\varkappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, \end{aligned} \tag{1.1.18}$$

where $\gamma = \sqrt{1 - u^2/c^2}$ It is easy to check that the energy $E = c\varkappa$, identified as the high-energy cutoff Λ_*, is an invariant as it is also the case for the fundamental length $l_{\Lambda_*} = \hbar c/E = \hbar/\varkappa$.

Remark 1.1.2. Note that the transformation (1.1.17) defined in p-space and the transformation (1.1.18) defined in x-space becomes Lorentz for small energies and momenta and defines a large invariant energy $l_{\Lambda_*}^{-1}$. The high-energy cutoff Λ_* is preserved by the modified action of the Lorentz group [9],[10].

This meant that the canonical concept of metric as quadratic invariant collapses at high energies, being replaced by the non-quadratic invariant [9]:

$$\|p\|^2 = \frac{\eta^{ab} p_a p_b}{(1 + l_{\Lambda_*} p_0)}, \tag{1.1.19}$$

or by the non-quadratic invariant

$$\|p\|^2 = \frac{\eta^{ab} p_a p_b}{(1 - l_{\Lambda_*} p_0)}, \tag{1.1.20}$$

where $l_{\Lambda_*} = \Lambda_*^{-1}, a, b = 0,1,2,3$.

Remark 1.1.3.Note that:

(i) the invariant (1.1.16) is infinite for the new negative invariant energy scale of the theory $\Lambda_* = -l_{\Lambda_*}^{-1}$, and it's not quadratic for energies close or above and

(ii) the invariant (1.1.17) is infinite for the new positive invariant energy scale of the theory $\Lambda_* = l_{\Lambda_*}^{-1}$.

Remark 1.1.4.It is also clear from Eq.(1.1.16) and Eq.(1.1.17) that the symmetry of positive and negative values of the energy is broken.The two theories with the two signs of l_Λ obviously are physically distinct; and we know of no theoretical argument which fixes
the sign of l_Λ

The massive particles have a positive invariant $\|p\|^2 > 0$ which can be identified with the square of the mass $\|p\|^2 = m^2, (c = 1)$.Thus in the case of the invariant (1.1.16) we

obtain

$$\frac{p_0^2 - p^2}{(1 + l_{\Lambda_*} p_0)^2} = m^2, p_0 \in (-l_{\Lambda_*}^{-1}, \infty) \tag{1.1.21}$$

From Eq.(1.1.18) we obtain

$$p_0 = \frac{m^2 l_{\Lambda_*}}{1 - m^2 l_{\Lambda_*}^2} + \frac{1}{\sqrt{1 - m^2 l_{\Lambda_*}^2}} \sqrt{\frac{m^4 l_{\Lambda_*}^2}{1 - m^2 l_{\Lambda_*}^2} + (p^2 + m^2)}\,. \qquad (1.1.22)$$

In the case of the invariant (1.1.17) we obtain

$$\frac{p_0^2 - p^2}{(1 - l_{\Lambda_*} p_0)^2} = m^2, p_0 \in (-\infty, l_{\Lambda_*}^{-1}). \qquad (1.1.23)$$

From Eq.(1.1.20) we obtain

$$p_0 = -\frac{m^2 l_{\Lambda_*}}{1 - m^2 l_{\Lambda_*}^2} - \frac{1}{\sqrt{1 - m^2 l_{\Lambda_*}^2}} \sqrt{\frac{m^4 l_{\Lambda_*}^2}{1 - m^2 l_{\Lambda_*}^2} + (p^2 + m^2)} \qquad (1.1.24)$$

The action for a scalar field φ must be invariant under the deformed Lorentz transformations.The invariant action reads [10]

$$S = \frac{1}{2} \int d^4x \frac{\eta^{ab}(\partial_a \varphi)(\partial_b \varphi)}{[1 + l_{\Lambda_*} \partial_0 \varphi]} + \frac{m^2}{2} \varphi^2. \qquad (1.1.25)$$

Thus there is no linear field equation.

I.2. Zel'dovich approuch by using Pauli-Villars regularization revisited.Ghosts as physical dark matter.

Remind that vacuum energy density for free scalar quantum field is

$$\varepsilon(\mu) = \frac{1}{2} \frac{c}{(2\pi\hbar)^3} \int_0^\infty 4\pi \sqrt{p^2 + \mu^2}\, p^2 dp = K \int_0^\infty \sqrt{p^2 + \mu^2}\, p^2 dp = KI(\mu), \qquad (1.2.1)$$

where $\mu = m_0 c$. From Eq.(1.2.1) one obtains [1]

$$p(\mu) = \frac{K}{3} \int_0^\infty \frac{p^4 dp}{\sqrt{p^2 + \mu^2}} = KF(\mu). \qquad (1.2.2)$$

For fermionic quantum field one obtains

$$\varepsilon(\mu) = KI(\mu), p(\mu) = -4KF(\mu). \qquad (1.2.3)$$

Thus free vacuum energy density ε and corresponding pressure p is

$$\varepsilon = \sum_i C_i I(\mu_i), P = \sum_i C_i F(\mu_i). \qquad (1.2.4)$$

Eq.(1.2.4) by using Pauli-Willars regularization [7],[8] in general case one obtains [3]

$$\varepsilon = \int f(\mu) I(\mu) d\mu, P = \int f(\mu) F(\mu) d\mu. \qquad (1.2.5)$$

Let us evaluate now the follwing quantities

$$I(\mu, p_0) = \int_0^{p_0} p^2 \sqrt{p^2 + \mu^2}\, dp = \int_0^{p_\mu} p^2 \sqrt{p^2 + \mu^2}\, dp + \int_{p_\mu}^{p_0} p^2 \sqrt{p^2 + \mu^2}\, dp =$$
$$= \int_0^{p_\mu} p^2 \sqrt{p^2 + \mu^2}\, dp = \int_{p_\mu}^{p_\mu} p^3 \sqrt{1 + \frac{\mu^2}{p^2}}\, dp + \int_{p_\mu}^{p_0} p^3 \sqrt{1 + \frac{\mu^2}{p^2}}\, dp \qquad (1.2.6)$$

and

$$F(\mu,p_0)=\frac{1}{3}\int\limits_0^{p_0}\frac{p^4dp}{\sqrt{p^2+\mu^2}}=\frac{1}{3}\int\limits_0^{p_\mu}\frac{p^4dp}{\sqrt{p^2+\mu^2}}+\frac{1}{3}\int\limits_{p_\mu}^{p_0}\frac{p^4dp}{\sqrt{p^2+\mu^2}}=$$
$$\frac{1}{3}\int\limits_0^{p_\mu}\frac{p^4dp}{\sqrt{p^2+\mu^2}}+\frac{1}{3}\int\limits_{p_\mu}^{p_0}\frac{p^3dp}{\sqrt{1+\frac{\mu^2}{p^2}}}, \tag{1.2.7}$$

where $p_\mu = r\mu, r>1, \mu/p<1/r<1.$ Note that

$$\sqrt{1+\frac{\mu^2}{p^2}}=1+\frac{1}{2}\frac{\mu^2}{p^2}-\frac{1}{8}\frac{\mu^4}{p^4}+\frac{1}{16}\frac{\mu^6}{p^6}+\ldots.$$
$$p^2\sqrt{p^2+\mu^2}=p^3\sqrt{1+\frac{\mu^2}{p^2}}=p^3+\frac{1}{2}\mu^2p-\frac{1}{8}\frac{\mu^4}{p}+\frac{1}{16}\frac{\mu^6}{p^3}+\ldots. \tag{1.2.8}$$

By inserting Eq.(1.2.8) into Eqs.(1.2.6) one obtains

$$I(\mu,p_0)=C_1\mu^4+\frac{1}{4}p_0^4+\frac{1}{4}\mu^2p_0^2-\frac{1}{8}\mu^4\ln\left(\frac{p_0}{\mu}\right)-\frac{1}{32}\frac{\mu^6}{p_0^2}+p_0^{-5}O(\mu^8), \tag{1.2.9}$$

where $C_1\mu^4=\int\limits_0^{p_\mu}p^2\sqrt{p^2+\mu^2}\,dp.$ Note that

$$\left(\sqrt{1+\frac{\mu^2}{p^2}}\right)^{-1}=1-\frac{1}{2}\frac{\mu^2}{p^2}+\frac{3}{8}\frac{\mu^4}{p^4}-\frac{5}{16}\frac{\mu^6}{p^6}+\ldots. \tag{1.2.10}$$

By inserting Eq.(1.2.10) into Eq.(1.2.7) one obtains

$$F(\mu,p_0)=C_2\mu^4+\frac{1}{12}p_0^4-\frac{1}{12}\mu^2p_0^2+\frac{1}{8}\mu^4\ln\left(\frac{p_0}{\mu}\right)+\frac{5}{32}\frac{\mu^6}{p_0^2}+p_0^{-5}O(\mu^8). \tag{1.2.11}$$

By inserting Eq.(1.2.9) and Eq.(1.2.11) into Eqs.(1.2.5) one obtains

$$\varepsilon=\frac{1}{4}p_0^4\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)d\mu+\frac{1}{4}p_0^2\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^2d\mu+\left(C_1-\frac{1}{8}\ln p_0\right)\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^4d\mu+$$
$$+\frac{1}{8}\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^4(\ln\mu)d\mu-\left(\frac{1}{p_0^2}\right)\frac{1}{32}\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^6d\mu+O\left(\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^8\right)p_0^{-5},$$
$$p=\frac{1}{12}p_0^4\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)d\mu-\frac{1}{12}p_0^2\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^2d\mu+\left(C_2+\frac{1}{8}\ln p_0\right)\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^4d\mu-$$
$$-\frac{1}{8}\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^4(\ln\mu)d\mu+\left(\frac{5}{p_0^2}\right)\frac{1}{32}\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^6d\mu+O\left(\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^8\right)p_0^{-5}. \tag{1.2.12}$$

We choose now

$$\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)d\mu=\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^2d\mu=\int\limits_0^{\mu_{\mathbf{eff}}}f(\mu)\mu^4d\mu=0. \tag{1.2.13}$$

By inserting Eq.(1.2.13) into Eqs.(1.2.12) one obtains

$$\begin{aligned}\varepsilon(\mu_{\mathbf{eff}}) &= \frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + O(p_0^{-2}),\\ p(\mu_{\mathbf{eff}}) &= -\frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + O(p_0^{-2}).\end{aligned} \tag{1.2.14}$$

Taking the limit $p \to \infty$ in Eq.(1.2.14) gives

$$\begin{aligned}\varepsilon(\mu_{\mathbf{eff}}) &= \frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu,\\ p(\mu_{\mathbf{eff}}) &= -\frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu.\end{aligned} \tag{1.2.15}$$

Thus finally we obtain [3]

$$\varepsilon(\mu_{\mathbf{eff}}) = -p(\mu_{\mathbf{eff}}) = \frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu = \frac{c^4\Lambda}{8\pi G}. \tag{1.2.16}$$

Remark 1.2.1. Remind that Pauli-Villars regularization consists of introducing a fictitious mass term. For example, we would replace a propagator $1/\left(k^2 - m_0^2 + i\epsilon\right)$, by the regulated propagator

$$\Delta(k^2) = \sum\nolimits_{i=0}^{N} \frac{a_i}{k^2 - m_i^2 + i\epsilon} = \frac{1}{k^2 - m_0^2 + i\epsilon} - \sum\nolimits_{i=1}^{N} \frac{a_i}{k^2 - m_i^2 + i\epsilon}, \tag{1.2.17}$$

where $a_0 = 1$ and $m_i, i = 1,2,\ldots N$ can be thought of as the mass of a fictitious heavy particle, whose contribution is subtracted from that of an ordinary particle. Assume that $m_i^2/k^2 < 1,$ if we expand each term of this sum (1.2.16) as a power series in $k^2 + i\epsilon$ we get

$$\Delta(k^2) = \sum\nolimits_{i=0}^{N} \frac{a_i}{k^2 + i\epsilon} + \sum\nolimits_{i=0}^{N} \frac{a_i m_i^2}{\left(k^2 + i\epsilon\right)^2} + \sum\nolimits_{i=0}^{N} O\left(\frac{1}{\left(k^2 + i\epsilon\right)^3}\right). \tag{1.2.18}$$

For a renormalizable theory the maximum supercriticial power of divergence of any integral is quadratic, so that the $O(1/k^6)$ terms are ultraviolet finite. The finiteness of the regulated integral is then guaranteed by requiring that

$$\sum\nolimits_{i=0}^{N} a_i = 0, \sum\nolimits_{i=0}^{N} a_i m_i^2 = 0. \tag{1.2.19}$$

Remark 1.2.2.Note that in order to aply Pauli-Villars regularization to QFT with Lagrangian $\mathcal{L}(\varphi,\psi,\partial_\mu\varphi,\partial_\mu\psi)$ we would replace the Lagrangian $\mathcal{L}(\varphi,\psi,\partial_\mu\varphi,\partial_\mu\psi)$ by Lagrangian $\underline{\mathcal{L}}\left(\underline{\varphi},\underline{\psi},\partial_\mu\underline{\varphi},\partial_\mu\underline{\psi}\right)$, where [7]:

$$\underline{\varphi}(x) = \varphi(x) + \sum\nolimits_n b_n\breve{\varphi}_n(x,\mu_n^2), \underline{\psi}(x) = \psi(x) + \sum\nolimits_n c_n\breve{\psi}_n(x,\varkappa_n^2), \tag{1.2.20}$$

where commutator for $\breve{\varphi}_n$ and anticommutator for $\breve{\psi}_n$ reads

$$\begin{aligned}[\breve{\varphi}_m(x,\mu_m^2),\breve{\varphi}_n(x',\mu_n^2)] &= -i\rho_n\Delta(x-x',\mu_n^2)\delta_{mn},\\ \{\breve{\psi}_m(x,\varkappa_m^2),\breve{\psi}_n(x',\varkappa_n^2)\} &= -i\varepsilon_n S(x-x',\varkappa_n^2)\delta_{mn}.\end{aligned} \tag{1.2.21}$$

From Eqs.(1.2.20)-Eqs.(1.2.21) one obtains

$$\left[\underline{\varphi}(x),\underline{\varphi}(x')\right] = i\sum_{n=0}^{N}\rho_n b_n^2\Delta(x-x',\mu_n^2),$$
$$\left[\underline{\psi}(x),\underline{\psi}(x')\right] = -i\sum_{n=0}^{N}\varepsilon_n\bar{c}_n c_n S(x-x',\varkappa_n^2). \tag{1.2.22}$$

Assume now that

$$\sum_{n=0}^{N}\rho_n b_n^2 = 0,\sum_{n=0}^{N}\rho_n b_n^2\mu_n^2 = 0,\sum_{n=0}^{N}\varepsilon_n\bar{c}_n c_n = 0,\sum_{n=0}^{N}\varepsilon_n\bar{c}_n c_n\varkappa_n^2 = 0. \tag{1.2.23}$$

From Eqs.(1.2.23) it follows directly that QFT with Lagrangian $\underline{\mathcal{L}}\left(\underline{\varphi},\underline{\psi},\partial_\mu\underline{\varphi},\partial_\mu\underline{\psi}\right)$ is finite QFT with indefinite metric [4],see Remark 1.2.1.

Remark 1.2.3. Note that "bad ghosts" represent general meaning of the word "ghost" in theoretical physics: states of negative norm [7] or fields with the wrong sign of the kinetic term, such as Pauli–Villars ghosts φ, whose existence allows the probabilities to be negative thus violating unitarity.The quadratic lagrangian $\mathcal{L}_\varphi^2$ for φ begins with a wrong sign kinetic term [in (+ – – –) signature]

$$\mathcal{L}_\varphi^2 = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi+\ldots \tag{1.2.24}$$

Remark 1.2.4. Note that in order to obtain Eqs.(1.2.14), the standard quantum fields do not need to couple directly to the ghost sector. In this paper the ghost sector is considered as physical mechanism which acts only on a function $f(\mu)$ in Eqs.(1.2.13).It means that there exist the ghost-driven acceleration of the univers hidden in cosmological constant Λ.

Remark 1.2.5. As pointed out in paper [12] even if the standard model fields have no direct couplings to the ghost sector, they will indirectly interact with it through gravity, and the propagation of gravity through the ghost condensate gives rise to a fascinating modification of gravity in the IR. However,no modifications of gravity can be seen directly, and no cosmological experiment can distinguish the ghost-driven acceleration from a cosmological constant.

Remark 1.2.6. In order to obtain disered physical result from Eqs.(1.2.15),i.e.,

$$\varepsilon_{\mathbf{vac}} = 0.7\times10^{-29}gcm^{-3} = 2.8\times10^{-47}Gev^4/\hbar^3c^5 \tag{1.2.25}$$

we assume that

$$f(\mu) = f_{s.m.}(\mu)+f_{g.m.}(\mu), \tag{1.2.26}$$

where $f_{s.m.}(\mu)$ corresponds to standard matter and where $f_{g.m.}(\mu)$ corresponds to a physical ghost matter.

Remark 1.2.7. We assume now that

$$|f(\mu)| = \begin{cases} O(\mu^{-n}),n>1 & \mu\leq\mu_{\mathbf{eff}} \\ 0 & \mu>\mu_{\mathbf{eff}} \end{cases} \tag{1.2.27}$$

From Eq.(1.2.27) and Eqs.(1.2.15) it follows directly that

$$|p(\mu_{\mathbf{eff}})| = |\varepsilon(\mu_{\mathbf{eff}})| = \frac{1}{8}\left|\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu\right| \leq O\left(\mu_{\mathbf{eff}}^{-n+5}\ln\mu_{\mathbf{eff}}\right). \tag{1.2.28}$$

Remark 1.2.8.However serious problem arises from non-renormalizability of canonical quantum gravity with Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R. \tag{1.2.29}$$

For example taking Λ^3 particles of energy a per unit volume gives the gravitational self-energy density of order Λ^6, i.e.,the density ε_Λ diverges as Λ^6

$$\varepsilon_\Lambda \simeq G\Lambda^6, \tag{1.2.30}$$

where Λ is a high-energy cutoff [5].

In order to avoid these difficulties we apply instead Einstein-Hilbert action (1.2.29) the gravitational action which include terms quadratic in the curvature tensor

$$\Im = -\int d^4x\sqrt{-g}\,(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + 2\kappa^{-2}R), \tag{1.2.31}$$

Remark 1.2.8.Gravitational actions (1.2.31) which include terms quadratic in the curvature tensor are renormalizable [13]. The requirement that the graviton propagator behave like p^{-4} for large momenta makes it necessary to choose the indefinite-metric vector space over the negative-energy states.These negative-norm states cannot be excluded from the physical sector of the vector space without destroying the unitarity of the $\mathbf{S}$ matrix, however, for their unphysical behavior may be restricted to arbitrarily large energy scales Λ_* by an appropriate limitation on the renormalized masses m_2 and m_0.

Remark 1.2.9.We assum that $m_0c \gg \mu_{\mathbf{eff}}, m_2c \gg \mu_{\mathbf{eff}}$.

Remark 1.2.10.The canonical Quantum Field Theory is widely believed to break down at some fundamental high-energy cutoff Λ_* and therefore the quantum fluctuations in the vacuum can be treated classically seriously only up to this high-energy cutoff, see for example [14]. In this paper we argue that Quantum Field Theory in fractal space-time with negative Hausdorff-Colombeau dimensions [15] gives high-energy cutoff on natural way.

II. Ghosts as physical dark matter.

II.1.Pauli-Villars ghosts as physical dark matter.

Before explaining the role of PV ghosts,etc. as physical dark matter remind the idea of PV regularization as a conventional UV regularization.We consider, as an example, the scalar field theory with the interaction $\lambda\varphi^4$. Lagrangian density of this theory reads

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{m_0^2}{2}\varphi^2 + \lambda\varphi^4. \tag{2.1.1}$$

This theory requires UV regularization (e.g. in (2+1) and (3+1) dimensions). Let us show that it is sufficient to introduce N extra fields with large mass playing the role of the regularization parameter. Lagrangian density can be rewritten as follows

$$\mathcal{L} = \sum\nolimits_{i=0}^{N} (-1)^i \left(\frac{1}{2} \partial^{\mu} \varphi \partial_{\mu} \varphi - \frac{m_i^2}{2} \varphi_i^2 \right) + \lambda : \varphi^4 :,$$
$$\varphi = \varphi_0 + \breve{\varphi} = \sum\nolimits_{i=0}^{N} \varphi_i, \breve{\varphi} = \sum\nolimits_{i=1}^{N} a_i \varphi_i. \tag{2.1.2}$$

Here the symbol ”::” means that in perturbation theory we drop Feynman diagrams with loops containing only one vertex. The φ_0 is usual field with mass m_0 and the $\varphi_i, i = 1,\ldots,N$ is the extra field with mass $m_i, i = 1,\ldots,N.$ It can be shown that in (3+1)-dimensional theory the introduction of one PV field is sufficient for the ultraviolet regularization of perturbation theory in λ. One can show that momentum space Feynman diagrams in the original theory with Lagrangian density (2.1.1) diverge no more than quadratically [16]-[18] (beside of vacuum diagrams) shown in Fig.2.1.1.

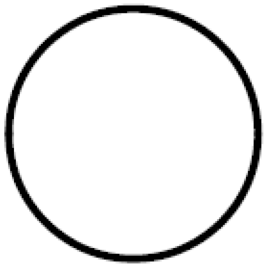

Fig.2.1.1.One-loop massive vacuum diagram.

