

Observations of structure of a possible unification algebra

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Abstract. A C-loop algebra, designated \mathbb{U} is assembled as the product: $M_4(C) \otimes \mathbb{T}$. When $M_4(C)$ is assigned to represent $Cl_{1,3}(R) \otimes \mathbb{C}$ and the principle of spatial equivalence is invoked, a sub-algebra designated \mathbb{W} is found to have features that suggest it could provide an underlying basis for the standard model of fundamental particles. \mathbb{U} is of the same order as $Cl_{0,10}(R)$, but has a “natural” partition into $Cl_{1,3}(R) \otimes \mathbb{C} \otimes \mathbb{W}$, suggesting that its use in string/M theories in the place of $Cl_{0,10}(R)$ may generate a description of reality.

1. Introduction

This paper describes algebraic structures using labels for unit elements which highlight features related to spatial equivalence, as set out in section 2.

Sections 3 to 6 document the algebraic structures. They have been investigated by using the multiplication tables for unit elements combined with random coefficients to generate random products, which are then used to check the properties of the algebras, testing for distributivity, associativity, flexibility, alternativity and power associativity. The Loops package[1] for GAP4[2] has also been used.

In sections 3 to 6, observations and postulates are presented relating the algebraic structures to physics. The author has limited understanding of subjects such as torsion, manifolds and fiber bundles, so the postulates are speculative, but they demonstrate the potential for the algebraic structures to provide a basis for the unification of general relativity with quantum mechanics.

Sections 7 to 10 present details relating to the assembly of the algebraic structures and their properties.

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2. Notation for algebras \mathbb{T} , \mathbb{M} , \mathbb{U} , \mathbb{W} and \mathbb{D}

Labels for algebras and their unit elements are based on patterns related