If we consider now Feynman diagrams in the theory with Lagrangian density (2.1.2) we see that propagators of fields φ_0 and $\breve{\varphi}$ sum up in corresponding diagrams so that we obtain the following expression which plays the role of regularized propagator

$$\Delta(k^2) = \sum\nolimits_{j=0}^{N} \frac{a_j}{k^2 - m_j^2 + i0} = \frac{1}{k^2 - m_0^2 + i0} - \sum\nolimits_{j=1}^{N} \frac{a_j}{k^2 - m_j^2 + i0}, \tag{2.1.3}$$

where $k^2 = k_0^2 - k_1^2 + k_2^2 + k_3^2$. Integral corresponding to vacuum diagram is

$$\Im = \int \frac{d^4k}{(2\pi)^4} \Delta(k^2) = \int \frac{d^4k}{(2\pi)^4} \sum\nolimits_{j=0}^{N} \frac{a_j}{k^2 - m_j^2 + i0}. \tag{2.1.4}$$

To do this integral, since it is convergent, we can Wick rotate. Then we get

$$\Im_E = \frac{i}{8\pi^2} \int_0^{\infty} dk_E \sum\nolimits_{j=0}^{N} \frac{a_j k_E^3}{k_E^2 + m_j^2}. \tag{2.1.5}$$

To do this integral, since it is convergent, we can dealing with regularized integral

$$\Im(\varepsilon, \Lambda) = \frac{i}{8\pi^2} \int_{\varepsilon}^{\Lambda} dk_E \sum\nolimits_{j=0}^{N} \frac{a_j k_E^3}{k_E^2 + m_j^2}, \tag{2.1.6}$$

where $\varepsilon \asymp 0, \Lambda \asymp \infty$, i.e. $\Im(\varepsilon, \Lambda) \approx \Im_E$. We assume now that Pauli-Villars conditions given by Eqs.(1.2.18) holds.Let us consider now the quantity

$$\Im_{\eta} \triangleq \Im_{\eta}(\varepsilon, \Lambda) = \frac{i}{8\pi^2} \int_{\varepsilon}^{\Lambda} dk_E \sum\nolimits_{j=0}^{N} \frac{a_j k_E^3}{k_E^2 + \eta m_j^2}, \tag{2.1.7}$$

where $\eta \in (0,1]$, and therefore from Eq.(2.1.7) we obtain

$$\Im_\eta|_{\eta=0} = \frac{i}{8\pi^2}\int_\varepsilon^\Lambda dk_E \sum\nolimits_{j=0}^N a_j k_E = \frac{i}{8\pi^2}\sum\nolimits_{j=0}^N a_j \int_\varepsilon^\Lambda k_E dk_E \equiv 0, \qquad (2.1.8)$$

since Eqs.(1.2.18) holds.From Eq.(2.1.7) by differentiation we obtain

$$\frac{d}{d\eta}\Im_\eta = \frac{i}{8\pi^2}\int_\varepsilon^\Lambda dk_E \sum\nolimits_{j=0}^N \frac{a_j m_j^2 k_E^3}{(k_E^2+\eta m_j^2)^2}, \qquad (2.1.9)$$

and therefore from Eq.(1.2.9) we obtain

$$\begin{aligned}\frac{d}{d\eta}\Im_\eta\Big|_{\eta=0} &= \frac{i}{8\pi^2}\int_\varepsilon^\Lambda dk_E \sum\nolimits_{j=0}^N \frac{a_j m_j^2 k_E^3}{(k_E^2+\eta m_j^2)^2}\Bigg|_{\eta=0} = \\ &= \frac{i}{8\pi^2}\sum\nolimits_{j=0}^N a_j m_j^2 \int_\varepsilon^\Lambda k_E^{-1} dk_E \equiv 0,\end{aligned} \qquad (2.1.10)$$

since Eqs.(1.2.18) holds.From Eq.(2.1.9) by differentiation we obtain

$$\begin{aligned}\frac{d^2}{d\eta^2}\Im_\eta &= \sum\nolimits_{j=0}^N \Re_j(\eta) = \frac{i}{4\pi^2}\int_\varepsilon^\Lambda dk_E \sum\nolimits_{j=0}^N \frac{a_j m_j^4 k_E^3}{(k_E^2+\eta m_j^2)^3}, \\ \Re_j(\eta) &= \frac{i a_j m_j^4}{4\pi^2}\int_\varepsilon^\Lambda dk_E \frac{k_E^3}{(k_E^2+\eta m_j^2)^3}.\end{aligned} \qquad (2.1.11)$$

Note that

$$\Re_j(\eta) \simeq \frac{i a_j m_j^4}{4\pi^2}\int_0^\infty dk_E \frac{k_E^3}{(k_E^2+\eta m_j^2)^3} = \frac{i a_j m_j^4}{4\pi^2}\frac{-i}{4\eta m_j^2} = \frac{a_j m_j^2}{16\pi^2\eta}. \qquad (2.1.12)$$

Thus

$$\frac{d}{d\eta}\Im_\eta = \sum\nolimits_{j=0}^N \int_0^1 \Re_j(\eta) d\eta = \sum\nolimits_{j=0}^N \frac{a_j m_j^2}{16\pi^2}\ln\eta \qquad (2.1.13)$$

and

$$\Im_\eta = \sum\nolimits_{j=0}^N \frac{a_j m_j^2}{16\pi^2}(\eta\ln\eta - \eta), \qquad (2.1.14)$$

Therefore

$$\Im(\varepsilon,\Lambda) = \Im_\eta|_{\eta=1} = -\sum\nolimits_{j=0}^N \frac{a_j m_j^2}{16\pi^2} \equiv 0, \qquad (2.1.15)$$

since Eqs.(1.2.18) holds.Thus integral (2.1.4) corresponding to vacuum diagram by using Pauli-Villars renormalization identically equal zero,i.e.

$$\mathbf{Ren}_{PV}(\Im) = \int \frac{d^4k}{(2\pi)^4}\Delta(k^2) = \int \frac{d^4k}{(2\pi)^4}\sum\nolimits_{j=0}^N \frac{a_j}{k^2-m_j^2+i0} \equiv 0. \qquad (2.1.16)$$

Let us consider now how this method works in the case of the simplest scalar diagram shown in Fig.2.1.2. The corresponding Feinman integral has the form

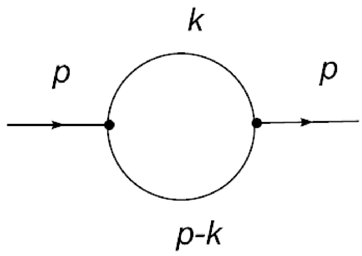

Fig.2.1.2

$$\Im(p^2) = \frac{1}{(2\pi)^4} \int \frac{d^4k}{(k^2 - m_0^2 + i0)[(p^2 - k^2) - m_0^2 + i0]}. \qquad (2.1.17)$$

Regularized Feinman integral (2.1.17) reads

$$\Im_{reg}(p^2) = \frac{1}{(2\pi)^4} \int \sum_{j=0}^{N} \frac{a_j d^4k}{(k^2 - m_j^2 + i0)[(p^2 - k^2) - m_j^2 + i0]}, \qquad (2.1.18)$$

where $N = 1$.To do this integral, since it is convergent, we can Wick rotate. Then we get

$$\Im_{reg}(p^2) = \frac{i}{(2\pi)^4} \int \sum_{j=0}^{N} \frac{a_j d^4k}{(k^2 + m_j^2)[(p^2 - k^2) + m_j^2]}. \qquad (2.1.19)$$

The integral (2.1.19) can be written as

$$\Im_{reg}(p^2) = \frac{i}{(2\pi)^4} \int_0^1 dx \int \sum_{j=0}^{N} \frac{a_j d^4k}{[k^2 + p^2x(1-x) + m_j^2]^2} = \frac{i}{8\pi^2} \int_0^1 dx \int \sum_{j=0}^{N} \frac{a_j k_E^3 dk_E}{[k_E^2 + p^2x(1-x) + m_j^2]^2}. \qquad (2.1.20)$$

To do this integral, since it is convergent, we can dealing with regularized integral

$$\Im_{reg}(p^2, \varepsilon, \Lambda) = \frac{i}{8\pi^2} \int_0^1 dx \int_\varepsilon^\Lambda \sum_{j=0}^{N} \frac{a_j k_E^3 dk_E}{[k_E^2 + p^2x(1-x) + m_j^2]^2}. \qquad (2.1.21)$$

Let us consider now the quantity

$$\Im_\eta(p^2, \varepsilon, \Lambda) = \frac{i}{8\pi^2} \int_0^1 dx \int_\varepsilon^\Lambda \sum_{j=0}^{N} \frac{a_j k_E^3 dk_E}{[k_E^2 + p^2x(1-x) + \eta m_j^2]^2}. \qquad (2.1.22)$$

where $\eta \in (0,1]$,and therefore from Eq.(2.1.22) we obtain $\Im_0(p^2, \varepsilon, \Lambda) \equiv 0$,since Eqs.(1.2.18) holds.From Eq.(2.1.22) by differentiation we obtain

$$\frac{d}{d\eta}\Im_{\eta}(p^2,\varepsilon,\Lambda) =$$

$$-\frac{i}{4\pi^2}\int_0^1 dx\int_\varepsilon^\Lambda \sum_{j=0}^N \frac{a_j m_j^2 k_E^3 dk_E}{[k_E^2+p^2x(1-x)+\eta m_j^2]^3} \simeq$$

$$-\frac{i}{4\pi^2}\sum_{j=0}^N a_j m_j^2 \Re_j(p^2,\eta,\Lambda,\varepsilon), \tag{2.1.23}$$

$$\Re_j(p^2,\eta,\Lambda,\varepsilon) \simeq \int_0^1 dx\int \frac{k_E^3 dk_E}{[k_E^2+p^2x(1-x)+\eta m_j^2]^3} = \frac{1}{4}\int_0^1 \frac{dx}{p^2x(1-x)+\eta m_j^2}.$$

From Eq.(2.1.23) we obtain

$$\frac{d}{d\eta}\Im_{\eta}(p^2,\varepsilon,\Lambda) \simeq -\frac{i}{4\pi^2}\sum_{j=0}^N a_j m_j^2 \Re_j(p^2,\eta,\varepsilon,\Lambda) =$$

$$-\frac{i}{16\pi^2}\sum_{j=0}^N a_j \int_0^1 \frac{dx}{m_j^{-2}p^2x(1-x)+\eta}. \tag{2.1.24}$$

From Eq.(2.1.24) we obtain

$$\Im_{reg}(p^2) = -\frac{i}{16\pi^2}\sum_{j=0}^N a_j \int_0^1 dx \int_0^1 \frac{d\eta}{m_j^{-2}p^2x(1-x)+\eta}. \tag{2.1.25}$$

Note that

$$\int_0^1 \frac{d\eta}{m_j^{-2}p^2x(1-x)+\eta} =$$

$$[m_j^{-2}p^2x(1-x)+\eta]\ln[m_j^{-2}p^2x(1-x)+\eta]\Big|_0^1 - 1 =$$

$$[m_j^{-2}p^2x(1-x)+1]\ln[m_j^{-2}p^2x(1-x)+1] -$$

$$-[m_j^{-2}p^2x(1-x)]\ln[m_j^{-2}p^2x(1-x)] - 1. \tag{2.1.26}$$

Thus

$$\Im_{reg}(p^2) = -\frac{i}{16\pi^2}\sum_{j=0}^{N=1} a_j \int_0^1 dx \int_0^1 \frac{d\eta}{m_j^{-2}p^2x(1-x)+\eta} =$$
$$-\frac{i}{16\pi^2}\sum_{j=0}^{N=1} a_j \int_0^1 dx\{[m_j^{-2}p^2x(1-x)+1]\ln[m_j^{-2}p^2x(1-x)+1]-$$
$$-[m_j^{-2}p^2x(1-x)]\ln[m_j^{-2}p^2x(1-x)]\} + \frac{i}{16\pi^2}\sum_{j=0}^{N=1} a_j =$$
$$-\frac{i}{16\pi^2}\sum_{j=0}^{N=1} a_j \int_0^1 dx\{[m_j^{-2}p^2x(1-x)+1]\ln[m_j^{-2}p^2x(1-x)+1]- \tag{2.1.27}$$
$$-[m_j^{-2}p^2x(1-x)]\ln[m_j^{-2}p^2x(1-x)]\} =$$
$$-\frac{i}{16\pi^2}\int_0^1 dx\{[m_0^{-2}p^2x(1-x)+1]\ln[m_0^{-2}p^2x(1-x)+1]-$$
$$-[m_0^{-2}p^2x(1-x)]\ln[m_0^{-2}p^2x(1-x)]\} +$$
$$+\frac{i}{16\pi^2}\int_0^1 dx\{[m_1^{-2}p^2x(1-x)+1]\ln[m_1^{-2}p^2x(1-x)+1]-$$
$$-[m_1^{-2}p^2x(1-x)]\ln[m_1^{-2}p^2x(1-x)]\}.$$

From Eq.(2.1.27) we obtain

$$\Im_{reg}(p^2) =$$
$$-\frac{i}{16\pi^2}\int_0^1 dx\{[m_0^{-2}p^2x(1-x)+1]\ln[m_0^{-2}p^2x(1-x)+1]-$$
$$-[m_0^{-2}p^2x(1-x)]\ln[m_0^{-2}p^2x(1-x)]\} + \tag{2.1.28}$$
$$+\frac{i}{16\pi^2}\int_0^1 dx\{[m_1^{-2}p^2x(1-x)+1]\ln[m_1^{-2}p^2x(1-x)+1]-$$
$$-[m_1^{-2}p^2x(1-x)]\ln[m_1^{-2}p^2x(1-x)]\}.$$

We assume now that $m_1^{-2}p^2 \ll 1$ and from Eq.(2.1.28) finally we obtain

$$\Im_{reg}(p^2) =$$
$$-\frac{i}{16\pi^2}\int_0^1 dx\{[m_0^{-2}p^2x(1-x)+1]\ln[m_0^{-2}p^2x(1-x)+1]- \tag{2.1.29}$$
$$-[m_0^{-2}p^2x(1-x)]\ln[m_0^{-2}p^2x(1-x)]\} + O(m_1^{-2}p^2).$$

Remark 2.1.1. The simple renormalizable models with finite mases $m_i, i = 1,\dots,N.$ which we have considered in this section many years regarded only as constructs for a study of the ultraviolet problem of QFT. The difficulties with unitarity appear to preclude their direct acceptability as canonical physical theories in locally Minkowski space-time. However, for their unphysical behavior may be restricted to *arbitrarily large energy scales* Λ_* mentioned above by an appropriate limitation on the finite masses m_i.

II.2.Renormalizability-of-Higher-Derivative-Quantum-Gravity

Gravitational actions which include terms quadratic in the curvature tensor are renormalizable. The necessary Slavnov identities are derived from Becchi-Rouet-Stora (BRS) transformations of the gravitational and Faddeev-Popov ghost fields. In general, non-gauge-invariant divergences do arise, but they may be absorbed by nonlinear renormalizations of the gravitational and ghost fields and of the BRS transformations [13].The geneic expression of the action reads

$$I_{sym} = -\int d^4x\sqrt{-g}\,(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + 2\kappa^{-2}R), \tag{2.2.1}$$

where the curvature tensor and the Ricci is defined by $R^{\lambda}_{\mu\alpha\nu} = \partial_\nu\Gamma^{\lambda}_{\mu\alpha}$ and $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$ correspondingly, $\kappa^2 = 32\pi G.$The convenient definition of the gravitational field variable in terms of the contravariant metric density reads

$$\kappa h^{\mu\nu} = g^{\mu\nu}\sqrt{-g} - \eta^{\mu\nu}. \tag{2.2.2}$$

Analysis of the linearized radiation shows that there are eight dynamical degrees of freedom in the field. Two of these excitations correspond to the familiar massless spin-2 graviton. Five more correspond to a massive spin-2 particle with mass m_2. The eighth corresponds to a massive scalar particle with mass m_0. Although the linearized field energy of the massless spin-2 and massive scalar excitations is positive definite, the linearized energy of the massive spin-2 excitations is negative definite. This feature is characteristic of higher-derivative models, and poses the major obstacle to their physical interpretation.

In the quantum theory, there is an alternative problem which may be substituted for the negative energy. It is possible to recast the theory so that the massive spin-2 eigenstates of the free-fieid Hamiltonian have positive-definite energy, but also negative norm in the state vector space.

These negative-norm states cannot be excluded from the physical sector of the vector space without destroying the unitarity of the $\mathbf{S}$ matrix. The requirement that the graviton propagator behave like p^{-4} for large momenta makes it necessary to choose the indefinite-metric vector space over the negative-energy states.

The presence of massive quantum states of negative norm which cancel some of the divergences due to the massless states is analogous to the Pauli-Villars regularization of other field theories. For quantum gravity, however, the resulting improvement in the ultraviolet behavior of the theory is sufficient only to make it renormalizable, but not finite.

The gauge choice which we adopt in order to defining the quantum theory is the canonical harmonic gauge: $\partial_\nu h^{\mu\nu} = 0.$ Corresponding Green's functions are then given by a generating functional

$$\begin{gathered} Z(T_{\mu\nu}) = N\int\Big[\prod\nolimits_{\mu\leq\nu} dh^{\mu\nu}\Big][dC^\sigma][d\overline{C}_\tau]\delta^4(F^\tau) \\ \exp\Big[\,i\Big(I_{sym} + \int d^4x\overline{C}_\tau\vec{F}^{\tau}_{\mu\nu}D^{\mu\nu}_{\alpha}C^\alpha + \kappa\int d^4xT_{\mu\nu}h^{\mu\nu}\Big)\Big]. \end{gathered} \tag{2.2.3}$$

Here $F^\tau = \vec{F}^\tau_{\mu\nu} h^{\mu\nu}, \vec{F}^\tau_{\mu\nu} = \delta^r_\mu \vec{\partial}_\nu$ and the arrow indicates the direction in which the derivative acts. N is an normalization constant. C^σ is the Faddeev-Popov ghost field, and $\overline{C}_\tau$ is the antighost field. Notice that both C^σ and $\overline{C}_\tau$ are anticommuting quantities. $D^{\mu\nu}_\alpha$ is the operator which generates gauge transformations in $h^{\mu\nu}$, given an arbitrary spacetime-dependent vector $\xi^\alpha(x)$ corresponding to $x^{\mu'} = x^\mu + \kappa\xi^\mu$ and where

$$D^{\mu\nu}_\alpha \xi^\alpha(x) = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\nu}\partial_\alpha \xi^\alpha + \kappa(\partial_\alpha \xi^\mu h^{\alpha\nu} + \partial_\alpha \xi^\nu h^{\alpha\mu} - \xi^\alpha \partial_\alpha h^{\mu\nu} - \partial_\alpha \xi^\alpha h^{\mu\nu}.) \quad (2.2.4)$$

In the functional integral (2.2.3), we have written the metric for the gravitational field as

$$\left[\prod_{\mu\leq\nu} dh^{\mu\nu}\right]$$

without any local factors of $g = \det(g_{\mu\nu})$. Such factors do not contribute to the Feynman rules because their effect is to introduce terms proportional to $\delta^4(0)\int d^4x \ln(-g)$ into the effective action and $\delta^4(0)$ is set equal to zero in dimensional regularization.

In calculating the generating functional (2.2.3.) by using the loop expansion, one may represent the δ function which fixes the gauge as the limit of a Gaussian, discarding an infinite normalization constant

$$\delta^4(F^\tau) \sim \lim_{\Delta\to 0} \exp\left[i\left(\tfrac{1}{2}\Delta^{-1}\int d^4x F_\tau F^\tau\right)\right]. \quad (2.2.5)$$

In this expression, the index τ has been lowered using the flat-space metric tensor $\eta_{\mu\nu}$. For the remainder of this paper, we shall adopt the standard approach to the covariant quantization of gravity, in which only Lorentz tensors occur, and all raising and lowering of indices is done with respect to flat space. The graviton propagator may be calculated from $I_{sym} + \frac{1}{2}\Delta^{-1}\int d^4x F_\tau F^\tau$ in the usual fashion, letting $\Delta \to 0$ after inverting. The expression $\frac{1}{2}\Delta^{-1}\int d^4x F_\tau F^\tau$ contains only two derivatives. Consequently, there are parts of the graviton propagator which behave like p^{-2} for large momenta. Specifically, the p^{-2} terms consist of everything but those parts of the propagator which are transverse in all indices. These terms give rise to unpleasant infinities already at the one-loop order. For example, the graviton self-energy diagram shown in Fig.2.2.1 has a divergent part with the general structure $(\partial^4 h)^2$. Such divergences do cancel when they are connected to tree diagrams whose outermost lines are on the mass shell, as they must if the **S** matrix is to be made finite without introducing counterterms for them. However, they greatly complicate the renormalization of Green's functions.

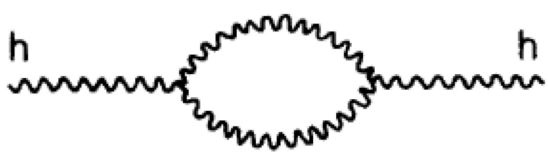

Fig.2.2.1.The one-loop graviton self-energy diagram.

We may attempt to extricate ourselves from the situation described in the last paragraph by picking a different weighting functional. Keeping in mind that we want no part of the graviton propagator to fall off slower than p^{-4} for large momenta, we now choose the weighting functional [12]

$$\omega_4(e^\tau) = \exp\Big[i\Big(\tfrac{1}{2}\Delta^{-1} \int d^4x e_\tau \Box^2 e^\tau \Big) \Big], \tag{2.2.6}$$

where e^τ is any four-vector function.The corresponding gauge-fixing term in the effective action is

$$-\tfrac{1}{2}\kappa^2\Delta^{-1} \int d^4x F_\tau \Box^2 F^\tau. \tag{2.2.7}$$

The graviton propagator resulting from the gauge-fixing term (2.2.7) is derived in [12]. For most values of the parameters α and β in I_{sym} it satisfies the requirement that all its leading parts fall off like p^{-4} for large momenta. There are, however, specific choices of these parameters which must be avoided. If $\alpha = 0,$ the massive spin-2 excitations disappear, and inspection of the graviton propagator shows that some terms then behave like k^{-2}. Likewise, if $3\beta - \alpha = 0,$ the massive scalar excitation disappears, and there are again terms in the propagator which behave like p^{-2}. However, even if we avoid the special cases $\alpha = 0$ and $3\beta - \alpha = 0,$ and if we use the propagator derived from (2.2.7), we still do not obtain a clean renormalization of the Green's functions. We now turn to the implications of gauge invariance.Before we write down the BRS transformations for gravity, let us first establish the commutation relation for gravitational gauge transformations, which reveals the group structure of the theory. Take the gauge transformation (2.2.4) of $h^{\mu\nu},$ generated by ξ^μ and perform a second gauge transformation, generated by $\eta^\mu,$ on the $h^{\mu\nu}$ fields appearing there. Then antisymmetrize in ξ^μ and η^μ.The result is

$$\frac{\delta D^{\mu\nu}_\alpha}{\delta h^{\rho\sigma}} D^{\rho\sigma}_\beta (\xi^\alpha\eta^\beta - \eta^\alpha\xi^\beta) = \kappa D^{\mu\nu}_\lambda (\partial_\alpha\xi^\lambda\eta^\alpha - \partial_\alpha\xi^\alpha\eta^\lambda), \tag{2.2.8}$$

where the repeated indices denote both summation over the discrete values of the indices and integration over the spacetime arguments of the functions or operators indexed.

The BRS transformations for gravity appropriate for the gauge-fixing term (2.2.6) are [12]

$$\begin{gathered}(a)\ \delta_{\mathbf{BRS}} h^{\mu\nu} = \kappa D^{\mu\nu}_\alpha C^\alpha \delta\lambda, (b)\ \delta_{\mathbf{BRS}} C^\alpha = -\kappa^2 \partial_\beta C^\alpha C^\beta \delta\lambda, \\ (c)\ \delta_{\mathbf{BRS}} \overline{C}_\tau = -\kappa^3 \Delta^{-1} \Box^2 F_\tau \delta\lambda, \end{gathered} \tag{2.2.9}$$

where $\delta\lambda$ is an infinitesimal anticommuting constant parameter.The importance of these transformations resides in the quantities which they leave invariant. Note that

$$\delta_{\mathbf{BRS}}(\partial_\beta C^\sigma C^\beta) = 0 \tag{2.2.10}$$

and

$$\delta_{\mathbf{BRS}}(D^{\mu\nu}_\alpha C^\alpha) = 0. \tag{2.2.11}$$

As a result of Eq. (2.2.11), the only part of the ghost action which varies under the BRS transformations is the antighost $\overline{C}_\tau$. Accordingly, the transformation (2.2.9c) has been chosen to make the variation of the ghost action just cancel the variation of the gauge-fixing term. Therefore, the entire effective action is BRS invariant:

$$\delta_{\mathbf{BRS}}\Big(I_{sim} - \tfrac{1}{2}\kappa^2\Delta^{-1} F_\tau \Box^2 F^\tau + \overline{C}_\tau F^\tau_{\mu\nu} D^{\mu\nu}_\alpha C^\alpha \Big) = 0. \tag{2.2.12}$$

Equations (2.2.9), (2.2.10), and (2.2.12) now enable us to write the Slavnov identities in an economical way. In order to carry out the renormalization program, we will need to have Slavnov identities for the proper vertices.

A. Slavnov identities for Green's functions

First consider the Slavnov identities for Green's functions.

$$Z(T_{\mu\nu},\overline{\beta}_\sigma,\beta^\tau,K_{\mu\nu},L_\sigma) = N\int\left[\prod_{\mu\leq\nu} dh^{\mu\nu}\right][dC^\sigma][d\overline{C}_\tau]$$
$$\exp\left[i\widetilde{\widetilde{\Sigma}}(h^{\mu\nu},C^\sigma,\overline{C}_\tau,K_{\mu\nu},L_\sigma,\overline{\beta}_\sigma C^\sigma)+\overline{\beta}_\sigma C^\sigma+\overline{C}_\tau\beta^\tau+\kappa T_{\mu\nu}h^{\mu\nu}\right]. \quad (2.2.13)$$

Anticommuting sources have been included for the ghost and antighost fields, and the effective action $\widetilde{\widetilde{\Sigma}}$ has been enlarged by the inclusion of BRS invariant couplings of the ghosts and gravitons to some external fields $K_{\mu\nu}$ (anticommuting) and L_σ (commuting),

$$\widetilde{\widetilde{\Sigma}} = I_{sim}-\tfrac{1}{2}\kappa^2\Delta^{-1}F_\tau\Box^2F^\tau+\overline{C}_\tau\vec{F}^\tau_{\mu\nu}D^{\mu\nu}_\alpha C^\alpha+\kappa K_{\mu\nu}D^{\mu\nu}_\alpha+\kappa^2L_\sigma\partial_\beta C^\sigma C^\beta. \quad (2.2.14)$$

$\widetilde{\widetilde{\Sigma}}$ is BRS invariant by virtue of Eq.(2.2.9), Eq.(2.2.10), and Eq.(2.2.12). We may use the new couplings to write this invariance as

$$\frac{\delta\widetilde{\widetilde{\Sigma}}}{\delta K_{\mu\nu}}\frac{\delta\widetilde{\widetilde{\Sigma}}}{\delta h^{\mu\nu}}+\frac{\delta\widetilde{\widetilde{\Sigma}}}{\delta L_\sigma}\frac{\delta\widetilde{\widetilde{\Sigma}}}{\delta C^\sigma}+\kappa^3\Delta^{-1}\Box^2F_\tau\frac{\delta\widetilde{\widetilde{\Sigma}}}{\delta\overline{C}_\tau}. \quad (2.2.15)$$

In this equation, and throughout this subsection, we use left variational derivatives with respect to anticommuting quantities: $\delta f(C^\sigma) = \delta C^\tau\delta f/\delta C^\tau$. Equation (2.2.15) may be simplified by rewriting it in terms of a reduced effective action,

$$\Sigma = \widetilde{\widetilde{\Sigma}}+\tfrac{1}{2}\kappa^2\Delta^{-1}F_\tau\Box^2F^\tau. \quad (2.2.16)$$

Substitution of (2.2.16) into (2.2.15) gives

$$\frac{\delta\Sigma}{\delta K_{\mu\nu}}\frac{\delta\Sigma}{\delta h^{\mu\nu}}+\frac{\delta\Sigma}{\delta L_\sigma}\frac{\delta\Sigma}{\delta C^\sigma} = 0, \quad (2.2.17)$$

where we have used the relation

$$\kappa^{-1}\vec{F}^\tau_{\mu\nu}\frac{\delta\Sigma}{\delta K_{\mu\nu}}-\frac{\delta\Sigma}{\delta\overline{C}_\tau} = 0. \quad (2.2.18)$$

Note that a measure

$$\left[\prod_{\mu\leq\nu} dh^{\mu\nu}\right][dC^\sigma][d\overline{C}_\tau] \quad (2.2.19)$$

is BRS invariant since for infinitesimal transformations, the Jacobian is $1,$ because of the trace relations

$$(a)\ \frac{\delta^2\widetilde{\widetilde{\Sigma}}}{\delta K_{(\mu\nu)}\delta h^{(\mu\nu)}} = 0,(b)\ \frac{\delta^2\widetilde{\widetilde{\Sigma}}}{\delta C^\sigma\delta L_\sigma} = 0, \quad (2.2.20)$$

both of which follow from $\int d^4x\partial_\alpha C^\alpha = 0.$ The parentheses surrounding the indices in (2.2.20a) indicate that the summation is to be carried out only for $\mu\leq\nu$.

Remark 2.2.1.Note that the Slavnov identity for the generating functional of Green's functions is obtained by performing the BRS transformations (2.2.9) on the integration variables in the generating functional (2.2.13). This transformation does not change the value of the generating functional and therefore we obtain

$$
\begin{gathered}
N\int\Big[\prod_{\mu\le\nu} dh^{\mu\nu}\Big][dC^\sigma][d\overline{C}_\tau]\times \\
\Big(\kappa^2 T_{\mu\nu}D^{\mu\nu}_\alpha - \kappa^2\overline{\beta}_\sigma\partial_\beta C^\sigma C^\beta + \kappa^3\Delta^{-1}\beta^\tau\Box^2\overrightarrow{F}_{\tau\mu\nu}h^{\mu\nu}\Big)\times \\
\exp\Big[i\Big(\widetilde{\Sigma} + \kappa T_{\mu\nu}h^{\mu\nu} + \overline{\beta}_\sigma C^\sigma + \overline{C}_\tau\beta^\tau\Big)\Big] = 0.
\end{gathered}
\tag{2.2.21}
$$

Another identity which we shall need is the ghost equation of motion. To derive this equation, we shift the antighost integration variable $\overline{C}_\tau$ to $\overline{C}_\tau + \delta\overline{C}_\tau$, again with no resulting change in the value of the generating functional:

$$
N\int\Big[\prod_{\mu\le\nu} dh^{\mu\nu}\Big][dC^\sigma][d\overline{C}_\tau]\Big(\frac{\delta\widetilde{\Sigma}}{\delta C^\sigma} + \beta^\tau\Big)\exp\Big[i\Big(\widetilde{\Sigma} + \kappa T_{\mu\nu}h^{\mu\nu} + \overline{\beta}_\sigma C^\sigma + \overline{C}_\tau\beta^\tau\Big)\Big] \tag{2.2.22}
$$

We define now the generating functional of connected Green's functions as the logarithm of the functional (2.2.13),

$$
W[T_{\mu\nu},\overline{\beta}_\sigma,\beta^\tau,K_{\mu\nu},L_\sigma] = -i\ln Z[T_{\mu\nu},\overline{\beta}_\sigma,\beta^\tau,K_{\mu\nu},L_\sigma]. \tag{2.2.23}
$$

and make use of the couplings to the external fields $K_{\mu\nu}$ and L_σ to rewrite (2.2.22) in terms of W

$$
\kappa T_{\mu\nu}\frac{\delta W}{\delta K_{\mu\nu}} - \overline{\beta}_\sigma\frac{\delta W}{\delta L_\sigma} + \kappa^2\Delta^{-1}\beta^\tau\Box^2\overrightarrow{F}_{\tau\mu\nu}\frac{\delta W}{\delta T_{\mu\nu}} = 0. \tag{2.2.24}
$$

Similarly, we get the ghost equation of motion:

$$
\kappa^{-1}\overrightarrow{F}^\tau_{\mu\nu}\frac{\delta W}{\delta K_{\mu\nu}} + \beta^\tau = 0. \tag{2.2.25}
$$

B. Proper vertices

A Legendre transformation takes us from the generating functional of connected Green's functions (2.2.23) to the generating functional of proper vertices. First, we define the expectation values of the gravitational, ghost, and antighost fields in the presence of the sources $T_{\mu\nu},\overline{\beta}_\sigma$, and β^τ and the external fields $K_{\mu\nu}$ and L_σ

$$
(a)\ h^{\mu\nu}(x) = \frac{\delta W}{\kappa\delta T_{\mu\nu}(x)},(b)\ C^\sigma(x) = \frac{\delta W}{\delta\overline{\beta}_\sigma(x)},(c)\ \overline{C}_\tau(x) = \frac{\delta W}{\delta\beta^\tau(x)}. \tag{2.2.26}
$$

We have chosen to denote the expectation values of the fields by the same symbols which were used for the fields in the effective action (2.2.14).

The Legendre transformation can now be performed, giving us the generating functional of proper vertices as a functional of the new variables (2.2.26) and the external fields $K_{\mu\nu}$ and L_σ

$$
\widetilde{\Gamma}[h^{\mu\nu},C^\sigma,\overline{C}_\tau,K_{\mu\nu},L_\sigma] = W[T_{\mu\nu},\overline{\beta}_\sigma,\beta^\tau,K_{\mu\nu},L_\sigma] - \kappa T_{\mu\nu}h^{\mu\nu} - \overline{\beta}_\sigma C^\sigma - \overline{C}_\tau\beta^\tau. \tag{2.2.27}
$$

In this equation, the quantities $T_{\mu\nu}$, $\overline{\beta}_\sigma$, and β^τ are given implicitly in terms of $h^{\mu\nu},C^\sigma,\overline{C}_\tau,K_{\mu\nu},$ and L_σ by Eq.(2.2.26).The relations dual to (2.2.26) are

$$
(a)\ \kappa T_{\mu\nu}(x) = -\frac{\delta\widetilde{\Gamma}}{\delta h^{\mu\nu}(x)},(b)\ \overline{\beta}_\sigma(x) = \frac{\delta\widetilde{\Gamma}}{\delta C^\sigma(x)},(c)\ \beta_\tau(x) = -\frac{\delta\widetilde{\Gamma}}{\delta\overline{C}_\tau(x)}. \tag{2.2.28}
$$

Since the external fields $K_{\mu\nu}$ and L_σ do not participate in the Legendre transformation (2.2.26), for them we have the relations

$$
(a)\ \frac{\delta\widetilde{\Gamma}}{\delta K_{\mu\nu}(x)} = \frac{\delta W}{\delta K_{\mu\nu}(x)},(b)\ \frac{\delta\widetilde{\Gamma}}{\delta L_\sigma(x)} = \frac{\delta W}{\delta L_\sigma(x)}. \tag{2.2.29}
$$

Finally, the Slavnov identity for the generating functional of proper vertices is obtained by transcribing (2.2.24) using the relations (2.2.26), (2.2.28), and (2.2.29)

$$\frac{\delta\widetilde{\Gamma}}{\delta K_{\mu\nu}}\frac{\delta\widetilde{\Gamma}}{\delta h^{\mu\nu}} + \frac{\delta\widetilde{\Gamma}}{\delta L_{\sigma}}\frac{\delta\widetilde{\Gamma}}{\delta C^{\sigma}} + \kappa^{3}\Delta^{-1}\Box^{2}\vec{F}_{\tau\mu\nu}h^{\mu\nu}\frac{\delta\widetilde{\Gamma}}{\delta C^{\sigma}} = 0. \tag{2.2.30}$$

We also have the ghost equation of motion,

$$\kappa^{-1}\vec{F}^{\tau}_{\mu\nu}\frac{\delta\widetilde{\Gamma}}{\delta K_{\mu\nu}} - \frac{\delta\widetilde{\Gamma}}{\delta C^{\sigma}} = 0. \tag{2.2.31}$$

Since Eq. (2.2.30) has exactly the same form as (2.2.15), we follow the example set by (2.2.16) and define a reduced generating functional of the proper vertices,

$$\Gamma = \widetilde{\Gamma} + \tfrac{1}{2}\kappa^{2}\Delta^{-1}\left(\vec{F}_{\tau\mu\nu}h^{\mu\nu}\right)\Box^{2}\left(\vec{F}^{\tau}_{\rho\sigma}h^{\rho\sigma}\right). \tag{2.2.32}$$

Substituting this into (2.2.30) and (2.2.31), the Slavnov identity becomes

$$\frac{\delta\Gamma}{\delta K_{\mu\nu}}\frac{\delta\Gamma}{\delta h^{\mu\nu}} + \frac{\delta\Gamma}{\delta L_{\sigma}}\frac{\delta\Gamma}{\delta C^{\sigma}} = 0. \tag{2.2.33}$$

and the ghost equation of motion becomes

$$\kappa^{-1}\vec{F}^{\tau}_{\mu\nu}\frac{\delta\Gamma}{\delta K_{\mu\nu}} - \frac{\delta\Gamma}{\delta\overline{C}_{\tau}} = 0. \tag{2.2.34}$$

Equations (2.2.33) and (2.2.34) are of exactly the same form as (5.5) and (5.6). This is as it should be, since at the zero-loop order

$$\Gamma^{(0)} = \Sigma. \tag{2.2.35}$$

C. Structure of the divergences and renormalization equation.

The Slavnov identity (2.2.33) is quadratic in the functional Γ. This nonlinearity is reflected in the fact that the renormalization of the effective action generally also involves the renormalization of the BRS transformations which must leave the effective action invariant.

The canonical approach uses the Slavnov identity for the generating functional of proper vertices to derive a linear equation for the divergent parts of the proper vertices. This equation is then solved to display the structure of the divergences. From this structure, it can be seen how to renormalize the effective action so that it remains invariant under a renormalized set of BRS transformations [13].

Suppose that we have successfully renormalized the reduced effective action up to $n-1$ loop order; that is, suppose we have constructed a quantum extension of Σ which satisfies Eqs. (2.2.17) and (2.2.18) exactly, and which leads to finite proper vertices when calculated up to order $n-1$. We will denote this renormalized quantity by $\Sigma^{(n-1)}$. In general, it contains terms of many different orders in the loop expansion, including orders greater than $n-1$. The $n-1$ loop part of the reduced generating functional of proper vertices will be denoted by $\Gamma^{(n-1)}$.

When we proceed to calculate $\Gamma^{(n)}$, we find that it contains divergences. Some of these come from n-loop Feynman integrals. Since all the subintegrals of an n-loop Feynman integral contain less than w loops, they are finite by assumption. Therefore, the divergences which arise from w-loop Feynman integrals come only from the overall divergences of the integrals, so the corresponding parts of $\Gamma^{(n)}$ are local in structure. In the dimensional regularization procedure, these divergences are of order $\epsilon^{-1} = (d-4)^{-1}$, where d is the dimensionality of spacetime in the Feynman integrals.

There may also be divergent parts of $\Gamma^{(n)}$ which do not arise from loop integrals, and which contain higher-order poles in the regulating parameter ϵ. Such divergences comes from n-loop order parts of $\Sigma^{(n-1)}$ which are necessary to ensure that (2.2.17) is satisfied. Consequently, they too have a local structure. We may separate the divergent and finite parts of $\Gamma^{(n)}$:

$$\Gamma^{(n)} = \Gamma^{(n)}_{\mathbf{div}} + \Gamma^{(n)}_{\mathbf{finite}}. \tag{2.2.36}$$

If we insert this breakup into Eq. (5.20), and keep only the terms of the equation which are of n-loop order, we get

$$\begin{gathered}\frac{\delta\Gamma^{(n)}_{\mathbf{div}}}{\delta K_{\mu\nu}}\frac{\delta\Gamma^{(0)}}{\delta h^{\mu\nu}} + \frac{\delta\Gamma^{(0)}}{\delta K_{\mu\nu}}\frac{\delta\Gamma^{(n)}_{\mathbf{div}}}{\delta h^{\mu\nu}} + \frac{\delta\Gamma^{(n)}_{\mathbf{div}}}{\delta L_\sigma}\frac{\delta\Gamma^{(0)}}{\delta C^\sigma} + \frac{\delta\Gamma^{(0)}}{\delta L_\sigma}\frac{\delta\Gamma^{(n)}_{\mathbf{div}}}{\delta C^\sigma} = \\ -\sum_{i=0}^{n}\left[\frac{\delta\Gamma^{(n-i)}_{\mathbf{finite}}}{\delta K_{\mu\nu}}\frac{\delta\Gamma^{(i)}_{\mathbf{finite}}}{\delta h^{\mu\nu}} + \frac{\delta\Gamma^{(n-i)}_{\mathbf{finite}}}{\delta L_\sigma}\frac{\delta\Gamma^{(i)}_{\mathbf{finite}}}{\delta C^\sigma}\right].\end{gathered} \tag{2.2.37}$$

Since each term on the right-hand side of (2.2.37) remains finite as $\epsilon \to 0$, while each term on the left-hand side contains a factor with at least a simple pole in e, each side of the equation must vanish separately. Remembering the Eq.(2.2.35), we can write the following equation, called the renormalization equation:

$$\Re\Gamma^{(n)}_{\mathbf{div}} = 0, \tag{2.2.38}$$

where

$$\Re = \frac{\delta\Sigma}{\delta h^{\mu\nu}}\frac{\delta}{\delta K_{\mu\nu}} + \frac{\delta\Sigma}{\delta C^\sigma}\frac{\delta}{\delta L_\sigma} + \frac{\delta\Sigma}{\delta K_{\mu\nu}}\frac{\delta}{\delta h^{\mu\nu}} + \frac{\delta\Sigma}{\delta L_\sigma}\frac{\delta}{\delta C^\sigma}. \tag{2.2.39}$$

Similarly by collecting the n-loop order divergences in the ghost equation of motion (2.2.34) we get

$$\kappa^{-1}\vec{F}^{\tau}_{\mu\nu}\frac{\delta\Gamma^{(n)}_{\mathbf{div}}}{\delta K_{\mu\nu}} - \frac{\delta\Gamma^{(n)}_{\mathbf{div}}}{\delta\overline{C}_\tau} = 0. \tag{2.2.40}$$

In order to construct local solutions to Eqs. (2.2.38) and (2.2.40) remind that the operator $\Re$ defined in (2.2.39) is nilpotent [13]:

$$\Re^2 = 0. \tag{2.2.41}$$

Equation (2.2.41) gives us the local solutions to Eq.(2.2.38) of the form

$$\Gamma^{(n)}_{\mathbf{div}} = \Im(h^{\mu\nu}) + \Re[X(h^{\mu\nu}, C^\sigma, \overline{C}_\tau, K_{\mu\nu}, L_\sigma)], \tag{2.2.42}$$

where $\Im$ is an arbitrary gauge-invariant local functional of $h^{\mu\nu}$ and its derivatives, and X is an arbitrary local functional of $h^{\mu\nu}, C_\sigma, \overline{C}_\tau, K_{\mu\nu}$ and L^σ and their derivatives. In order to satisfy the ghost equation of motion (2.2.40) we require that

$$\Gamma^{(n)}_{\mathbf{div}} = \Gamma^{(n)}_{\mathbf{div}}\left(h^{\mu\nu}, C^\sigma, K_{\mu\nu} - \kappa^{-1}\overline{C}_\tau\vec{F}^{\tau}_{\mu\nu}, L_\sigma\right). \tag{2.2.43}$$

D. Ghost number and power counting

Structure of the effective action (2.2.14) shows that we may define the following conserved quantity, called ghost number [13]:

$$N_{\mathbf{ghost}}[h^{\mu\nu}] = 0, N_{\mathbf{ghost}}[C_\sigma] = +1, N_{\mathbf{ghost}}[\overline{C}_\tau] = -1,$$
$$N_{\mathbf{ghost}}[K_{\mu\nu}] = -1, N_{\mathbf{ghost}}[L_\sigma] = -2. \tag{2.2.44}$$

From Eqs.(2.2.44) follows that

$$N_{\mathbf{ghost}}[\Sigma] = N_{\mathbf{ghost}}[\Gamma] = 0. \tag{2.2.45}$$

Since

$$N_{\mathbf{ghost}}[\Re] = +1, \tag{2.2.46}$$

we require of the functional $X(\cdot)$ that

$$N_{\mathbf{ghost}}[X] = -1. \tag{2.2.47}$$

In order to complete analysis of the structure of $\Gamma^{(n)}_{\mathbf{div}}$, we must supplement the symmetry equations (2.2.42), (2.2.43), and (2.2.47) with the constraints on the divergences which arise from power counting. Accordingly, we introduce the following notations:

n_E = number of graviton vertices with two derivatives,
n_G = number of antighost-graviton-ghost vertices,
n_K = number of K-graviton-ghost vertices,
n_L = number of L-ghost-ghost vertices,
I_G = number of internal-ghost propagators,
E_C=number of external ghosts,
$E_{\overline{C}}$=number of external antighosts.

Since graviton propagators behave like p^{-4}, and ghost propagators like p^{-2}, we are led by standard power counting to the degree of divergence of an arbitrary diagram,

$$D = 4 - 2n_E + 2I_G - 2n_G - 3n_L - 3n_K - E_{\overline{C}}. \tag{2.2.48}$$

The last term in (2.2.48) arises because each external antighost line carries with it a factor of external momentum. We can make use of the topological relation

$$2I_G - 2n_G = 2n_L + n_K - E_C - E_{\overline{C}} \tag{2.2.49}$$

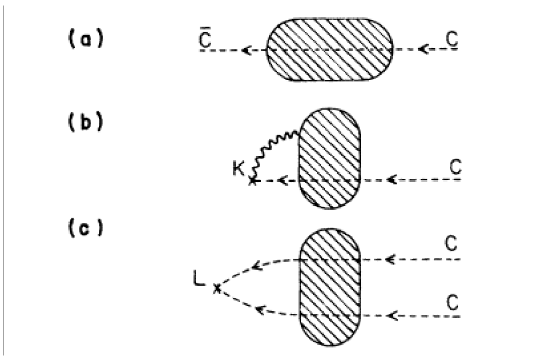

Fig.2.2.2.The three types of divergent diagram which involve external ghost lines. Arbitrarily many gravitons may emerge from each of the central regions,(a) Ghost action type,(b) K type, (c) L type.

to write the degree of divergence as

$$D = 4 - 2n_E - n_L - 2n_K - E_C - 2E_{\overline{C}}. \tag{2.2.50}$$

Together with conservation of ghost number,Eq. (2.2.50) enables us to catalog three different types of divergent structures involving ghosts. These are illustrated in Fig.2.2.2. Each of the three types has degree of divergence $D = 1 - 2n_E$. Consequently, all the divergences which involve ghosts have $n_E = 0$. Since the degree of divergence is then 1, the associated divergent structures in $\Gamma^{(n)}_{\mathbf{div}}$ have an extra derivative appearing on one of the fields. Diagrams whose external lines are all gravitons have degree of divergence $D = 4 - 2n_E$. Combining (2.2.50) with (2.2.47), (2.2.43), and (2.2.42), we can finally write the most general expression for $\Gamma^{(n)}_{\mathbf{div}}$ which satisfies all the constraints of symmetries and power counting:

$$\Gamma^{(n)}_{\mathbf{div}} = \Im(h^{\mu\nu}) + \Re\Big[\Big(K_{\mu\nu} - \kappa^{-1}\overline{C}_\tau \vec{F}^\tau_{\mu\nu}\Big)P^{\mu\nu}(h^{\alpha\beta}) + L_\sigma Q^\sigma_\tau(h^{\alpha\beta})C^\tau\Big], \tag{2.2.51}$$

where $P^{\mu\nu}(h^{\alpha\beta})$ and $Q^\sigma_\tau(h^{\alpha\beta})$ are arbitrary Lorentz-covariant functions of the gravitational field $h^{\mu\nu}$, but not of its derivatives, at a single spacetime point. $\Im(h^{\mu\nu})$ is a local gauge-invariant functional of $h^{\mu\nu}$ containing terms with four, two, and zero derivatives.Expanding (2.2.51), we obtain an array of possible divergent structures:

$$\begin{gathered}\Gamma^{(n)}_{\mathbf{div}} = \Im(h^{\mu\nu}) + \frac{\delta I_{sym}}{\delta h^{\mu\nu}}P^{\mu\nu} + \Big(\kappa K_{\rho\sigma} - \overline{C}_\tau \vec{F}^\tau_{\rho\sigma}\Big)\Big(\frac{\delta D^{\rho\sigma}_\alpha}{\delta h^{\mu\nu}}C^\alpha\Big)P^{\mu\nu} - \\ -\Big(\kappa K_{\rho\sigma} - \overline{C}_\tau \vec{F}^\tau_{\rho\sigma}\Big)\frac{\delta P^{\rho\sigma}}{\delta h^{\mu\nu}}D^{\mu\nu}_\alpha C^\alpha - \Big(\kappa K_{\mu\nu} - \overline{C}_\tau \vec{F}^\tau_{\mu\nu}\Big)D^{\mu\nu}_\sigma(Q^\sigma_\epsilon C^\epsilon) - \kappa^2 L_\sigma \partial_\beta(Q^\sigma_\tau C^\tau)C^\beta \\ -\kappa^2 L_\sigma \partial_\beta C^\sigma Q^\beta_\tau C^\tau - \kappa L_\sigma \frac{\delta Q^\sigma_\tau}{\delta h^{\mu\nu}}C^\tau D^{\mu\nu}_\alpha C^\alpha + \kappa^2 L_\sigma Q^\sigma_\tau \partial_\beta C^\tau C^\beta.\end{gathered} \tag{2.2.52}$$

The breakup between the gauge-invariant divergences S and the rest of (2.2.52) is determined only up to a term of the form [13]

$$\int d^4x(\eta^{\mu\nu} + \kappa h^{\mu\nu})\frac{\delta I_{sym}}{\kappa\delta h^{\mu\nu}}, \tag{2.2.53}$$

which can be generated by adding to $P^{\mu\nu}$ a term proportional to $\eta^{\mu\nu} + \kappa h^{\mu\nu} = \sqrt{g}\, g^{\mu\nu}$. The profusion of divergences allowed by (2.2.52) appears to make the task of renormalizing the effective action rather complicated. Although the many divergent structures do pose a considerable nuisance for practical calculations, the situation is still reminiscent in principle of the renormalization of Yang-Mills theories. There, the non-gauge-invariant divergences may be eliminated by a number of field renormalizations. We shall find the same to be true here, but because the gravitational field $h^{\mu\nu}$ carries no weight in the power counting, there is nothing to prevent the field renormalizations from being nonlinear, or from mixing the gravitational and ghost fields. The corresponding renormalizations procedure considered in [13].

Remark 2.2.2. We assume now that:

(i) The local Poincaré group of momentum space is deformed at some fundamental high-energy cutoff Λ_* [9],[10].

(ii) The canonical quadratic invariant $\|p\|^2 = \eta^{ab}p_a p_b$ collapses at high-energy cutoff Λ_* and being replaced by the non-quadratic invariant:

$$\|p\|^2 = \frac{\eta^{ab}p_a p_b}{(1 + l_{\Lambda_*}p_0)}. \tag{2.2.1}$$

(iii) The canonical concept of Minkowski space-time collapses at a small distances

$l_{\Lambda_*} = \Lambda_*^{-1}$ to fractal space-time with Hausdorff-Colombeau negative dimension and therefore the canonical Lebesgue measure d^4x being replaced by the Colombeau-Stieltjes measure

$$(d\eta(x,\varepsilon))_\varepsilon = (v_\varepsilon(s(x))d^4x)_\varepsilon, \tag{5.1.2}$$

where

$$\begin{gathered}(v_\varepsilon(s(x)))_\varepsilon = \left(\left(|s(x)|^{|D^-|}+\varepsilon\right)^{-1}\right)_\varepsilon,\\ s(x) = \sqrt{x_\mu x^\mu},\end{gathered} \tag{5.1.3}$$

see subsection IV.2.

(iv) The canonical concept of local momentum space collapses at fundamental high-energy cutoff Λ_* to fractal momentum space with Hausdorff-Colombeau negative dimension and therefore the canonical Lebesgue measure $d^3\mathbf{k}$, where $\mathbf{k} = (k_x,k_y,k_z)$ being replaced by the Hausdorff-Colombeau measure

$$d^{D^+,D^-}\mathbf{k} \triangleq \frac{\Delta(D^-)d^{D^+}\mathbf{k}}{\left(|\mathbf{k}|^{|D^-|}+\varepsilon\right)_\varepsilon} = \frac{\Delta(D^+)\Delta(D^-)p^{D^+-1}dp}{(p^{|D^-|}+\varepsilon)_\varepsilon}, \tag{5.1.4}$$

see subsection III.3-III.4. Note that the integral over measure $d^{D^+,D^-}\mathbf{k}$ is given by formula (3.3.16).

Remark 2.2.3.(**I**)The renormalizable models which we have considered in this section many years regarded only as constructs for a study of the ultraviolet problem of quantum gravity. The difficulties with unitarity appear to preclude their direct acceptability as canonical physical theories in locally Minkowski space-time. In canonical case they do have only some promise as phenomenological models.

(**II**) However, for their unphysical behavior may be restricted to *arbitrarily large energy scales* Λ_* mentioned above by an appropriate limitation on the renormalized masses m_2 and m_0. Actually, it is only the massive spin-two excitations of the field which give the trouble with unitarity and thus require a very large mass. The limit on the mass m_0 is determined only by the observational constraints on the static field.

III. Hausdorff-Colombeau measure and associated negative Hausdorff-Colombeau dimension.

III.1.Fractional Integration in negative dimensions.

Let $\mu_H^{D_+}$ be a Hausdorff measure [19] and $X \subset \mathbb{R}^n$ is measurable set. Let $s(x)$ be a function $s : X \to \mathbb{R}$ such that is symmetric with respect to some centre $x_0 \in X$, i.e. $s(x) =$ constant for all x satisfying $d(x,x_0) = r$ for arbitrary values of r. Then the integral in

respect to Hausdorff measure over n-dimensional metric space X is then given by [19]:

$$\int_X s(x)d\mu_H^{D_+} = \frac{2\pi^{D_+/2}}{\Gamma(D_+/2)}\int_0^\infty s(r)r^{D_+-1}dr. \tag{3.1.1}$$

The integral in RHS of the Eq.(3.1.1) is known in the theory of the Weyl fractional calculus where, the Weyl fractional integral $W^Df(x)$, is given by

$$W^{D}f(x) = \frac{1}{\Gamma(D)}\int_0^{\infty}(t-x)^{D-1}f(t)dt. \tag{3.1.2}$$

Remark 3.1.1. In order to extend the Weyl fractional integral (3.1.1) in negative dimensions we apply the Colombeau generalized functions [20] and Colombeau generalized numbers [21].

Recall that Colombeau algebras $\mathcal{G}(\Omega)$ of the Colombeau generalized functions defined as follows.Let Ω be an open subset of $\mathbb{R}^n$. Throughout this paper, for elements of the space $C^{\infty}(\Omega)^{(0,1]}$ of sequences of smooth functions indexed by $\varepsilon \in (0,1]$ we shall use the canonical notations $(\zeta_\varepsilon(x))_\varepsilon$ and $(u_\varepsilon)_\varepsilon$ so $u_\varepsilon \in C^{\infty}(\Omega),\ \varepsilon \in (0,1]$.

Definition 3.1.1.We set $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, where

$$\begin{gathered}\mathcal{E}_M(\Omega) = \left\{(u_\varepsilon)_\varepsilon \in C^{\infty}(\Omega)^{(0,1]} \,\middle|\, \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}^n \exists p \in \mathbb{N} \text{ with}\right.\\ \left.\sup_{x\in K}|u_\varepsilon(x)| = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\right\},\\ \mathcal{N}(\Omega) = \left\{(u_\varepsilon)_\varepsilon \in C^{\infty}(\Omega)^{(0,1]} \,\middle|\, \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}^n \forall q \in \mathbb{N}\right.\\ \left.\sup_{x\in K}|u_\varepsilon(x)| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\right\}.\end{gathered} \tag{3.1.3}$$

Notice that $\mathcal{G}(\Omega)$ is a differential algebra.Equivalence classes of sequences $(u_\varepsilon)_\varepsilon$ will be denoted by $\mathbf{cl}[(u_\varepsilon)_\varepsilon]$.is a differential algebra containing $D'(\Omega)$ as a linear subspace and $C^{\infty}(\Omega)$ as subalgebra.

Definition 3.1.1. Weyl fractional integral $\left(W_\varepsilon^{D^{-}} f(x)\right)_\varepsilon$ in negative dimensions $D^- < 0$, $D^- \neq 0,-1,\dots,-n,\dots, n \in \mathbb{N}$ is given by

$$\begin{gathered}W^{D^-}f(x) = \frac{1}{\Gamma(D^-)}\left(\int_\varepsilon^{\infty}(t-x)^{D^--1}f(t)dt\right)_\varepsilon\\ \text{or}\\ \left(W_\varepsilon^{D^-}f(x)\right)_\varepsilon = \frac{1}{\Gamma(D^-)}\left(\int_0^{\infty}\frac{1}{\varepsilon+(t-x)^{|D^-|+1}}f(t)dt\right)_\varepsilon,\end{gathered} \tag{3.1.4}$$

where $\varepsilon \in (0,1]$ and $\int_0^{\infty}|f(t)dt| < \infty$.Note that $(W_\varepsilon^{D^-}f(x))_\varepsilon \in \mathcal{G}(\mathbb{R})$.Thus in order to obtain apropriate extension of the Weyl fractional integral $W^{D^+}f(x)$ on the negative dimensions $D_- < 0$ the notion of the Colombeau generalized functions is essentially importent.

Remark 3.1.2.Thus in negative dimensions from Eq.(3.1.1) we formally obtain

$$\left(\int_X s(x)d\mu_{HC,\varepsilon}^{D^-}\right)_\varepsilon = \frac{2\pi^{D^-/2}}{\Gamma(D^-/2)}\left(\int_0^{\infty}\frac{s(r)}{\varepsilon + r^{|D^-|+1}}dr\right)_\varepsilon = (I_\varepsilon^{D^-})_\varepsilon, \tag{3.1.5}$$

where $\varepsilon \in (0,1]$ and $D^- \neq 0,-2,\dots,-2n,\dots, n \in \mathbb{N}$ and where $(\mu_{HC,\varepsilon}^{D^-})_\varepsilon$ is apropriate generalized Colombeau outer measure.Namely Hausdorff-Colombeau outer measure.

Remark 3.1.3. Note that: if $s(0) \neq 0$ the quantity $(I_\varepsilon^{D^+,D^-})_\varepsilon$ takes infinite large value in sense of Colombeau generalized numbers ,i.e., $(I_\varepsilon^{D^+,D^-})_\varepsilon =_{\widetilde{\mathbb{R}}} \widetilde{\infty}$, see Definition 3.3.2 and Definition 3.3.3.

Remark 3.1.4.We apply throught this paper more general definition then (3.1.4):

$$\left(\int_X s(x)d\mu_{HC,\varepsilon}^{D^+,D^-}\right)_\varepsilon = \frac{4\pi^{D^+/2}\pi^{D^-/2}}{\Gamma(D^+/2)\Gamma(D^-/2)}\left(\int_0^{\infty}\frac{r^{D^+-1}s(r)}{\varepsilon + r^{|D^-|}}dr\right)_\varepsilon = (I_\varepsilon^{D^+,D_-})_\varepsilon, \tag{3.1.5}$$

where $\varepsilon \in (0,1]$ and $D^+ \geq 1,\ D^- \neq 0,-2,\dots,-2n,\dots, n \in \mathbb{N}$ and where $\left(\mu_{HC,\varepsilon}^{D^+,D^-}\right)_\varepsilon$ is apropriate generalized Colombeau outer measure.Namely Hausdorff-Colombeau outer measure. In subsection 3.3 we pointed out that there exist Colombeau generalized measure $\left(d\mu_{HC,\varepsilon}^{D^+,D^-}\right)_\varepsilon$ and therefore Eq.(3.1.4) gives apropriate extension of the Eq.(3.1.1) on the negative Hausdorff-Colombeau dimensions.

III.2.Hausdorff measure and associated positive Hausdorff dimension.

Recall that the classical Hausdorff measure [19],[22] originate in Caratheodory's construction, which is defined as follows: for each metric space X, each set $F = \{E_i\}_{i\in\mathbb{N}}$ of subsets E_i of X, and each positive function $\zeta^+(E)$, such that $0 \leq \zeta^+(E_i) \leq \infty$ whenever $E_i \in F$, a preliminary measure $\phi_\delta^+(E)$ can be constructed corresponding to $0 < \delta \leq +\infty$, and then a final measure $\mu^+(E)$, as follows: for every subset $E \subset X$, the preliminary measure $\phi_\delta^+(E)$ is defined by

$$\phi_\delta^+(E) = \inf_{\{E_i\}_{i\in\mathbb{N}}} \left\{\sum_{i\in\mathbb{N}} \zeta^+(E_i) | E \subset \bigcup_{i\in\mathbb{N}} E_i, diam(E_i) \leq \delta\right\}. \qquad (3.2.5)$$

Since $\phi_{\delta_1}^+(E) \geq \phi_{\delta_2}^+(E)$ for $0 < \delta_1 < \delta_2 \leq +\infty$, the limit

$$\mu^+(E) = \lim_{\delta\to 0_+} \phi_\delta^+(E) = \sup_{\delta>0} \phi_\delta^+(E) \qquad (3.2.6)$$

exists for all $E \subset X$. In this context, $\mu^+(E)$ can be called the result of Caratheodory's construction from $\zeta^+(E)$ on F. $\phi_\delta^+(E)$ can be referred to as the size δ approximating positive measure. Let $\zeta^+(E_i,d^+)$ be for example

$$\zeta^+(E_i,d^+) = \Theta(d^+)[diam(E_i)]^{d^+}, 0 < \Theta(d^+), \qquad (3.2.7)$$

for non-empty subsets $E_i, i \in \mathbb{N}$ of X. Where $\Theta(d^+)$ is some geometrical factor,depends on the geometry of the sets E_i, used for covering. When F is the set of all non-empty subsets of X, the resulting measure $\mu_H^+(E,d^+)$ is called the d^+-dimensional Hausdorff measure over X; in particular, when F is the set of all (closed or open) balls in X,

$$\Theta(d^+) \triangleq \Omega(d^+) = \pi^{\frac{d^+}{2}}(2^{-d^+})\Gamma\left(1 + \frac{d^+}{2}\right). \qquad (3.2.8)$$

Consider a measurable metric space $(X,\mu_H(d)), X \subseteqq \mathbb{R}^n, d \in (-\infty,+\infty)$. The elements of X are denoted by $x,y,z,\ldots,$ and represented by n-tuples of real numbers $x = (x_1,x_2,\ldots,x_n)$

The metric $d(x,y)$ is a function $d : X \times X \to R_+$ is defined in n dimensions by

$$d(x,y) = \sum_{ij}[\delta_{ij}(y_i - x_i)(y_j - x_j)]^{1/2} \qquad (3.2.9)$$

and the diameter of a subset $E \subset X$ is defined by

$$diam(E) = \sup\{d(x,y)|x,y \in E\}. \qquad (3.2.10)$$

Definition 3.2.1. The Hausdorff measure $\mu_H^+(E,D^+)$ of a subset $E \subset X$ with the associated Hausdorff positive dimension $D^+ \in \mathbb{R}_+$ is defined by canonical way

$$\mu_H^+(E,D^+) = \lim_{\delta\to 0}\left[\inf_{\{E_i\}_{i\in\mathbb{N}}} \left\{\sum_{i\in\mathbb{N}} \zeta^+(E_i,D^+) | E \subset \bigcup_i E_i, \forall i(diam(E_i) < \delta)\right\}\right], \qquad (3.2.11)$$

$$D^+(E) = \sup\{d^+ \in \mathbb{R}_+ | d^+ > 0, \mu_H^+(E,d^+) = +\infty\}.$$

Definition 3.2.2. Remind that a function $f : X \to \mathbb{R}$ defined in a measurable space

(X,Σ,μ), is called a simple function if there is a finite disjoint set of sets $\{E_1,,\ldots,E_n\}$ of measurable sets and a finite set $\{\alpha_1,,\ldots,\alpha_n\}$ of real numbers such that $f(x) = \alpha_i$ if $x \in E_i$ and $f(x) = 0$ if $x \notin E_i$. Thus $f(x) = \sum_{i=1}^n \alpha_i \chi_{E_i}(x)$, where $\chi_{E_i}(x)$ is the characteristic function of E_i. A simple function f on a measurable space (X,Σ,μ) is integrable if $\mu(E_i) < \infty$ for every index i for which $\alpha_i \neq 0$. The Lebesgue-Stieltjes integral of f is defined by

$$\int f d\mu = \sum_{i=1}^n \alpha_i \mu(E_i). \tag{3.2.12}$$

A continuous function is a function for which $\lim_{x\to y} f(x) = f(y)$ whenever $\lim_{x\to y} d(x,y) = 0$.

The Lebesgue-Stieltjes integral over continuous functions can be defined as the limit of infinitesimal covering diameter: when $\{E_i\}_{i\in\mathbb{N}}$ is a disjoined covering and $x_i \in E_i$ by definition (3.2.12) one obtains

$$\int_X f(x) d\mu_H^+(x,D^+) = \lim_{diam(E_i)\to 0} \left[\sum_{\cup E_i = X} f(x_i) \inf_{\{E_{ij}\} \text{ with } \cup_j E_{ij} \supset E_i} \sum_j \zeta^+(E_{ij}, D^+(E_{ij})) \right]. \tag{3.2.13}$$

From now on, X is assumed metrically unbounded, i.e. for every $x \in X$ and $r > 0$ there exists a point y such that $d(x,y) > r$. The standard assumption that D^+ is uniquely defined in all subsets E of X requires X to be regular (homogeneous, uniform) with respect to the measure,i.e. $\mu_H^+(B_r(x),D^+) = \mu_H^+(B_r(y),D^+)$ for all elements $x,y \in X$ and (convex) balls $B_r(x)$ and $B_r(y)$ of the form $B_{r>0}(x) = \{z | d(x,z) \leq r; x,z \in X\}$. In the limit $diam(E_i) \to 0$, the infimum is satisfied by the requirement that the variation over all coverings $\{E_{ij}\}_{ij\in\mathbb{N}}$ is replaced by one single covering E_i, such that $\cup_j E_{ij} \to E_i \ni x_i$. Hence

$$\int_X f(x) d\mu_H^+(x,D^+) = \lim_{diam(E_i)\to 0} \sum_{\cup E_i = X} f(x_i) \zeta^+(E_i, D^+). \tag{3.2.14}$$

The range of integration X may be parametrised by polar coordinates with $r = d(x,0)$ and angle Ω. $\{E_{r_i,\Omega_i}\}_{i\in\mathbb{N}}$ can be thought of as spherically symmetric covering around a centre at the origin. In the limit, the function $\zeta^+(E_{r,\Omega},D^+)$ defined by Eq.(3.2.7) is given by

$$d\mu_H^+(x,D^+) = \lim_{diam(E_{r,\omega})\to 0} \zeta^+(E_{r,\Omega},D^+) = d\Omega^{D^+-1} r^{D^+-1} dr. \tag{3.2.15}$$

Let us assume now for simplification that $f(x) = f(\|x\|) = f(r)$ and.$\lim_{r\to\infty} f(r) = 0$. The integral over a D^+-dimensional metric space X is then given by

$$\int_X f(x) d\mu_H^+(x,D^+) = \int_X f(x) d^{D^+} x = \frac{2\pi^{\frac{D^+}{2}}}{\Gamma\left(1+\frac{D^+}{2}\right)} \int_0^\infty f(r) r^{D^+-1} dr. \tag{3.2.16}$$

The integral defined in (3.2.14) satisfies the following conditions.

(i) Linearity:

$$\int_X [a_1 f_1(x) + a_2 f_2(x)] d\mu_H^+(x,D^+) = a_1 \int_X f_1(x) d\mu_H^+(x,D^+) + a_2 \int_X f_2(x) d\mu_H^+(x,D^+). \tag{3.2.17}$$

(ii) Translational invariance:

$$\int_X f(x+x_0) d\mu_H^+(x,D^+) = \int_X f(x) d\mu_H^+(x,D^+) \tag{3.2.18}$$

since $d\mu_H^+(x-x_0,D^+) = d\mu_H^+(x,D^+)$.

(iii) Scaling property:

$$\int_X f(ax)d\mu_H^+(x,D^+) = a^{-D^+}\int_X f(x)d\mu_H^+(x,D^+) \tag{3.2.19}$$

since $d\mu_H^+(x/a,D^+) = a^{-D^+}d\mu_H^+(x,D^+)$.

(iv) The generalised $\delta^{D^+}(x)$ function:
The generalised $\delta^{D^+}(x)$ function for sets with non-integer Hausdorff dimension exists and can be defined by formula

$$\int_X f(x)\delta^{D^+}(x-x_0)d\mu_H^+(x,D^+) = f(x_0). \tag{3.2.20}$$

III.3.Hausdorff-Colombeau measure and associated negative Hausdorff-Colombeau dimensions.

During last 20 years the notion of negative dimension in geometry was many developed, see [15],[23]-[27].

Remind that canonical difenitions of noninteger positive dimension alwais equipped with an measure. Hausdorff–Besicovich dimension equipped with Hausdorff measure $d\mu_H^+(x,D^+)$.

Let us consider example of a space of noninteger positive dimension equipped with the Haar measure.On the closed interval $0 \leq x \leq 1$ there is a scale $0 \leq \sigma \leq 1$ of Cantor dust with the Haar measure equal to x^σ for any interval $(0,x)$ similar to the entire given set of the Cantor dust. The direct product of this scale by the Euclidean cube of dimension $k-1$ gives the entire scale $k+\sigma$, where $k \in \mathbb{Z}$ and $\sigma \in (0,1)$ [24].

In this subsection we define generalized Hausdorff-Colombeau measure.In subsection III.4 we will prove that negative dimensions of fractal equipped with the Hausdorff-Colombeau measure in natural way.

Let Ω be an open subset of $\mathbb{R}^n$, let X be metric space $X \subseteqq \mathbb{R}^n$ and let F be a set $F = \{E_i\}_{i\in\mathbb{N}}$ of subsets E_i of X. Let $\zeta(E,x,\check{x})$ be a function $\zeta : F\times\Omega\times\Omega \to \mathbb{R}$. Let $C_F^\infty(\Omega)$ be a set of the all functions $\zeta(E,x)$ such that $\zeta(E,x) \in C^\infty(\Omega)$ whenever $E \in F$. Throughout this paper, for elements of the space $C_F^\infty(\Omega)^{(0,1]}$ of sequences of smooth functions indexed by $\varepsilon \in (0,1]$ we shall use the canonical notations $(\zeta_\varepsilon(E,x))_\varepsilon$ and $(\zeta_\varepsilon)_\varepsilon$ so $\zeta_\varepsilon \in C_F^\infty(\Omega)$, $\varepsilon \in (0,1]$.

Definition 3.3.1. We set $\mathcal{G}_F(\Omega) = \mathcal{E}_M(F,\Omega)/\mathcal{N}(F,\Omega)$, where

$$\begin{gathered}\mathcal{E}_M(F,\Omega) = \Big\{(\zeta_\varepsilon)_\varepsilon \in C_F^\infty(\Omega)^{(0,1]} \,\Big|\, \forall K \subset\subset \Omega, \forall\alpha \in \mathbb{N}^n \exists p \in \mathbb{N} \text{ with} \\ \sup_{E\in F; x\in K}|\zeta_\varepsilon(E,x)| = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\Big\}, \\ \mathcal{N}(F,\Omega) = \Big\{(\zeta_\varepsilon)_\varepsilon \in C_F^\infty(\Omega)^{(0,1]} \,\Big|\, \forall K \subset\subset \Omega, \forall\alpha \in \mathbb{N}^n \forall q \in \mathbb{N} \\ \sup_{E\in F; x\in K}|\zeta_\varepsilon(E,x)| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\Big\}.\end{gathered} \tag{3.3.1}$$

Notice that $\mathcal{G}_F(\Omega)$ is a differential algebra.Equivalence classes of sequences $(\zeta_\varepsilon)_\varepsilon$ will be denoted by $\mathbf{cl}[(\zeta_\varepsilon)_\varepsilon]$ or simply $[(\zeta_\varepsilon)_\varepsilon]$.

Definition 3.3.2. We denote by $\widetilde{\mathbb{R}}$ the ring of real,Colombeau generalized numbers. Recall that by definition $\widetilde{\mathbb{R}} = \mathcal{E}_M(\mathbb{R})/\mathcal{N}(\mathbb{R})$ [21], where

$$\begin{aligned}\mathcal{E}_M(\mathbb{R}) &= \{(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} | (\exists\alpha \in \mathbb{R})(\exists\varepsilon_0 \in (0,1])\forall\varepsilon \leq \varepsilon_0[|x_\varepsilon| \leq \varepsilon^\alpha]\},\\ \mathcal{N}(\mathbb{R}) &= \{(x_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} | (\forall\alpha \in \mathbb{R})(\exists\varepsilon_0 \in (0,1])\forall\varepsilon \leq \varepsilon_0[|x_\varepsilon| \leq \varepsilon^\alpha]\}.\end{aligned} \quad (3.3.2)$$

Notice that the ring $\widetilde{\mathbb{R}}$ arises naturally as the ring of constants of the Colombeau algebras $\mathcal{G}(\Omega)$. Recall that there exists natural embedding $\widetilde{r} : \mathbb{R} \hookrightarrow \widetilde{\mathbb{R}}$ such that for all $r \in \mathbb{R}, \widetilde{r} = (r_\varepsilon)_\varepsilon$ where $r_\varepsilon \equiv r$ for all $\varepsilon \in (0,1]$. We say that r is standard number and abbreviate $r \in \mathbb{R}$ for short. The ring $\widetilde{\mathbb{R}}$ can be endowed with the structure of a partially ordered ring: for $r,s \in \widetilde{\mathbb{R}}$ $\eta \in \mathbb{R}_+, \eta \leq 1$ we abbreviate $r \leq_{\widetilde{\mathbb{R}},\eta} s$ or simply $r \leq_{\widetilde{\mathbb{R}}} s$ if and only if there are representatives $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ with $r_\varepsilon \leq s_\varepsilon$ for all $\varepsilon \in (0,\eta]$. Colombeau generalized number $r \in \widetilde{\mathbb{R}}$ with representative $(r_\varepsilon)_\varepsilon$ we abbreviate $\mathbf{cl}[(r_\varepsilon)_\varepsilon]$.

Definition 3.3.3. (i) Let $\delta \in \widetilde{\mathbb{R}}$. We say that δ is infinite small Colombeau generalized number and abbreviate $\delta \approx_{\widetilde{\mathbb{R}}} \widetilde{0}$ if there exists representative $(\delta_\varepsilon)_\varepsilon$ and some $q \in \mathbb{N}$ such that $|\delta_\varepsilon| = O(\varepsilon^q)$ as $\varepsilon \to 0$. (ii) Let $\Delta \in \widetilde{\mathbb{R}}$. We say that Δ is infinite large Colombeau generalized number and abbreviate $\Delta =_{\widetilde{\mathbb{R}}} \widetilde{\infty}$ if $\Delta^{-1}_{\widetilde{\mathbb{R}}} \approx_{\widetilde{\mathbb{R}}} \widetilde{0}$. (iii) Let $\mathbb{R}_\infty$ be $\mathbb{R} \cup \{\infty\}$ We say that $\Theta \in \widetilde{\mathbb{R}}_\infty$ is infinite Colombeau generalized number and abbreviate $\Theta =_{\widetilde{\mathbb{R}}} \infty_{\widetilde{\mathbb{R}}}$ if there exists representative $(\Theta_\varepsilon)_\varepsilon$ where $\Theta_\varepsilon = \infty$ for all $\varepsilon \in (0,1]$. Here we set $\mathcal{E}_M(\mathbb{R}_\infty) = \mathcal{E}_M(\mathbb{R}) \cup \{(\Theta_\varepsilon)_\varepsilon\}$, $\mathcal{N}(\mathbb{R}_\infty) = \mathcal{N}(\mathbb{R})$ and $\widetilde{\mathbb{R}}_\infty = \mathcal{E}_M(\mathbb{R}_\infty)/\mathcal{N}(\mathbb{R}_\infty)$.

Definition 3.3.4.The singular Hausdorff-Colombeau measure originate in Colombeau generalization of canonical Caratheodory's construction, which is defined as follows: for each metric space X, each set $F = \{E_i\}_{i\in\mathbb{N}}$ of subsets E_i of X, and each Colombeau generalized function $(\zeta_\varepsilon(E,x,\check{x}))_\varepsilon$, such that: (i) $0 \leq (\zeta_\varepsilon(E,x,\check{x}))_\varepsilon$, (ii) $(\zeta_\varepsilon(E,\check{x},\check{x}))_\varepsilon =_{\widetilde{\mathbb{R}}} \widetilde{\infty}$, whenever $E \in F$, a preliminary Colombeau measure $(\phi_\delta(E,x,\check{x},\varepsilon))_\varepsilon$ can be constructed corresponding to $0 < \delta \leq +\infty$, and then a final Colombeau measure $(\mu_\varepsilon(E,x,\check{x}))_\varepsilon$, as follows: for every subset $E \subset X$, the preliminary Colombeau measure $(\phi_\delta(E,x,\check{x},\varepsilon))_\varepsilon$ is defined by

$$\phi_\delta(E,x,\check{x},\varepsilon) = \sup_{\{E_i\}_{i\in\mathbb{N}}} \left\{\sum_{i\in\mathbb{N}} \zeta_\varepsilon(E_i,x,\check{x}) | E \subset \bigcup_{i\in\mathbb{N}} E_i, diam(E_i) \leq \delta\right\}. \quad (3.3.3)$$

Since for all $\varepsilon \in (0,1] : \phi_{\bar{\delta}_1}(E,x,\check{x},\varepsilon) \geq \phi_{\bar{\delta}_2}(E,x,\check{x},\varepsilon)$ for $0 < \delta_1 < \delta_2 \leq +\infty$, the limit

$$(\mu(E,x,\check{x},\varepsilon))_\varepsilon = \left(\lim_{\delta\to 0_+} \phi_\delta(E,x,\check{x},\varepsilon)\right)_\varepsilon = \left(\inf_{\delta>0} \phi_\delta(E,x,\check{x},\varepsilon)\right)_\varepsilon \quad (3.3.4)$$

exists for all $E \subset X$. In this context, $(\mu(E,x,\check{x},\varepsilon))_\varepsilon$ can be called the result of Caratheodory's construction from $(\zeta_\varepsilon(E,x,\check{x}))_\varepsilon$ on F and $(\phi_\delta(E,x,\check{x},\varepsilon))_\varepsilon$ can be referred to as the size δ approximating Colombeau measure.

Definition 3.2.5. Let $(\zeta_\varepsilon(E_i,D^+,D^-,x,\check{x}))_\varepsilon$ be

$$(\zeta_\varepsilon(E_i,D^+,D^-,x,\check{x}))_\varepsilon = \begin{cases} \left(\dfrac{\Theta_1(D^+)\Theta_2(D^-)[diam(E_i)]^{D^+}}{[d(x,\check{x})]^{|D^-|} + \varepsilon}\right)_\varepsilon & \text{if } x \in E_i \\ 0 & \text{if } x \notin E_i \end{cases} \quad (3.3.5)$$

where $\varepsilon \in (0,1], \Theta_1(D^+), \Theta_2(D^-) > 0$. In particular, when F is the set of all (closed or

open)

balls in X,

$$\Theta_1(D^+) = \frac{2^{-D^+}\pi^{\frac{D^+}{2}}}{\Gamma\left(1+\frac{D^+}{2}\right)} \tag{3.3.6}$$

and

$$\Theta_2(D^-) = \frac{2^{-D^-}\pi^{\frac{D^-}{2}}}{\left|\Gamma\left(1+\frac{D^-}{2}\right)\right|}, \tag{3.3.7}$$

$$D^- \neq -2,-4,-6,\ldots,-2(n+1),\ldots$$

Definition 3.2.6. The Hausdorff-Colombeau singular measure $(\mu_H(E,D^+,D^-,x,\check{x},\varepsilon))_\varepsilon$ of a subset $E \subset X$ with the associated Hausdorff-Colombeau dimension $\check{D}^+(D^-) \in \mathbb{R}_+, D^- \in \mathbb{R}_+,$ is defined by

$$(\mu_{HC}(E,\check{D}^+,D^-,x,\check{x},\varepsilon))_\varepsilon =$$

$$\left(\lim_{\delta\to 0}\left[\sup_{\{E_i\}_{i\in\mathbb{N}}}\left\{\sum_{i\in\mathbb{N}}(\zeta_\varepsilon(E_i,\check{D}^+,D^-,x,\check{x}))_\varepsilon | E \subset \bigcup_i E_i, \forall i(diam(E_i) < \delta)\right\}\right]\right)_\varepsilon, \tag{3.3.8}$$

$$\check{D}^+ = \sup\left\{D^+ > 0 | (\mu_{HC}(E,D^+,D^-,x,\check{x},\varepsilon))_\varepsilon = \infty_{\widetilde{\mathbb{R}}}\right\},$$

The Colombeau-Lebesgue-Stieltjes integral over continuous functions $f : X \to \mathbb{R}$ can be evaluated similarly as in subsection III.3,(but using the limit in sense of Colombeau generalized functions) of infinitesimal covering diameter when $\{E_i\}_{i\in\mathbb{N}}$ is a disjoined covering and $x_i \in E_i$:

$$\left(\int_X f(x)d\mu_{HC}(E,D^+,D^-,x,\check{x},\varepsilon)\right)_\varepsilon =$$

$$\left(\lim_{diam(E_i)\to 0}\left[\sum_{\cup E_i = X} f(x_i) \inf_{\{E_{ij}\}\text{ with }\cup_j E_{ij}\supset E_i} \sum_j \zeta_\varepsilon(E_i,D^+,D^-,x_i,\check{x})\right]\right)_\varepsilon. \tag{3.3.9}$$

We assume now that X is metrically unbounded, i.e. for every $x \in X$ and $r > 0$ there exists a point y such that $d(x,y) > r.$ The standard assumption that $\check{D}^+$ and $\check{D}^-$ is uniquely defined in all subsets E of X requires X to be regular (homogeneous, uniform) with respect to the measure,i.e. $(\mu_{HC}^-(B_r(\check{x}),\check{D}^+,\check{D}^-,x,\check{x},\varepsilon))_\varepsilon = (\mu_{HC}^-(B_r(\check{y}),\check{D}^+,\check{D}^-,x',\check{y},\varepsilon))_\varepsilon,$ where $d(x,\check{x}) = d(x',\check{y})$ for all elements $\check{x},\check{y},x,x' \in X$ and convex balls $B_r(\check{x})$ and $B_r(\check{y})$ of the form $B_r(\check{x}) = \{z | d(\check{x},z) \leq r; \check{x},z \in X\}$ and $B_r(\check{y}) = \{z | d(\check{y},z) \leq r; \check{y},z \in X\}$. In the limit $diam(E_i) \to 0$, the infimum is satisfied by the requirement that the variation over all coverings $\{E_{ij}\}_{ij\in\mathbb{N}}$ is replaced by one single covering $E_i,$ such that $\cup_j E_{ij} \to E_i \ni x_i$. Therefore

$$\left(\int_X f(x)d\mu_{HC}(E,\check{D}^+,\check{D}^-,x,\check{x},\varepsilon)\right)_\varepsilon =$$

$$\left(\lim_{diam(E_i)\to 0}\sum_{\cup E_i = X} f(x_i)\zeta_\varepsilon(E_i,\check{D}^+,\check{D}^-,x_i,\check{x})\right)_\varepsilon. \tag{3.3.10}$$

Assume that $f(x) = f(r), r = \|r\|$. The range of integration X may be parametrised by polar coordinates with $r = d(x,0)$ and angle ω. $\{E_{r_i,\omega_i}\}$ can be thought of as spherically symmetric covering around a centre at the origin. Thus

$$\left(\int_X f(r) d\mu_{HC}(E, \check{D}^+, \check{D}^-, x, \check{x}, \varepsilon)\right)_\varepsilon = \left(\lim_{diam(E_i)\to 0} \sum_{\cup E_i = X} f(r_i)\zeta_\varepsilon(E_i, \check{D}^+, \check{D}^-, x_i, \check{x})\right)_\varepsilon . \qquad (3.3.11)$$

Notice that the metric set $X \subset \mathbb{R}^n$ can be tesselated into regular polyhedra; in particular it is always possible to divide $\mathbb{R}^n$ into parallelepipeds of the form

$$\Pi_{i_1,\ldots,i_n} = \{x = (x_1,\ldots,x_n) \in X | x_j = (i_j - 1)\Delta x_j + \gamma_j, 0 \leq \gamma_j \leq \Delta x_j, j = 1,\ldots,n\}. \qquad (3.3.12)$$

For $n = 2$ the polyhedra $\Pi_{i_1,i_2}$is shown in figure 3.3.1.Since X is uniform

$$(d\mu_{HC}(x, \check{D}^+, \check{D}^-, x, \check{x}, \varepsilon))_\varepsilon = \left(\lim_{diam(\Pi_{i_1,\ldots,i_n})} \zeta_\varepsilon(\Pi_{i_1,\ldots,i_n}, \check{D}^+, \check{D}^-, x, \check{x})\right)_\varepsilon = \left(\lim_{diam(\Pi_{i_1,\ldots,i_n})} \prod_{j=1}^n \left(\frac{\Delta x_j}{|x_j - \check{x}_j|^{|\check{D}^-|} + \varepsilon}\right)^{\frac{\check{D}^+}{n}}\right)_\varepsilon \triangleq \left(\prod_{j=1}^n \frac{d^{\frac{\check{D}^+}{n}} x_j}{\left(|x_j - \check{x}_j|^{|\check{D}^-|} + \varepsilon\right)^{\frac{\check{D}^+}{n}}}\right)_\varepsilon . \qquad (3.3.13)$$

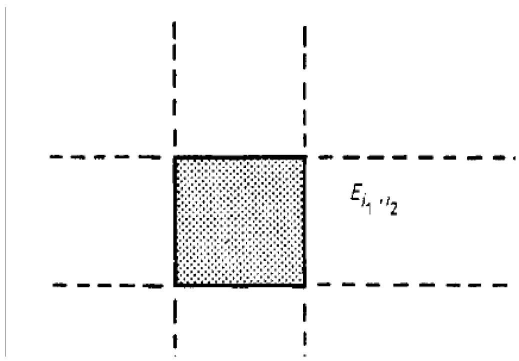

Fig.3.3.1. The polyhedra covering for $n = 2$.

Notice that the range of integration X may also be parametrised by polar coordinates with

$r = d(x,0)$ and angle Ω. $E_{r,\Omega}$ can be thought of as spherically symmetric covering around a centre at the origin (see figure 3.3.2 for the two-dimensional case). In the limit,

the Colombeau generaliza function $(\zeta_\varepsilon(E_{r,\Omega}, \check{D}^+, \check{D}^-, r, 0))_\varepsilon$ is given by

$$(d\mu_{HC}(r, \Omega, \check{D}^+, \check{D}^-, \varepsilon))_\varepsilon = \left(\lim_{diam(\Pi_{i_1,\ldots,i_n})} \zeta_\varepsilon(E_{r,\Omega}, \check{D}^+, \check{D}^-, \{r,\Omega\}, 0)\right)_\varepsilon \triangleq \frac{d\Omega^{\check{D}^+-1} r^{\check{D}^+-1} dr}{\left(r^{|\check{D}^-|} + \varepsilon\right)_\varepsilon} \qquad (3.3.14)$$

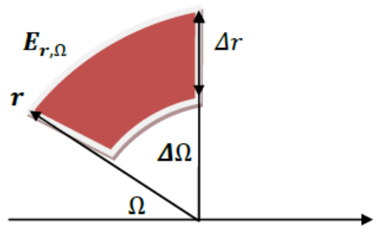

Fig.3.3.2.The spherical covering $E_{r,\Omega}$.

When $f(x)$ is symmetric with respect to some centre $\check{x} \in X$, i.e. $f(x) =$ constant for all x satisfying $d(x,\check{x}) = r$ for arbitrary values of r, then chainge of the variable

$$x \to z = x - \check{x} \tag{3.3.15}$$

can be performed to shift the centre of symmetry to the origin (since X is not a linear space, (3.3.15) need not be a map of X onto itself and (3.3.15) is measure preseming). The integral over metric space X is then given by formula

$$\left(\int_X f(x)d\mu_{HC}(E,\check{D}^+,\check{D}^-,x,\check{x},\varepsilon)\right)_\varepsilon = \frac{4\pi^{D^+/2}\pi^{D^-/2}}{\Gamma(D^+/2)\Gamma(D^-/2)}\left(\int_0^\infty \frac{r^{D^+-1}f(r)}{\varepsilon + r^{|D^-|}}dr\right)_\varepsilon. \tag{3.3.16}$$

III.4.Main properties of the Hausdorff-Colombeau metric measures with associated negative Hausdorff-Colombeau dimensions.

Definition 3.4.1. An outer Colombeau metric measure on a set $X \subseteq \mathbb{R}^n$ is a Colombeau

generalized function $[(\phi_\varepsilon(E))_\varepsilon] \in \mathcal{G}_F(\Omega)$ (see Definition 3.3.1) defined on all

subsets of X satisfies the following properties:.

(i) Null empty set:The empty set has zero Colombeau outer measure

$$[(\phi_\varepsilon(\varnothing))_\varepsilon] = 0. \tag{3.4.1}$$

(ii) Monotonicity: For any two subsets A and B of X

$$A \subseteq B \Rightarrow [(\phi_\varepsilon(A))_\varepsilon] \leq_{\widetilde{\mathbb{R}}} [(\phi_\varepsilon(B))_\varepsilon]. \tag{3.4.2}$$

(iii) Countable subadditivity: For any sequence $\{A_j\}$ of subsets of X pairwise disjoint or not

$$\left[(\phi_\varepsilon(\cup_{j=1}^\infty A_j))_\varepsilon\right] \leq_{\widetilde{\mathbb{R}}} \left[\left(\sum_{j=1}^\infty \phi_\varepsilon(A_j)\right)_\varepsilon\right]. \tag{3.4.3}$$

(iv) Whenever $d(A,B) = \inf\{d_n(x,y)|x \in A, y \in B\} > 0$

$$[(\phi_\varepsilon(A \cup B))_\varepsilon] = [(\phi_\varepsilon(A))_\varepsilon] + [(\phi_\varepsilon(B))_\varepsilon], \tag{3.4.4}$$

where $d_n(x,y)$ is the usual Euclidean metric:$d_n(x,y) = \sqrt{\sum(x_i - y_i)^2}$.

Definition 3.4.2. We say that outer Colombeau metric measure $(\mu_\varepsilon)_\varepsilon, \varepsilon \in (0,1]$ is a Colombeau measure on σ-algebra of subests of $X \subseteq \mathbb{R}^n$ if $(\mu_\varepsilon)_\varepsilon$ satisfies the following property:

$$\left[(\phi_\varepsilon(\cup_{j=1}^{\infty} A_j))_\varepsilon\right] = \left[\left(\sum_{j=1}^{\infty} \phi_\varepsilon(A_j)\right)_\varepsilon\right]. \tag{3.4.5}$$

Definition 3.4.3. If U is any non-empty subset of n-dimensional Euclidean space, $\mathbb{R}^n$, the diamater $|U|$ of U is defined as

$$|U| = \sup\{d(x,y) | x,y \in U\} \tag{3.4.6}$$

If $F \subseteq \mathbb{R}^n$, and a collection $\{U_i\}_{i\in\mathbb{N}}$ satisfies the following conditions:

(i) $|U_i| < \delta$ for all $i \in \mathbb{N}$, (ii) $F \subseteq \cup_{i\in\mathbb{N}} U_i$, then we say the collection $\{U_i\}_{i\in\mathbb{N}}$ is a δ-cover of F.

Definition 3.4.4. If $F \subseteq \mathbb{R}^n$ and $s,q,\delta > 0$, we define Hausdorff-Colombeau content:

$$(H_\delta^{s,q}(F,\varepsilon))_\varepsilon = \left(\inf\left\{\sum_{i=1}^{\infty} \frac{|U_i|^s}{\|x_i\|^q + \varepsilon}\right\}\right)_\varepsilon \tag{3.4.7}$$

where the infimum is taken over all δ-covers of F and where $x_i = (x_{i,1},\ldots x_{i,n}) \in U_i$ for all $i \in \mathbb{N}$ and $\|x\|$ is the usual Euclidean norm: $\|x\| = \sqrt{\sum_{j=1}^{n} x_j^2}$.

Note that for $0 < \delta_1 < \delta_2 < 1, \varepsilon \in (0,1]$ we have

$$H_{\delta_1}^{s,q}(F,\varepsilon) \geq H_{\delta_2}^{s,q}(F,\varepsilon) \tag{3.4.8}$$

since any δ_1 cover of F is also a δ_2 cover of F, i.e. $H_{\delta_1}^{s,q}(F,\varepsilon)$ is increasing as δ decreases.

Definition 3.4.4. We define the (s,q)-dimensional Hausdorff-Colombeau (outer) measure as:

$$(H^{s,q}(F,\varepsilon))_\varepsilon = \left(\lim_{\delta\to 0} H_\delta^{s,q}(F,\varepsilon)\right)_\varepsilon. \tag{3.4.9}$$

Theorem 3.4.1. For any δ-cover, $\{U_i\}_{i\in\mathbb{N}}$ of F, and for any $\varepsilon \in (0,1]$, $t > s$:

$$H^{t,q}(F,\varepsilon) \leq \delta^{t-s} H^{s,q}(F,\varepsilon). \tag{3.4.10}$$

Proof. Consider any δ-cover $\{U_i\}_{i\in\mathbb{N}}$ of F. Then each $|U_i|^{t-s} \leq \delta^{t-s}$ since $|U_i| \leq \delta$, so:

$$|U_i|^t = |U_i|^{t-s}|U_i|^s \leq \delta^{t-s}|U_i|^s. \tag{3.4.11}$$

From (3.4.11) follows that

$$\frac{|U_i|^t}{\|x_i\|^q + \varepsilon} \leq \frac{\delta^{t-s}|U_i|^s}{\|x_i\|^q + \varepsilon} \tag{3.4.12}$$

and summing (3.4.11) over all $i \in \mathbb{N}$ we obtain

$$\sum_{i=1}^{\infty} \frac{|U_i|^t}{\|x_i\|^q + \varepsilon} \leq \delta^{t-s} \sum_{i=1}^{\infty} \frac{|U_i|^s}{\|x_i\|^q + \varepsilon}. \tag{3.4.13}$$

Thus (3.4.10) follows by taking the infimum.

Theorem 3.4.2. (1) If $(H^{s,q}(F,\varepsilon))_\varepsilon <_{\widetilde{\mathbb{R}}} \infty_{\widetilde{\mathbb{R}}}$, and if $t > s$, then $(H^{t,q}(F,\varepsilon))_\varepsilon = 0_{\widetilde{\mathbb{R}}}$.

(2) If $0_{\widetilde{\mathbb{R}}} <_{\widetilde{\mathbb{R}}} (H^{s,q}(F,\varepsilon))_\varepsilon$, and if $t < s$, then $(H^{t,q}(F,\varepsilon))_\varepsilon = \infty_{\widetilde{\mathbb{R}}}$.

Proof. (1) The result follows from (3.4.10) after taking limits, since $\forall\varepsilon \in (0,1]$ by definitions follows that $H^{s,q}(F,\varepsilon) < \infty$.

(2) From (3.4.10) $\forall\varepsilon \in (0,1], \forall\delta > 0$ follows that

$$\frac{1}{\delta^{s-t}} H^{s,q}(F,\varepsilon) \le H^{t,q}(F,\varepsilon). \tag{3.4.14}$$

After taking limit $\delta \to 0$, we obtain $H^{t,q}(F,\varepsilon) = \infty$, since $\lim_{\delta\to 0}(\delta^{s-t})^{-1} = \infty$ and $\lim_{\delta\to 0} H^{s,q}_{\delta}(F,\varepsilon) = H^{s,q}(F,\varepsilon) > 0$.

Definition 3.4.5. We define now the Hausdorff-Colombeau dimension $\dim_{HC}(F,q)$ of a set F (relative to $q > 0$) as

$$\begin{gathered}\dim_{HC}(F,q) = \\ \sup\left\{s = s(q) | (H^{s,q}(F,\varepsilon))_{\varepsilon} = \infty_{\widetilde{\mathbb{R}}}\right\} = \inf\left\{s = s(q) | (H^{s,q}(F,\varepsilon))_{\varepsilon} = 0_{\widetilde{\mathbb{R}}}\right\}.\end{gathered} \tag{3.4.15}$$

Remark 3.4.1. From theorem 3.4.2 follows that for any fixed $q = \breve{q}$: $(H^{s,\breve{q}}(F,\varepsilon))_{\varepsilon} = 0_{\widetilde{\mathbb{R}}}$ or $(H^{s,\breve{q}}(F,\varepsilon))_{\varepsilon} = \infty_{\widetilde{\mathbb{R}}}$ everywhere except at a unique value s, where this value may be finite. As a function of $s, H^{s,\breve{q}}(F,\varepsilon)$ is decreasing function. Therefore, the graph of $H^{s,\breve{q}}(F,\varepsilon)$ will have a unique value where it jumps from ∞ to 0.

Remark 3.4.2. Note that the graph of $(H^{s,\breve{q}}(F,\varepsilon))_{\varepsilon}$ for a fixed $q = \breve{q}$ is

$$(H^{s,\breve{q}}(F,\varepsilon))_{\varepsilon} = \begin{cases} \infty_{\widetilde{\mathbb{R}}} & \text{if} \quad s < \dim_{HC}(F,\breve{q}) \\ 0_{\widetilde{\mathbb{R}}} & \text{if} \quad s > \dim_{HC}(F,\breve{q}) \\ \text{undetermined} & \text{if} \quad s = \dim_{HC}(F,\breve{q}) \end{cases} \tag{3.4.16}$$

Definition 3.4.6. We say that fractal $\mathcal{F} \subseteqq \mathbb{R}^n$ has a negative dimension relative to $q > 0$ if $\dim_{HC}(F,q) - q < 0$.

IV. Scalar quantum field theory in spacetime with Hausdorff- Colombeau negative dimensions.

IV.1.Equation of motion and Hamiltonian.

Scalar quantum field theory and quantum gravity in spacetime with noninteger positive Hausdorff dimensions developed in papers [29]-[32].Quantum mechanics in negative dimensions developed in papers [33],[34] Scalar quantum field theory and quantum gravity in spacetime with Hausdorff-Colombeau negative dimensions originally developed in paper [15].In this section only free scalar quantum field in spacetime with negative dimensions briefly is considered.

A negative-dimensional spacetime structures is a desirable feature of superrenormalizable spacetime models of quantum gravity, and the most simply way to obtain it is to let the effective dimensionality of the multifractal universe to change at different scales. A simple realization of this feature is via suitable extended fractional calculus and the definition of a fractional action. Note that below we use canonical isotropic scaling such that:

$$[x^{\mu}] = -1, \mu = 0,1,\ldots,D_{\mathbf{t}} - 1 \tag{4.1.1}$$

while replacing the standard measure with a nontrivial Colombeau-Stieltjes measure,

$$\begin{gathered} d^{D_{\mathbf{t}}}x \to d^{D_{\mathbf{f}}}x = (d\eta(x,\varepsilon))_{\varepsilon}, \\ [\eta] = D_{\mathbf{t}} \bullet \alpha, \alpha \in [1,-\infty). \end{gathered} \tag{4.1.2}$$

Here $D_{\mathfrak{t}}$ is the topological (positive integer) dimension of embedding spacetime and α is a parameter.Any Colombeau integrals on net multifractals can be approximated by the left-sided Colombeau-Riemann–Liouville complex milti-fractional integral of a function $\mathcal{L}(t)$:

$$\left(\int_0^{\bar{t}} d\eta(x,\varepsilon)\mathcal{L}(t)\right)_\varepsilon \propto \left(I_{\bar{t},\varepsilon}^{\{z_i(\bar{t})\}}\right)_\varepsilon \triangleq \left(\sum_{i=1}^m \int_\varepsilon^{\bar{t}} \frac{[(\bar{t}-t)+i\varepsilon]^{z_i(\bar{t})-1}}{\Gamma(z_i(\bar{t}))}\mathcal{L}(t)dt\right)_\varepsilon, \quad (4.1.3)$$
$$(\eta(t,\varepsilon))_\varepsilon = \left(\frac{\bar{t}^{\,z_i(\bar{t})} - [(\bar{t}-t)+i\varepsilon]^{z_i(\bar{t})}}{\Gamma(z_i(\bar{t})+1)}\right)_\varepsilon,$$

where $\varepsilon \in (0,1]$, $\bar{t}$ is fixed and the order $z(\bar{t})$ is (related to) the complex Hausdorff-Colombeau dimensions of the set. In particular if $z_i \in \mathbb{C}, i = 1,2,\ldots,m$ is a complex parameters an integrals on net multifractals can be approximated by finite sum of the left-sided Colombeau-Riemann–Liouville complex fractional integral of a function $\mathcal{L}(t)$

$$\left(\int_0^{\bar{t}} d\eta(x,\varepsilon)\mathcal{L}(t)\right)_\varepsilon \propto \left(I_{\bar{t},\varepsilon}^{\{z_i\}_{i=1}^m}\right)_\varepsilon =$$
$$\sum_{i=1}^m \left(I_{\bar{t},\varepsilon}^{z_i}\right)_\varepsilon \triangleq \sum_{i=1}^m \left(\frac{1}{\Gamma(z_i)}\int_\varepsilon^{\bar{t}} d[(\bar{t}-t)+i\varepsilon]^{z_i-1}\mathcal{L}(t)\right)_\varepsilon, \quad (4.1.4)$$
$$(\eta(t,\varepsilon))_\varepsilon = \sum_{i=1}^m \left(\frac{\bar{t}^{\,z_i} - [(\bar{t}-t)+i\varepsilon]^{z_i}}{\Gamma(z_i+1)}\right)_\varepsilon.$$

Note that a change of variables $t \to \bar{t}-t$ trasforms Eq. (4.1.4) into the form

$$\left(\int_0^{\bar{t}} d\eta(x,\varepsilon)\mathcal{L}(t)\right)_\varepsilon = \sum_{i=1}^m \left(\int_0^{\bar{t}} dt\frac{[t+i\varepsilon]^{z_i-1}}{\Gamma(z(\bar{t}))}\mathcal{L}(\bar{t}-t)\right)_\varepsilon. \quad (4.1.5)$$

The Colombeau-Riemann–Liouville multifractional integral (5.1.5) can be mapped onto a Colombeau-Weyl multifractional integral in the formal limit $\bar{t} \to +\infty$.We assume otherwise,so that there exists $\lim_{\bar{t}\to+\infty} z(\bar{t})$ and $\lim_{\bar{t}\to+\infty} \mathcal{L}(\bar{t}-t) = \mathcal{L}[q(t),\dot{q}(t)]$. In particular if

$z \in \mathbb{C}$ is a complex parameter a change of variables $t \to \bar{t}-t$ trasforms eq. (5.1.5) into the form

$$\sum_{i=1}^m \left(I_{\bar{t},\varepsilon}^{z_i}\right)_\varepsilon = \sum_{i=1}^m \left(\int_\varepsilon^{\bar{t}} dt\frac{[t+i\varepsilon]^{z_i-1}}{\Gamma(z_i)}\mathcal{L}[q(t),\dot{q}(t)]\right)_\varepsilon. \quad (4.1.6)$$

This form will be the most convenient for defining a Colombeau-Stieltjes field theory action.In $D_{\mathfrak{t}}$ dimensions, we consider now the action

$$(\mathbf{S}_\varepsilon)_\varepsilon = \left(\int_M d\eta(x,\varepsilon)\mathcal{L}[\varphi_\varepsilon(x),\partial_\mu\varphi_\varepsilon(x)]\right)_\varepsilon, \quad (4.1.7)$$

where $\mathcal{L}[\varphi,\partial_\mu\varphi]$ is the Lagrangian density of the scalar field $(\varphi_\varepsilon(x))_\varepsilon$ and where

$$(d\eta(x,\varepsilon))_\varepsilon = \sum_{i=1}^m \prod_{\mu=0}^{D_{\mathfrak{t}}-1}\left(f_{\mu,i}(x,\varepsilon)\right)_\varepsilon dx^\mu, \left(f_{\mu,i}(x,\varepsilon)\right)_\varepsilon : M \to \widetilde{\mathbb{R}}, \quad (4.1.8)$$

is some Colombeau–Stieltjes measure. We denote with pair $(M,(d\eta(x,\varepsilon))_\varepsilon)$ the metric spacetime M equipped with Colombeau-Stieltjes measure $(d\eta(x,\varepsilon))_\varepsilon$. The former can be taken to be the canonical scalar field Lagrangian,

$$(\mathcal{L}[\varphi_\varepsilon(x),\partial_\mu\varphi_\varepsilon(x)])_\varepsilon = -\frac{1}{2}(\partial_\mu\varphi_\varepsilon\partial^\mu\varphi_\varepsilon)_\varepsilon - (V(\varphi_\varepsilon))_\varepsilon, \quad (4.1.9)$$

where $V(\varphi)$ is a potential and contraction of Lorentz indices is done via the Minkowski metric $\eta_{\mu\nu} = (-,+,\ldots+)_{\mu\nu}$. As for the Colombeau-Stieltjes measure, we make the multifractal spacetime isotropic choice

$$\left(f_{(\mu,i),\varepsilon}\right)_\varepsilon = (f_{i,\varepsilon})_\varepsilon, \mu = 1,\ldots,D_{\mathbf{t}}-1; i = 1,\ldots,m. \tag{4.1.10}$$

Hence the scalar field action (4.1.7) reads

$$\begin{gathered}(\mathbf{S}_\varepsilon)_\varepsilon = \left(\int_M d\eta(x,\varepsilon)\mathcal{L}[\varphi_\varepsilon(x),\partial_\mu\varphi_\varepsilon(x)]\right)_\varepsilon = \\ \sum_{j=1}^m \left(\int d^{D_{\mathbf{t}}}x v_{\varepsilon,j}(x)\left[\frac{1}{2}\partial_\mu\varphi_\varepsilon\partial^\mu\varphi_\varepsilon + V(\varphi_\varepsilon)\right]\right)_\varepsilon,\end{gathered} \tag{4.1.11}$$

where $(v_\varepsilon(x))_\varepsilon$ is a coordinate-dependent Lorentz scalar

$$(v_{\varepsilon,j}(x))_\varepsilon = \left(\frac{1}{[s_j(x)]^{D_{\mathbf{t}}(|\alpha-1|)}+\varepsilon}\right)_\varepsilon. \tag{4.1.12}$$

We define now the Dirac distribution as Colombeau generalized function by equation

$$\sum_{j=1}^m \left(\int d\eta_j(x,\varepsilon)\delta_{\{v_j\}}^{(D_{\mathbf{f},j})}(x,\varepsilon)\right)_\varepsilon = m. \tag{4.1.13}$$

In particular for for the case $m = 1$

$$\left(\int d\eta(x,\varepsilon)\delta_{\{v\}}^{(D_{\mathbf{f}})}(x,\varepsilon)\right)_\varepsilon = 1. \tag{4.1.14}$$

Invariance of the action under the infinitesimal shift $\varphi(x) \to \varphi(x) + \delta\varphi(x)$ gives the equation of motion for a generic weight $(v_{i,\varepsilon})_\varepsilon, i = 1,\ldots,m$:

$$\left(\frac{\partial\mathcal{L}}{\partial\varphi_\varepsilon}\right)_\varepsilon - \sum_{i=1}^m \left(\left[\left(\frac{\partial_\mu v_{i,\varepsilon}}{v_{i,\varepsilon}}\right) + \frac{d}{dx^\mu}\right]\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\varepsilon)}\right)_\varepsilon = 0. \tag{4.1.15}$$

In particular for for the case $m = 1$ we obtain

$$\left(\frac{\partial\mathcal{L}}{\partial\varphi_\varepsilon}\right)_\varepsilon - \left(\left[\frac{\partial_\mu v_\varepsilon}{v_\varepsilon} + \frac{d}{dx^\mu}\right]\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\varepsilon)}\right)_\varepsilon = 0. \tag{4.1.16}$$

From Eq.(4.1.11) and Eq.(4.1.15) we obtain

$$(\Box\varphi_\varepsilon)_\varepsilon + \sum_{i=1}^m \left\{\left(\left[\frac{\partial_\mu v_{i,\varepsilon}}{v_{i,\varepsilon}}\right]\partial^\mu\varphi_\varepsilon\right)_\varepsilon\right\} - \left(\frac{d}{d\varphi_\varepsilon}V(\varphi_\varepsilon)\right)_\varepsilon = 0. \tag{4.1.17}$$

where $\Box = \partial_\mu\partial^\mu$. In particular for for the case $m = 1$ we obtain

$$(\Box\varphi_\varepsilon)_\varepsilon + \left(\left[\frac{\partial_\mu v_\varepsilon}{v_\varepsilon}\right]\partial^\mu\varphi_\varepsilon\right)_\varepsilon - \left(\frac{d}{d\varphi_\varepsilon}V(\varphi_\varepsilon)\right)_\varepsilon = 0. \tag{4.1.18}$$

IV.2.Propagator in configuration space with negative-dimensions.

We define the canonical vacuum-to-vacuum amplitude by

$$(Z[J,\varepsilon])_\varepsilon = \left(\int \mathbf{D}\varphi_\varepsilon \exp\left[i\sum_{j=1}^m \int d\eta_{j,\varepsilon}(\mathcal{L}+\varphi_\varepsilon J)\right]\right)_\varepsilon, \tag{4.2.1}$$

where J is a source. Integration by parts in the exponent leads to the Lagrangian density for a free field as

$$(\mathcal{L}_\varepsilon)_\varepsilon = \frac{1}{2}\left(\varphi_\varepsilon\left(\square + \sum_{j=1}^m \frac{\partial_\mu v_{j,\varepsilon}}{v_{j,\varepsilon}}\partial^\mu - m^2\right)\varphi_\varepsilon\right)_\varepsilon = \frac{1}{2}(\varphi_\varepsilon \mathfrak{I}_\varepsilon \varphi_\varepsilon)_\varepsilon, \tag{4.2.2}$$

where

$$\mathfrak{I}_\varepsilon = \square + \sum_{j=1}^m \frac{\partial_\mu v_{j,\varepsilon}}{v_{j,\varepsilon}}\partial^\mu - m^2; j = 1,\ldots,m. \tag{4.2.3}$$

In particular for for the case $m = 1$ we obtain

$$\mathfrak{I}_\varepsilon = \square + \frac{\partial_\mu v_\varepsilon}{v_\varepsilon}\partial^\mu - m^2. \tag{4.2.4}$$

The propagator is the Green function $(G_\varepsilon(x))_\varepsilon$ solving the equation

$$(\mathfrak{I}_\varepsilon G_\varepsilon(x))_\varepsilon = (\delta_\nu^{D^-}(x,\varepsilon))_\varepsilon, \tag{4.2.5}$$

where $D^- = D_{\mathbf{t}}(\alpha - 1) < 0$. By virtue of Lorentz covariance, the Green function $G_\varepsilon(x)$ must depend only on the Lorentz interval $s^2 = x_\mu x^\mu = x_i x^i - t^2$, where $x^0 = t$ and $i = 1,\ldots,D_{\mathbf{t}} - 1$. In particular, $(v_\varepsilon)_\varepsilon = (v_\varepsilon(s(x)))_\varepsilon$ with the correct scaling property is

$$(v_\varepsilon(s(x)))_\varepsilon = \left(\left(|s(x)|^{|D^-|} + \varepsilon\right)^{-1}\right)_\varepsilon, s(x) = \sqrt{x_\mu x^\mu}. \tag{4.2.6}$$

Note that

$$\partial_\mu = \frac{x_\mu}{(s+\varepsilon)_\varepsilon}\partial_s, \square = \partial_s^2 + \frac{D_{\mathbf{t}} - 1}{(s+\varepsilon)_\varepsilon}\partial_s. \tag{4.2.7}$$

Hence the inhomogeneous equation (4.2.5) reads

$$\left(\partial_s^2 + \frac{D_{\mathbf{t}}\alpha - 1}{(s+\varepsilon)_\varepsilon}\partial_s - m^2\right)(G_\varepsilon(x))_\varepsilon = (\delta_\nu^{D^-}(x,\varepsilon))_\varepsilon. \tag{4.2.8}$$

We first consider the Euclidean propagator and denote with $r = \sqrt{x_i x^i + t^2}$ the Wick-rotated Lorentz invariant. In the massless case, the solution of the homogeneous equations for any $\alpha < 0$ is

$$(G_\varepsilon(r))_\varepsilon = Cr^{2\beta}, \beta = \tfrac{2 + D_{\mathbf{t}}|\alpha|}{2}. \tag{4.2.9}$$

Let us now consider the massive case.The solution of the homogeneous equation $(\mathfrak{I}_\varepsilon G_\varepsilon(r))_\varepsilon = 0$ for any $\alpha < 0$ is

$$(G_\varepsilon(r))_\varepsilon = \left(\frac{r}{m}\right)^{\frac{2+D_{\mathbf{t}}\cdot|\alpha|}{2}}\left[C_1\mathbf{K}_{-\frac{2+D_{\mathbf{t}}\cdot|\alpha|}{2}}(mr) + C_2\mathbf{I}_{-\frac{2+D_{\mathbf{t}}\cdot|\alpha|}{2}}(mr)\right], \tag{4.2.10}$$

where C_1, C_2 are constants and $\mathbf{K}_\lambda$ and $\mathbf{I}_\lambda$ are the modified Bessel functions. The function $\mathbf{I}_\nu(z)$ is

$$\mathbf{I}_\nu(z) = \sum_{k=0}^\infty \frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}. \tag{4.2.11}$$

Formula (4.2.11) is valid providing $\nu \neq -1,-2,-3,\ldots$.

$$\mathbf{I}_{-|\nu|}(z) = \sum_{k=0}^\infty \frac{(z/2)^{-|\nu|+2k}}{k!\Gamma(-|\nu|+k+1)} \tag{4.2.12}$$

Formula (4.2.12) is obtained by replacing ν in (4.2.13) with a $-\nu$.

$$\mathbf{K}_{-|\nu|}(z) = -\frac{\pi}{2\sin|\nu|\pi}[\mathbf{I}_{|\nu|}(z) - \mathbf{I}_{-|\nu|}(z)]. \tag{4.2.13}$$

The modified Bessel functions $\mathbf{I}_{-|\nu|}(z)$ and $\mathbf{K}_{-|\nu|}(z)$ have the following asymptotic forms for $z \to 0$:

$$\mathbf{K}_{-|\nu|}(z) \simeq \frac{1}{2}\Gamma(-|\nu|)\left(\frac{2}{z}\right)^{-|\nu|}, \mathbf{I}_{-|\nu|}(z) \simeq \frac{1}{\Gamma(-|\nu|+1)}\left(\frac{z}{2}\right)^{-|\nu|},$$
$$\nu \neq -1,-2,-3,\ldots. \tag{4.2.14}$$

Since for small $m \simeq 0$ the solution must agree with the massless case (4.2.9), we can set $C_2 = 0$. To find the solution of the inhomogeneous equation, one exploits the fact that the mass term does not contribute near the origin. Expanding Eq. (4.2.10) at $mr \simeq 0$ when $\alpha < 0$ ($C_2 = 0$), we find

$$(G_\varepsilon(r))_\varepsilon = C_1 2^{-\frac{4+D_{\mathbf{t}}\cdot|\alpha_i|}{2}}\Gamma\left(-\frac{2+D_{\mathbf{t}}\cdot|\alpha_i|}{2}\right)(r^2)^{\frac{2+D_{\mathbf{t}}\cdot|\alpha_i|}{2}} \tag{4.2.15}$$

which must coincide with Eq.(5.3.17). This gives the coefficient C_1 and the propagator reads

$$G(r) = -\frac{1}{2\pi^{\frac{D_{\mathbf{t}}}{2}}}\frac{\Gamma\left(\frac{D_{\mathbf{t}}}{2}\right)}{\Gamma\left(-\frac{D_{\mathbf{t}}|\alpha|}{2}\right)}\left(\frac{m}{2r}\right)^{-\frac{2+D_{\mathbf{t},i}\cdot|\alpha_i|}{2}}\mathbf{K}_{-\frac{2+D_{\mathbf{t}}|\alpha|}{2}}(mr). \tag{4.2.16}$$

V.The solution cosmological constant problem

V.1.Einstein-Gliner-Zel'dovich vacuum with tiny Lorentz invariance violation.

We assume now that:

(i) Poincaré group of momentum space is deformed at some fundamental high-energy cutoff Λ_* [9],[10].

(ii) The canonical quadratic invariant $\|p\|^2 = \eta^{ab}p_a p_b$ collapses at high-energy cutoff Λ_* and being replaced by the non-quadratic invariant:

$$\|p\|^2 = \frac{\eta^{ab}p_a p_b}{(1+l_{\Lambda_*}p_0)}. \tag{5.1.1}$$

(iii) The canonical concept of Minkowski space-time collapses at a small distances $l_{\Lambda_*} = \Lambda_*^{-1}$ to fractal space-time with Hausdorff-Colombeau negative dimension and therefore the canonical Lebesgue measure d^4x being replaced by the Colombeau-Stieltjes measure

$$(d\eta(x,\varepsilon))_\varepsilon = (\nu_\varepsilon(s(x))d^4x)_\varepsilon, \tag{5.1.2}$$

where

$$(\nu_\varepsilon(s(x)))_\varepsilon = \left(\left(|s(x)|^{|D^-|}+\varepsilon\right)^{-1}\right)_\varepsilon,$$
$$s(x) = \sqrt{x_\mu x^\mu}, \tag{5.1.3}$$

see subsection IV.2.

(iv) The canonical concept of momentum space collapses at fundamental high-energy cutoff Λ_* to fractal momentum space with Hausdorff-Colombeau negative dimension and therefore the canonical Lebesgue measure $d^3\mathbf{k}$, where $\mathbf{k} = (k_x,k_y,k_z)$ being replaced by the Hausdorff-Colombeau measure

$$d^{D^+,D^-}\mathbf{k} \triangleq \frac{\Delta(D^-)d^{D^+}\mathbf{k}}{\left(|\mathbf{k}|^{|D^-|}+\varepsilon\right)_{\varepsilon}} = \frac{\Delta(D^+)\Delta(D^-)p^{D^+-1}dp}{(p^{|D^-|}+\varepsilon)_{\varepsilon}}, \quad (5.1.4)$$

where $\Delta(D^{\pm}) = \dfrac{2\pi^{D^{\pm}/2}}{\Gamma(D^{\pm}/2)}$ and $p = |\mathbf{k}| = \sqrt{k_x + k_y + k_z}$.

Remark 5.1.1. Note that the integral over measure $d^{D^+,D^-}\mathbf{k}$ is given by formula(3.3.16). Thus vacuum energy density $\varepsilon(D^+,D^-,\mu_{\mathbf{eff}},p_*)$ for free quantum fields is

$$\varepsilon(D^+,D^-,\mu_{\mathbf{eff}},p_*) = \varepsilon(\mu_{\mathbf{eff}}) + \varepsilon(\mu_{\mathbf{eff}},p_*) + \breve{\varepsilon}(D^+,D^-,\mu_{\mathbf{eff}},p_*). \quad (5.1.5)$$

Here the quantity $\varepsilon(\mu_{\mathbf{eff}})$ is given by formula

$$\varepsilon(\mu_{\mathbf{eff}}) = \frac{1}{2(2\pi\hbar)^3}\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int\limits_{\|\mathbf{k}\|\leq\mu} \sqrt{\mathbf{k}^2+\mu^2}\, d^3\mathbf{k} =$$
$$K\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int\limits_{p\leq\mu} \sqrt{p^2+\mu^2}\, p^2 dp = K\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int\limits_0^{\mu} \sqrt{p^2+\mu^2}\, p^2 dp \quad (5.1.6)$$

where $K = \dfrac{2\pi}{(2\pi\hbar)^3}, c = 1$. The quantity $\varepsilon(\mu_{\mathbf{eff}},p_*)$ is given by formula

$$\varepsilon(\mu_{\mathbf{eff}},p_*) = \frac{1}{2(2\pi\hbar)^3}\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int\limits_{\mu<\|\mathbf{k}\|<p_*} \sqrt{\mathbf{k}^2+\mu^2}\, d^3\mathbf{k} =$$
$$K\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int\limits_{\mu<\|\mathbf{k}\|<p_*} \sqrt{p^2+\mu^2}\, p^2 dp. \quad (5.1.7)$$

The quantity $\breve{\varepsilon}(D^+,D^-,\mu_{\mathbf{eff}},p_*)$ (since Eq.(1.1.18) holds) is given by formula

$$\breve{\varepsilon}(D^+,D^-,\mu_{\mathbf{eff}},p_*) =$$
$$K'\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int\limits_{\|\mathbf{k}\|\geq p_*} \left[\frac{\mu^2 l_{\Lambda_*}}{1-\mu^2 l_{\Lambda_*}^2} + \frac{1}{\sqrt{1-\mu^2 l_{\Lambda_*}^2}}\sqrt{\frac{\mu^4 l_{\Lambda_*}^2}{1-\mu^2 l_{\Lambda_*}^2} + (|\mathbf{k}|^2+\mu^2)}\right] d^{D^+,D^-}\mathbf{k}, \quad (5.1.8)$$

where $K' = \dfrac{1}{2(2\pi\hbar)^3}, c = 1$.

Remark 5.1.2. We assume now that $\mu^2 l_{\Lambda_*}^2 \ll 1, \mu^4 l_{\Lambda_*}^2 \ll 1$ and therefore from Eq.(5.1.8)
we obtain

$$\varepsilon(D^+,D^-,\mu_{\mathbf{eff}},p_*) =$$
$$K' l_{\Lambda}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu \int\limits_{\|\mathbf{k}\|\geq p_*} d^{3,D-}\mathbf{k} + K'\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int\limits_{\|\mathbf{k}\|\geq p_*} \sqrt{\mathbf{k}^2+\mu^2}\, d^{D^+,D^-}\mathbf{k}. \quad (5.1.9)$$

From Eq.(5.1.9) and Eq.(5.1.4) we obtain

$$
\begin{gathered}
\varepsilon(D^{+},D^{-},\mu_{\mathbf{eff}},p_{*}) = \\
K' l_{\Lambda} \int_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{2} d\mu \int\limits_{\|\mathbf{k}\|\geq p_{*}} d^{D^{+},D^{-}}\mathbf{k} + K' \int_{0}^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int\limits_{\|\mathbf{k}\|\geq p_{*}} \sqrt{\mathbf{k}^{2}+\mu^{2}}\, d^{D^{+},D^{-}}\mathbf{k} = \\
\left(K' l_{\Lambda}\Delta(D^{+})\Delta(D^{-}) \int_{0}^{\mu_{\mathbf{eff}}} f d\mu(\mu)\mu^{2}\right) \int_{p_{*}}^{\infty} \frac{p^{D^{+}-1}dp}{(p^{|D^{-}|}+\varepsilon)_{\varepsilon}} + \\
+K'\Delta(D^{+})\Delta(D^{-}) \int_{0}^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int_{p_{*}}^{\infty} \frac{\sqrt{p^{2}+\mu^{2}}\, p^{D^{+}-1}dp}{(p^{|D^{-}|}+\varepsilon)_{\varepsilon}} = \\
\left(K' l_{\Lambda}\Delta(D^{+})\Delta(D^{-}) \int_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{2} d\mu\right) \int_{p_{*}}^{\infty} p^{D^{-}+D^{+}-1}dp + \\
+K'\Delta(D^{+})\Delta(D^{-}) \int_{0}^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int_{p_{*}}^{\infty} \sqrt{p^{2}+\mu^{2}}\, p^{D^{-}+D^{+}-1}dp.
\end{gathered}
\tag{5.1.10}
$$

Remark 5.1.2.We assume now that:

$$
D^{-}+D^{+}+2 \leq -6. \tag{5.1.11}
$$

Note that

$$
\begin{gathered}
\int_{0}^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int_{p_{*}}^{\infty} \sqrt{p^{2}+\mu^{2}}\, p^{D^{-}+D^{+}-1}dp = \int_{0}^{\mu_{\mathbf{eff}}} d\mu f(\mu) \int_{p_{*}}^{\infty} \sqrt{1+\frac{\mu^{2}}{p^{2}}}\, p^{D^{-}+D^{+}}dp = \\
\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)d\mu \int_{p_{*}}^{\infty} p^{D^{-}+D^{+}}dp + \frac{1}{2}\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{2}d\mu \int_{p_{*}}^{\infty} p^{D^{-}+D^{+}-1}dp - \\
-\frac{1}{8}\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{4}d\mu \int_{p_{*}}^{\infty} p^{D^{-}+D^{+}-3}dp + O(p_{*}^{D^{-}+D^{+}-4}) = \\
\frac{p_{*}^{D^{-}+D^{+}+1}}{D^{-}+D^{+}+1}\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)d\mu + \frac{p_{*}^{D^{-}+D^{+}}}{2(D^{-}+D^{+})}\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{2}d\mu - \\
-\frac{p_{*}^{D^{-}+D^{+}-1}}{8(D^{-}+D^{+}-1)}\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{4}d\mu + O(p_{*}^{D^{-}+D^{+}-4}).
\end{gathered}
\tag{5.1.12}
$$

Thus finally we obtain

$$
\begin{gathered}
\varepsilon(D^{+},D^{-},\mu_{\mathbf{eff}},p_{*}) = \\
\frac{K' p_{*}^{D^{-}+D^{+}+1}}{D^{-}+D^{+}+1}\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)d\mu + \left([K' l_{\Lambda}\Delta(D^{+})\Delta(D^{-})+0.5]\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{2}d\mu\right)\frac{p_{*}^{D^{-}+D^{+}}}{D^{-}+D^{+}} - \\
-\frac{K' p_{*}^{D^{-}+D^{+}-2}}{8(D^{-}+D^{+}-1)}\int_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{4}d\mu + O(p_{*}^{D^{-}+D^{+}-4}).
\end{gathered}
\tag{5.1.13}
$$

Remark 5.1.3.Note that (see Eqs.(1.2.12)):

$$
\begin{gathered}
\widetilde{\varepsilon}(\mu_{\mathbf{eff}},p_{*}) = \varepsilon(\mu_{\mathbf{eff}}) + \varepsilon(\mu_{\mathbf{eff}},p_{*}) = \\
\frac{1}{4}p_{*}^{4}\int\limits_{0}^{\mu_{\mathbf{eff}}} f(\mu)d\mu + \frac{1}{4}p_{*}^{2}\int\limits_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{2}d\mu + \left(C_{1}-\frac{1}{8}\ln p_{*}\right)\int\limits_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{4}d\mu + \\
+\frac{1}{8}\int\limits_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{4}(\ln\mu)d\mu - \left(\frac{1}{p_{*}^{2}}\right)\frac{1}{32}\int\limits_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{6}d\mu + O\left(\int\limits_{0}^{\mu_{\mathbf{eff}}} f(\mu)\mu^{8}\right)p_{*}^{-5}.
\end{gathered}
\tag{5.1.14}
$$

From Eq.(5.1.5),Eq.(5.1.13) and Eq.(5.1.14) finally we obtain

$$\varepsilon(D^+,D^-,\mu_{\mathbf{eff}},p_*) = \varepsilon(\mu_{\mathbf{eff}}) + \varepsilon(\mu_{\mathbf{eff}},p_*) + \breve{\varepsilon}(D^+,D^-,\mu_{\mathbf{eff}},p_*) =$$
$$\frac{1}{4}p_*^4\int_0^{\mu_{\mathbf{eff}}} f(\mu)d\mu + \frac{1}{4}p_*^2\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu + \left(C_1 - \frac{1}{8}\ln p_*\right)\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu +$$
$$+\frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu - \left(\frac{1}{p_*^2}\right)\frac{1}{32}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^6 d\mu + O\left(\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^8\right)p_*^{-5} + \tag{5.1.15}$$
$$+O(p_*^{D^-+D^++2}).$$

The pressure $p(D^+,D^-,\mu_{\mathbf{eff}},p_*)$ for free scalar quantum field is

$$p(D^+,D^-,\mu_{\mathbf{eff}},p_*) = p(\mu_{\mathbf{eff}}) + p(\mu_{\mathbf{eff}},p_*) + \breve{p}(D^+,D^-,\mu_{\mathbf{eff}},p_*). \tag{5.1.16}$$

Here the quantity $p(\mu_{\mathbf{eff}})$ is given by formula

$$p(\mu_{\mathbf{eff}}) = \frac{K}{3}\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu)\int_{\|p\|<\mu}\frac{p^4}{\sqrt{p^2+\mu^2}}dp. \tag{5.1.17}$$

The quantity $p(\mu_{\mathbf{eff}},p_*)$ is given by formula

$$p(\mu_{\mathbf{eff}},p_*) = \frac{K}{3}\int_0^{\mu_{\mathbf{eff}}} d\mu f(\mu)\int_{\mu\leq\|p\|\leq p_*}\frac{p^4}{\sqrt{p^2+\mu^2}}dp. \tag{5.1.18}$$

The quantity $\breve{p}(D^+,D^-,\mu_{\mathbf{eff}},p_*)$ is given by formula

$$\breve{p}(D^+,D^-,\mu_{\mathbf{eff}},p_*) \simeq \frac{K'}{3}\int_0^{\mu_{\mathbf{eff}}} d\mu\int_{\|p\|>p_*} f(\mu)\frac{p^4}{\sqrt{p^2+\mu^2}}dp, \tag{5.1.19}$$

where $K' = \dfrac{1}{2(2\pi\hbar)^3}, c = 1$.

Remark 5.1.4.Note that (see Eqs.(1.2.12)):

$$\widetilde{p}(\mu_{\mathbf{eff}},p_*) = p(\mu_{\mathbf{eff}}) + p(\mu_{\mathbf{eff}},p_*) =$$
$$\frac{1}{12}p_*^4\int_0^{\mu_{\mathbf{eff}}} f(\mu)d\mu - \frac{1}{12}p_*^2\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu + \left(C_2 + \frac{1}{8}\ln p_*\right)\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu - \tag{5.1.20}$$
$$-\frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + \left(\frac{5}{p_*^2}\right)\frac{1}{32}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^6 d\mu + O\left(\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^8\right)p_*^{-5}.$$

From Eq.(5.1.15),Eq.(5.1.19) and Eq.(5.1.20) similarly as above finally we get

$$p(D^+,D^-,\mu_{\mathbf{eff}},p_*) =$$
$$\frac{1}{12}p_*^4\int_0^{\mu_{\mathbf{eff}}} f(\mu)d\mu - \frac{1}{12}p_*^2\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu + \left(C_2 + \frac{1}{8}\ln p_*\right)\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu -$$
$$-\frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + \left(\frac{5}{p_*^2}\right)\frac{1}{32}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^6 d\mu + O\left(\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^8\right)p_*^{-5} + \tag{5.1.21}$$
$$+O(p_*^{D^-+D^++2}).$$

Remark 5.1.5.We assume now that:

$$\int_0^{\mu_{\mathbf{eff}}} f(\mu)d\mu = \int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu = \int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu = 0. \qquad (5.1.22)$$

From Eq.(5.1.15),Eq.(5.1.21) and Eq.(5.1.22) finally we get

$$\begin{aligned} \varepsilon &\triangleq \varepsilon(D^+, D^-, \mu_{\mathbf{eff}}, p_*) = \frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + O(p_*^{-2}), \\ p &\triangleq (D^+, D^-, \mu_{\mathbf{eff}}, p_*) = -\frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + O(p_*^{-2}). \end{aligned} \qquad (5.1.23)$$

Remark 5.1.5. Note that the Eq.(5.1.23) can be obtained without fine-tuning

(5.1.22) which was ussumed in Zel'dovich paper [1].
In order to obtain Eq.(5.1.23) ander strictly weaker conditions we assume now that:
(i)

$$|f(\mu)| = |f_{s.m.}(\mu) + f_{g.m.}(\mu)| = \mu_{\mathbf{eff}}^{-n}, \qquad (5.1.24)$$

where $n > 0$ is an parametr, $f_{s.m.}(\mu)$ corresponds to standard matter and where $f_{g.m.}(\mu)$ corresponds to physical ghost matter,see Eq.(1.2.2).
(ii)

$$I_1 = p_*^4 \int_0^{\mu_{\mathbf{eff}}} f(\mu)d\mu \approx 0, I_2 = p_*^2 \int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu \approx 0, I_3 = \ln p_* \int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu \approx 0 \qquad (5.1.25)$$

(iii)

$$\left| I_1 + I_2 + I_3 \right| \ll \left| \int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu \right|. \qquad (5.1.26)$$

V.2. Zeropoint energy density corresponding to a non-singular Gliner cosmology.

We assume now that

$$\begin{gathered} \int_0^{\mu_{\mathbf{eff}}} f(\mu)d\mu = 0, \int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu < 0, \int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu > 0, \\ p_* \gg \mu_{\mathbf{eff}}. \end{gathered} \qquad (5.2.1)$$

From Eq.(5.1.15),Eq.(5.1.21) and (5.2.1) we obtain

$$\begin{gathered}\varepsilon \triangleq \varepsilon(D^+,D^-,\mu_{\mathbf{eff}},p_*) = \\ \frac{1}{4}p_*^2\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu - \left(C_1 - \frac{1}{8}\ln p_*\right)\left|\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu\right| + \\ +\frac{1}{8}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu - \left(\frac{1}{p_*^2}\right)\frac{1}{32}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^6 d\mu + O\left(\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^8\right)p_*^{-5} + \\ +O(p_*^{D^-+D^++2}),\end{gathered} \tag{5.2.2}$$

and

$$\begin{gathered}p \triangleq p(D^+,D^-,\mu_{\mathbf{eff}},p_*) = \\ -\frac{1}{12}p_*^2\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu - \left(C_2 + \frac{1}{8}\ln p_*\right)\left|\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu\right| - \\ -\frac{1}{8}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + \left(\frac{5}{p_*^2}\right)\frac{1}{32}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^6 d\mu + O\left(\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^8\right)p_*^{-5} + \\ +O(p_*^{D^-+D^++2})\end{gathered} \tag{5.2.3}$$

correspondingly.From Eq.(5.2.2) and Eq.(5.2.3) we obtain

$$\begin{gathered}3p+\varepsilon = \\ -\frac{1}{4}p_*^2\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu - \left(3C_2 + \frac{3}{8}\ln p_*\right)\left|\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu\right| - \\ -\frac{3}{8}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + \left(\frac{5}{p_*^2}\right)\frac{3}{32}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^6 d\mu + \\ \frac{1}{4}p_*^2\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^2 d\mu - \left(C_1 - \frac{1}{8}\ln p_*\right)\left|\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu\right| + \\ +\frac{1}{8}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu - \left(\frac{1}{p_*^2}\right)\frac{1}{32}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^6 d\mu = \\ -\frac{1}{4}\ln p_*\left|\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu\right| - (3C_2 + C_1)\left|\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4 d\mu\right| - \frac{1}{4}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + \\ +\left(\frac{5}{p_*^2}\right)\frac{1}{16}\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^6 d\mu < 0.\end{gathered} \tag{5.2.4}$$

Therefore under conditions (5.2.1) the inequality

$$-2\varepsilon < 3p+\varepsilon < 0 \tag{5.2.5}$$

corresponding to Gliner non-singular cosmology [2],[4] is satisfied.

V.3. Zeropoint energy density in models with supermassive

physical ghost fields.

We assume now that:

(i) ghost fields corresponding to massive spin-2 particle with mass m_2 and to massive scalar particle with mass m_0 appears (see subsection II.2) as real physical fields in action ()

Remark 5.3.1. Note that their unphysical behavior may be restricted to arbitrarily high-energy cutoff Λ by an appropriate limitation on the renormalized masses m_2 and m_0.

Actually, it is only the massive spin-two excitations of the field which give the problem with unitarity and thus require a very large mass (see subsection II.2).

(ii) Poincaré group is deformed at some fundamental high-energy cutoff Λ_*

$$\Lambda_* = \Lambda_*(m_0, m_2) \ll m_0 c^2 < m_2 c^2. \tag{5.3.1}$$

The canonical quadratic invariant $\|p\|^2 = \eta^{ab} p_a p_b$ collapses at high-energy cutoff Λ_* and being replaced by the non-quadratic invariant:

$$\|p\|^2 = \frac{\eta^{ab} p_a p_b}{(1 + l_{\Lambda_*} p_0)}. \tag{5.3.2}$$

(iii) The canonical concept of Minkowski space-time collapses at a small distances to fractal space-time with Hausdorff-Colombeau negative dimension and therefore the canonical Lebesgue measure d^4x being replaced by the Colombeau-Stieltjes measure

$$(d\eta(x,\varepsilon))_\varepsilon = (v_\varepsilon(s(x)) d^4x)_\varepsilon, \tag{5.3.3}$$

where

$$(v_\varepsilon(s(x)))_\varepsilon = \left(\left(|s(x)|^{|D^-|} + \varepsilon\right)^{-1}\right)_\varepsilon, s(x) = \sqrt{x_\mu x^\mu}, \tag{5.3.4}$$

(iv) we assume that

$$f(\mu) = f_{s.m.}(\mu) + f_{g.m.}(\mu), \tag{5.3.5}$$

where $f_{s.m.}(\mu)$ corresponds to standard matter and where $f_{g.m.}(\mu)$ corresponds to physical ghost matter.

Remark 5.3.2. We assume now that

$$|f(\mu)| = \begin{cases} O(\mu^{-n}), n > 1 & m_0 c \ll \mu^1_{\mathbf{eff}} \leq \mu \leq \mu^2_{\mathbf{eff}} \ll m_2 c \\ 0 & \mu^1_{\mathbf{eff}} > \mu > \mu^2_{\mathbf{eff}} \end{cases} \tag{5.3.6}$$

Thus vacuum energy density $\varepsilon\left(D^+, D^-, \mu^1_{\mathbf{eff}}, \mu^2_{\mathbf{eff}}\right)$ for free quantum fields is

$$\varepsilon\left(D^+, D^-, \mu^1_{\mathbf{eff}}, \mu^2_{\mathbf{eff}}\right) = \varepsilon\left(\mu^1_{\mathbf{eff}}, \mu^2_{\mathbf{eff}},\right) + \check{\varepsilon}\left(D^+, D^-, \mu^1_{\mathbf{eff}}, \mu^2_{\mathbf{eff}}\right). \tag{5.3.7}$$

Here the quantity $\varepsilon\left(\mu^1_{\mathbf{eff}}, \mu^2_{\mathbf{eff}},\right)$ is given by formula

$$\varepsilon\left(\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right) = \frac{1}{2(2\pi\hbar)^3}\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\int\limits_{\|\mathbf{k}\|\leq\sqrt{\mu}}\sqrt{\mathbf{k}^2+\mu^2}\,d^3\mathbf{k} =$$
$$= K\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\int\limits_{p\leq\sqrt{\mu}}\sqrt{p^2+\mu^2}\,p^2dp, \quad (5.3.8)$$

where $K = \frac{2\pi}{(2\pi\hbar)^3}, c = 1$. The quantity $\breve{\varepsilon}\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right)$ is given by formula

$$\breve{\varepsilon}\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right) =$$
$$K'\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\int\limits_{\|\mathbf{k}\|>\sqrt{\mu}}\left[\frac{\mu^2 l_\Lambda}{1-\mu^2 l^2_\Lambda}+\frac{1}{\sqrt{1-\mu^2 l^2_{\Lambda_*}}}\sqrt{\frac{\mu^4 l^2_{\Lambda_*}}{1-\mu^2 l^2_{\Lambda_*}}+(|\mathbf{k}|^2+\mu^2)}\right]d^{D^+,D^-}\mathbf{k}, \quad (5.3.9)$$

where $K' = \frac{1}{2(2\pi\hbar)^3}, c = 1$.

Remark 5.3.2. We assume now that $\mu^2 l^2_{\Lambda_*} < 1$, and therefore from Eq.(5.3.9) we obtain

$$\varepsilon\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right) \simeq$$
$$K' l_\Lambda\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\mu^2\int\limits_{\|\mathbf{k}\|>\sqrt{\mu}} d^{3,D_-}\mathbf{k} + K'\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\int\limits_{\|\mathbf{k}\|>\sqrt{\mu}}\sqrt{\mathbf{k}^2+\mu^2}\,d^{D^+,D^-}\mathbf{k}. \quad (5.3.10)$$

From Eq.(5.3.10) and Eq.(5.1.4) we obtain

$$\varepsilon\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right) \simeq$$
$$K' l_\Lambda\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\mu^2\int\limits_{\|\mathbf{k}\|>\sqrt{\mu}} d^{D^+,D^-}\mathbf{k} + K'\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\int\limits_{\|\mathbf{k}\|>\sqrt{\mu}}\sqrt{\mathbf{k}^2+\mu^2}\,d^{D^+,D^-}\mathbf{k} =$$
$$K'\Delta(D^+)\Delta(D^-)l_\Lambda\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\mu^2\left[\int_{\sqrt{\mu}}^{\infty}\frac{p^{D^+-1}dp}{\left(p^{|D^-|}+\varepsilon\right)_\varepsilon}\right]+$$
$$+K'\Delta(D^+)\Delta(D^-)\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\left[\int_{\sqrt{\mu}}^{\infty}\frac{\sqrt{p^2+\mu^2}\,p^{D^+-1}dp}{\left(p^{|D^-|}+\varepsilon\right)_\varepsilon}\right] = \quad (5.3.11)$$
$$K'\Delta(D^+)\Delta(D^-)l_\Lambda\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\mu^2\left[\int_{\sqrt{\mu}}^{\infty}p^{D^-+D^+-1}dp\right]+$$
$$+K'\Delta(D^+)\Delta(D^-)\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\left[\int_{\sqrt{\mu}}^{\infty}\sqrt{p^2+\mu^2}\,p^{D^-+D^+-1}dp\right].$$

Note that

$$\sqrt{p^2+\mu^2} = \mu\sqrt{1+\frac{p^2}{\mu^2}} = \mu\left(1+\frac{1}{2}\frac{p^2}{\mu^2}-\frac{1}{8}\frac{p^4}{\mu^4}+\frac{1}{16}\frac{p^6}{\mu^6}+\ldots\right) =$$
$$= \mu+\frac{1}{2}\frac{p^2}{\mu}-\frac{1}{8}\frac{p^4}{\mu^3}+\frac{1}{16}\frac{p^6}{\mu^5}+\ldots \quad (5.3.12)$$

By inserting the Eq.(5.3.12) into the Eq.(5.3.8) we get

$$
\begin{aligned}
&\varepsilon\left(\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right) = \\
&K\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu) \int\limits_{p\leq\sqrt{\mu}} \left(\mu+\frac{1}{2}\frac{p^2}{\mu}-\frac{1}{8}\frac{p^4}{\mu^3}+\frac{1}{16}\frac{p^6}{\mu^5}+\ldots\right)p^2dp = \\
&K\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\left[\int_0^{\sqrt{\mu}}\left(\mu p^2+\frac{1}{2}\frac{p^4}{\mu}-\frac{1}{8}\frac{p^6}{\mu^3}+\frac{1}{16}\frac{p^8}{\mu^5}+\ldots\right)dp\right] = \\
&K\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\left[\mu\frac{p^3}{3}+\frac{1}{2}\frac{p^5}{5\mu}-\frac{1}{8}\frac{p^7}{7\mu^3}+\frac{1}{16}\frac{p^9}{9\mu^5}+\ldots\right]_0^{\sqrt{\mu}} = \qquad (5.3.13)\\
&K\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\left[\mu\frac{\mu^{\frac{3}{2}}}{3}+\frac{1}{2}\frac{\mu^{\frac{5}{2}}}{5\mu}-\frac{1}{8}\frac{\mu^{\frac{1}{2}}}{7\mu^3}+\frac{1}{16}\frac{\mu^{\frac{9}{2}}}{9\mu^5}+\ldots\right] = \\
&K\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} f(\mu)d\mu\left[\frac{1}{3}\mu^{\frac{5}{2}}+\frac{1}{10}\mu^{\frac{3}{2}}-\frac{1}{56}\mu^{\frac{1}{2}}+\frac{1}{144}\mu^{-\frac{1}{2}}+\ldots\right] = \\
&K\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} f(\mu)d\mu\left[\frac{1}{3}\mu^{\frac{5}{2}}+\frac{1}{10}\mu^{\frac{3}{2}}-\frac{1}{56}\mu^{\frac{1}{2}}+\frac{1}{144}\mu^{-\frac{1}{2}}\right]+o\left(\left(\mu^1_{\mathbf{eff}}\right)^{-n+1/2}\right).
\end{aligned}
$$

The pressure $p\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right)$ for free quantum fields is

$$
p\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right) = p\left(\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}},\right)+\breve{p}\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right). \qquad (5.3.14)
$$

Here the quantity $p\left(\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}},\right)$ is given by formula

$$
\begin{aligned}
p\left(\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right) &= \frac{1}{2(2\pi\hbar)^3}\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\int\limits_{\|\mathbf{k}\|\leq\sqrt{\mu}}\frac{\|\mathbf{k}\|^2}{\sqrt{\mathbf{k}^2+\mu^2}}d^3\mathbf{k} = \\
&= \frac{K}{3}\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\int\limits_{p\leq\sqrt{\mu}}\frac{p^4}{\sqrt{p^2+\mu^2}}dp.
\end{aligned} \qquad (5.3.15)
$$

The quantity $\breve{p}\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right)$ is given by formula

$$
\breve{p}\left(D^+,D^-,\mu^1_{\mathbf{eff}},\mu^2_{\mathbf{eff}}\right) \simeq \frac{K'}{3}\int_{\mu^1_{\mathbf{eff}}}^{\mu^2_{\mathbf{eff}}} d\mu f(\mu)\int\limits_{\|p\|>\sqrt{\mu}}\frac{\|\mathbf{k}\|^2}{\sqrt{\mathbf{k}^2+\mu^2}}d^{D^+,D^-}\mathbf{k}, \qquad (5.3.16)
$$

where $K' = \frac{1}{2(2\pi\hbar)^3}, c = 1$.Note that

$$
\begin{aligned}
&\frac{1}{\sqrt{p^2+\mu^2}} = \mu^{-1}\left(\sqrt{1+\frac{p^2}{\mu^2}}\right)^{-1} = \\
&\mu^{-1}\left(1-\frac{1}{2}\frac{p^2}{\mu^2}+\frac{3}{8}\frac{p^4}{\mu^4}-\frac{5}{16}\frac{p^6}{\mu^6}+\ldots\right) = \qquad (5.3.17)\\
&= \frac{1}{\mu}-\frac{1}{2}\frac{p^2}{\mu^3}+\frac{3}{8}\frac{p^4}{\mu^5}-\frac{5}{16}\frac{p^6}{\mu^7}+\ldots
\end{aligned}
$$

By inserting Eq.(5.3.17) into Eq.(5.3.15) we get

$$
\begin{aligned}
&p\left(\mu_{\mathbf{eff}}^{1},\mu_{\mathbf{eff}}^{2}\right)=\\
&\frac{K}{3}\int_{\mu_{\mathbf{eff}}^{1}}^{\mu_{\mathbf{eff}}^{2}}d\mu f(\mu)\int\limits_{p\leq\sqrt{\mu}}\left[\frac{1}{\mu}-\frac{1}{2}\frac{p^{2}}{\mu^{3}}+\frac{3}{8}\frac{p^{4}}{\mu^{5}}-\frac{5}{16}\frac{p^{6}}{\mu^{7}}+\ldots\right]p^{4}dp=\\
&\frac{K}{3}\int_{\mu_{\mathbf{eff}}^{1}}^{\mu_{\mathbf{eff}}^{2}}d\mu f(\mu)\int\limits_{p\leq\sqrt{\mu}}\left[\frac{p^{4}}{\mu}-\frac{1}{2}\frac{p^{6}}{\mu^{3}}+\frac{3}{8}\frac{p^{8}}{\mu^{5}}-\frac{5}{16}\frac{p^{10}}{\mu^{7}}+\ldots\right]dp=\\
&\frac{K}{3}\int_{\mu_{\mathbf{eff}}^{1}}^{\mu_{\mathbf{eff}}^{2}}d\mu f(\mu)\left[\frac{p^{5}}{5\mu}-\frac{1}{2}\frac{p^{7}}{7\mu^{3}}+\frac{3}{8}\frac{p^{9}}{9\mu^{5}}-\frac{5}{16}\frac{p^{11}}{10\mu^{7}}+\ldots\right]_{0}^{\sqrt{\mu}}= \qquad (5.3.18)\\
&\frac{K}{3}\int_{\mu_{\mathbf{eff}}^{1}}^{\mu_{\mathbf{eff}}^{2}}d\mu f(\mu)\left[\frac{\mu^{\frac{5}{2}}}{5\mu}-\frac{1}{2}\frac{\mu^{\frac{7}{2}}}{7\mu^{3}}+\frac{3}{8}\frac{\mu^{\frac{9}{2}}}{9\mu^{5}}-\frac{5}{16}\frac{\mu^{\frac{11}{2}}}{10\mu^{7}}+\ldots\right]=\\
&\frac{K}{3}\int_{\mu_{\mathbf{eff}}^{1}}^{\mu_{\mathbf{eff}}^{2}}d\mu f(\mu)\left[\frac{1}{5}\mu^{\frac{3}{2}}-\frac{1}{14}\mu^{\frac{1}{2}}+\frac{1}{24}\mu^{-\frac{1}{2}}-\frac{1}{32}\mu^{-\frac{3}{2}}+\ldots\right]=\\
&\frac{K}{3}\int_{\mu_{\mathbf{eff}}^{1}}^{\mu_{\mathbf{eff}}^{2}}d\mu f(\mu)\left[\frac{1}{5}\mu^{\frac{3}{2}}-\frac{1}{14}\mu^{\frac{1}{2}}+\frac{1}{24}\mu^{-\frac{1}{2}}-\frac{1}{32}\mu^{-\frac{3}{2}}\right]+o\left(\left(\mu_{\mathbf{eff}}^{1}\right)^{-n-1/2}\right).
\end{aligned}
$$

VI.Discussion and conclusion

We will now briefly review the canonical assumptions that are made in the usual formulation of the cosmological constant problem.

The canonical assumptions:

1.**The physical dark matter**.

Dark matter is a hypothetical form of matter that is thought to account for approximately 85% of the matter in the universe, and about a quarter of its total energy density. The majority of dark matter is thought to be non-baryonic in nature, possibly being composed of some as-yet undiscovered subatomic particles.Its presence is implied in a variety of astrophysical observations, including gravitational effects that cannot be explained unless more matter is present than can be seen. For this reason, most experts think dark matter to be ubiquitous in the universe and to have had a strong influence on its structure and evolution. The name dark matter refers to the fact that it does not appear to interact with observable electromagnetic radiation, such as light, and is thus invisible (or ’dark’) to the entire electromagnetic spectrum, making it extremely difficult to detect using usual astronomical equipment.Because dark matter has not yet been observed directly, it must barely interact with ordinary baryonic matter and radiation. The primary candidate for dark matter is some new kind of elementary particle that has not yet been discovered, in particular, weakly-interacting massive particles (WIMPs), or gravitationally-interacting massive particles (GIMPs).Many experiments to directly detect and study dark matter particles are being actively undertaken, but none has yet succeeded.

2.The total effective cosmological constant $\lambda_{\mathbf{eff}}$ is on at least the order of magnitude of the vacuum energy density generated by zero-point fluctuations of the standard particle fields.

3.Canonical QFT is an effective field theory description of a more fundamental theory, which becomes significant at some high-energy scale Λ_*.

4.The vacuum energy-momentum tensor is Lorentz invariant.

5.The Moller-Rosenfeld approach [35],[36] to semiclassical gravity by using an expectation value for the energy-momentum tensor is sound.

6.The Einstein equations for the homogeneous Friedmann-Robertson-Walker metric accurately describes the large-scale evolution of the Universe.

Remark 6.**1**.**1**.Note that obviously there is a strong inconsistency between Assumptions **2** and **3**: the vacuum state cannot be Lorentz invariant if modes are ignored above some high-energy cutoff Λ_*, because a mode that is high energy in one reference frame will be low energy in another appropriately boosted frame. In this paper Assumption 3 is not used and this contradiction is avoided.

Remark 6.**1**.**2**.Note that also, Assumptions 1, 3,4 and 5 is modifed, which we denote as Assumptions 4 and 5 respectively

Modified assumptions

1$'$.The physical dark matter.

2$'$.The total effective cosmological constant $\lambda_{\mathbf{eff}}$ is on at least the order $|\mu_{\mathbf{eff}}|^{-n+5}\ln|\mu_{\mathbf{eff}}|$ of magnitude of the*renormalized* vacuum energy density generated by zero-point fluctuations of standard particle fields and ghost particle fields,see subsection I.2.

4$'$.The vacuum energy-momentum tensor is not Lorentz invariant.

VI.1.The physical ghost matter and dark matter nature

In the contemporary quantum field theory, a ghost field, or gauge ghost is an unphysical state in a gauge theory. Ghosts are necessary to keep gauge invariance in theories where the local fields exceed a number of physical degrees of freedom.For example in quantum electrodynamics, in order to maintain manifest Lorentz invariance, one uses a four component vector potential $A_\mu(x)$, whereas the photon has only two polarizations. Thus, one needs a suitable mechanism in order to get rid of the unphysical degrees of freedom. Introducing fictitious fields, the ghosts, is one way of achieving this goal. Faddeev-Popov ghosts are extraneous fields which are introduced to maintain the consistency of the path integral formulation. Faddeev-Popov ghosts are sometimes referred to as "good ghosts".

"Bad ghosts" represent another, more general meaning of the word "ghost" in theoretical physics: states of negative norm,or fields with the wrong sign of the kinetic term, such as Pauli-Villars ghosts, whose existence allows the probabilities to be negative thus violating unitarity.

(**VI**.**1**) In contrary with standard Assumption1 in the case of the new approach introduced in this paper we assume that:

(**VI**.**1**.**1**.a) The ghosts fields and ghosts particles with masses at a scale less then an fixed scale $m_{\mathbf{eff}}$ really exist in the universe and formed dark matter sector of the universe,in particular:

(**VI**.**1**.**1**.b) these ghosts fields gives additive contribution to a full zero-point fluctuation (i.e. also to effective cosmological constant $\lambda_{\mathbf{eff}}$ [5],see subsection I.2).

(**VI**.**1**.**1**.c) Pauli-Villars renormalization of zero-point fluctuations (see subsection I.2) is no longer considered as an intermediate mathematical construct but obviously has rigorous physical meaning supported by assumption (**I**.a-b).

(**VI**.**1**.**2**) The physical dark matter formed by ghosts particles;

(**VI**.**1**.**3**) The standard model fields do not to couple directly to the ghost sector in the ultraviolet region of energy at a scale less then an fixed large energy scale Λ_*,in particular:

(**VI**.**1**.**3**.a) The "bad" ghosts fields with masses at a scale less then an fixed scale $m_{\mathbf{eff}}$, where $m_{\mathbf{eff}}c^2 \ll \Lambda_*$,cannot appear in any effective physycal lagrangian which contain also the standard particles fields.

In additional though not necessary we assume that:

(**VI**.**1**.**4**) The "bad" ghosts fields with masses at a scale m_*,where $m_*c^2 \gg \Lambda_*$can appear in any effective physycal lagrangian which contain also the standard particles fields,in particular:

(**VI**.**1**.**4**.a) Pauli-Villars finite renormalization with masses of ghosts fields at a scale m_* of the S-matrix in QFT (see subsection II.I-2) is no longer considered as an intermediate mathematical construct but obviously has rigorous physical meaning supported by assumption (**IV**).

(**VI**.**1**.**4**.b) If the "bad" ghosts fields coupled to matter directly, it gives rise to small and controlable violetion of the unitarity condition.

Remark VI.**1**.**3**.We emphazize that in universe standard matter coupled with a *physical*

ghost matter has the equation of state [3]:

$$\varepsilon_{\mathbf{vac}}(\mu_{\mathbf{eff}}) = -p(\mu_{\mathbf{eff}}) = \frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu = \frac{c^4\lambda_{\mathbf{vac}}}{8\pi G}, \tag{6.1.1}$$

where

$$|f(\mu)| = \begin{cases} O(\mu^{-n}), n > 1 & \mu \leq \mu_{\mathbf{eff}} \\ 0 & \mu > \mu_{\mathbf{eff}} \end{cases} \tag{6.1.2}$$

and where $\mu_{\mathbf{eff}} = m_{\mathbf{eff}}c$ (see subsection I.2,Eq.(1.2.16)) and therefore gives rise to a de Sitter phase of the universe even if bare cosmological constant $\lambda = 0$.

VI.2.Different contributions to $\lambda_{\mathbf{eff}}$

The total effective cosmological constant $\lambda_{\mathbf{eff}}$ is on at least the order of magnitude of the vacuum energy density generated by zero-point fluctuations of *standard* particle fields.

Assumption 2 is well justified in the case of the traditional approach, because the contribution from zero-point fluctuations is on the order of 1 in Planck units and no other known contributions are as large thus, assuming no significant cancellation of terms (e.g. fine tuning of the bare cosmological constant λ), the total $\lambda_{\mathbf{eff}}$ should be at least on the order of the largest contribution [14].

(**VI.2**) In contrary with standard Assumption1 in the case of the new approach introduced

in this paper we assume that:

(**VI.2.1**) For simplisity though not necessary bare cosmological constant $\lambda = 0$.

(**VI.2.2**) The total effective cosmological constant $\lambda_{\mathbf{eff}}$ depend only on mass distribution $f(\mu)$ and constant $\mu_{\mathbf{eff}} = m_{\mathbf{eff}}c$ but cannot depend on large energy scale $\sim \Lambda_*$

Remark VI.2.1.Note that in subsection we pointed out that under Assumption **VI.1** if bare

cosmological constant $\lambda = 0$ the total cosmological constant $\lambda_{\mathbf{vac}}$ is on at least the order

$|\mu_{\mathbf{eff}}|^{-n+5}$ of magnitude of the *renormalized* vacuum energy density

generated by zero-point fluctuations of standard particle fields and ghost particle fields

$$\begin{gathered} \varepsilon_{\mathbf{vac}}(\mu_{\mathbf{eff}}) = \frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + O(\Lambda_*^{-2}), \\ p_{\mathbf{vac}}(\mu_{\mathbf{eff}}) = -\frac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\mu^4(\ln\mu)d\mu + O(\Lambda_*^{-2}). \end{gathered} \tag{6.2.1}$$

VI.3. Effective field theory and Lorentz invariance violetion

To prevent the vacuum energy density from diverging,the traditional approach also assumes that performing a high-energy cutoff is acceptable. This type of regularization is a common step in renormalization procedures, which aim to eventually arrive at a physical, cutoff-independent result. However,in the case of the vacuum energy density, the result is inherently cutoff dependent, scaling quartically with the cutoff Λ_*.

Remark VI.3.1. By restricting to modes with particle energy a certain cutoff energy $\omega_{\mathbf{k}} \leq \Lambda_*$a finite, regularized result for the energy density can be obtained. The result is proportional to Λ_*^4.Any other fields will contribute similarly, so that if there are n_b bosonic fields and n_f fermionic fields, the density scales with $(n_b - 4n_f)\,\Lambda_*^4$. Typically, the cutoff is taken to be near $=1$ in Planck units (i.e.the Planck energy), so the vacuum energy gives a contribution to the cosmological constant on the order of at least unity according to Eq. (6.2.4). Thus we see the extreme ne-tuning problem: the original cosmological constant λ must cancel this large vacuum energy density $\varepsilon_{\mathbf{vac}} \simeq 1$ to a precision of 1 in 10^{120} -but not completely- to result in the observed value $\lambda_{\mathbf{eff}} = 10^{-120}$[5].

Remark VI.3.2. As it pointed out in this paper that a highenergy theory, i.e. QFT in fractal space-time with Hausdorff-Colombeau negative dimension would not display the zero-point fluctuations that are characteristic of QFT, and hence that the divergence caused by oscillations above the corresponding cutoff frequency is unphysical. In this case, the cutoff Λ_* is no longer an intermediate mathematical construct, but instead a physical scale at which the smooth,continuous behavior of QFT breaks down.

Poincaré group of the momentum space is deformed at some fundamental high-energy cutoff Λ_*The canonical quadratic invariant $\|p\|^2 = \eta^{ab}p_ap_b$ collapses at high-energy cutoff Λ_* and being replaced by the non-quadratic invariant:

$$\|p\|^2 = \frac{\eta^{ab}p_ap_b}{(1 + l_{\Lambda_*}p_0)}. \tag{6.3.1}$$

Remark VI.3.3. In contrary with canonical approuch the total effective cosmological constant $\lambda_{\mathbf{eff}}$ depend only on mass distribution $f(\mu)$ and constant $\mu_{\mathbf{eff}} = m_{\mathbf{eff}}c$ but cannot depend on large energy scale $\sim \Lambda_*$.

VI.4. Semiclassical Moller-Rosenfeld gravity

Assumption 5 means that it is valid to replace the right-hand side of the Einstein equation $T_{\mu\nu}$ with its expectation $\langle T_{\mu\nu}\rangle$. It requires that either gravity is not in fact quantum, and the Moller-Rosenfeld approach is a complete description of reality, or at least a valid approximation in the weak field limit.The usual argument states that the vacuum state $|0\rangle$ should be locally Lorentz invariant so that observers agree on the vacuum state. This means that the expectation value of the energy-momentum tensor on the vacuum, $\langle 0|\widehat{T}_{\mu\nu}|0\rangle$,must be a scalar multiple of the metric tensor $g_{\mu\nu}$ which is the only Lorentz invariant rank $(0,2)$ tensor. By using Moller-Rosenfeld approach the Einstein field equations of general relativity, a term representing the curvature of spacetime $R_{\mu\nu}$ is related to a term describing the energy-momentum of matter $\langle 0|\widehat{T}_{\mu\nu}|0\rangle$, as well as the cosmological constant λ and metric tensor $g_{\mu\nu}$ reads:

$$R_{\mu\nu} - \frac{1}{2}R_{\upsilon}^{\upsilon}g_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi\langle 0|\widehat{T}_{\mu\nu}|0\rangle. \quad (6.4.1)$$

The $\widehat{T}_{00}$ component is an energy density, we label $\langle 0|\widehat{T}_{\mu\nu}|0\rangle = \varepsilon_{\mathbf{vac}}$, so that the vacuum contribution to the right-hand side of Eq.(6.4.1) can be written as

$$8\pi\langle 0|\widehat{T}_{\mu\nu}|0\rangle = 8\pi\varepsilon_{\mathbf{vac}}g_{\mu\nu}. \quad (6.4.2)$$

Subtracting this from the right-hand side of Eq.(6.4.1) and grouping it with the cosmological constant term replaces with an "effective" cosmological constant [5]:

$$\lambda_{\mathbf{eff}} = \lambda + 8\pi\varepsilon_{\mathbf{vac}}. \quad (6.4.3)$$

Note that in flat spacetime, where $g_{\mu\nu} = diag(-1,+1,+1,+1)$, Eq.(6.4.2) implies $\varepsilon_{\mathbf{vac}} = -p_{\mathbf{vac}}$, where $p_{\mathbf{vac}} = \langle 0|\widehat{T}_{ii}|0\rangle$ for any $i = 1,2,3$ is the pressure. Obviously this implies that if the energy density is positive as is usually assumed, then the pressure must be negative, a conclusion which extends to any metric $g_{\mu\nu}$ with a $(-1,+1,+1,+1)$ signature.

Remark VI.4.1.In this paper we assume that the vacuum state $|0\rangle$ should be locally invariant under modified Lorentz boost (1.1.18) but not locally Lorentz invariant. Obviously this assumption violate the Eq.(6.4.2). However modified Lorentz boosts (1.1.18) becomes Lorentz boosts for a sufficiently small energies and therefore in IR region one obtain in a good aproximation

$$8\pi\langle 0|\widehat{T}_{\mu\nu}|0\rangle \approx 8\pi\varepsilon_{\mathbf{vac}}g_{\mu\nu} \quad (6.4.4)$$

and

$$\lambda_{\mathbf{eff}} \approx \lambda + 8\pi\varepsilon_{\mathbf{vac}}. \quad (6.4.5)$$

Thus Moller-Rosenfeld approach holds in a good approximation.

VI.5. Quantum gravity at energy scale $\Lambda \leq \Lambda_*$Controlable violetion of the unitarity condition.

Gravitational actions which include terms quadratic in the curvature tensor are renormalizable. The necessary Slavnov identities are derived from Becchi-Rouet-Stora (BRS) transformations of the gravitational and Faddeev-Popov ghost fields. In general, non-gauge-invariant divergences do arise, but they may be absorbed by nonlinear renormalizations of the gravitational and ghost fields and of the BRS transformations [13].The geneic expression of the action reads

$$I_{sym} = -\int d^4x\sqrt{-g}\,(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + 2\kappa^{-2}R), \quad (6.5.1)$$

where the curvature tensor and the Ricci is defined by $R^{\lambda}_{\mu\alpha\nu} = \partial_{\nu}\Gamma^{\lambda}_{\mu\alpha}$ and $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$ correspondingly, $\kappa^2 = 32\pi G$.The convenient definition of the gravitational field variable in terms of the contravariant metric density reads

$$\kappa h^{\mu\nu} = g^{\mu\nu}\sqrt{-g} - \eta^{\mu\nu}. \quad (6.5.2)$$

Analysis of the linearized radiation shows that there are eight dynamical degrees of freedom in the field. Two of these excitations correspond to the familiar massless spin-2 graviton. Five more correspond to a massive spin-2 particle with mass m_2. The eighth corresponds to a massive scalar particle with mass m_0. Although the linearized field energy of the massless spin-2 and massive scalar excitations is positive definite, the linearized energy of the massive spin-2 excitations is negative definite. This feature is characteristic of higher-derivative models, and poses the major obstacle to their physical interpretation.

In the quantum theory, there is an alternative problem which may be substituted for the negative energy. It is possible to recast the theory so that the massive spin-2 eigenstates of the free-fieid Hamiltonian have positive-definite energy, but also negative norm in the state vector space.These negative-norm states cannot be excluded from the physical sector of the vector space without destroying the unitarity of the $\mathbf{S}$ matrix. The requirement that the graviton propagator behave like p^{-4} for large momenta makes it necessary to choose the indefinite-metric vector space over the negative-energy states.The presence of massive quantum states of negative norm which cancel some of the divergences due to the massless states is analogous to the Pauli-Villars regularization of other field theories. For quantum gravity, however, the resulting improvement in the ultraviolet behavior of the theory is sufficient only to make it renormalizable,but not finite.

Remark 6.**5**.**1**.(**I**)The renormalizable models which we have considered in this paper many years mistakenly regarded only as constructs for a study of the ultraviolet problem of quantum gravity. The difficulties with unitarity appear to preclude their direct acceptability as canonical physical theories in locally Minkowski space-time. In canonical case they do have only some promise as phenomenological models.

(**II**) However, for their unphysical behavior may be restricted to *arbitrarily large energy scales* Λ_* mentioned above by an appropriate limitation on the renormalized masses m_2 and m_0. Actually, it is only the massive spin-two excitations of the field which give the trouble with unitarity and thus require a very large mass. The limit on the mass m_0 is determined only by the observational constraints on the static field.

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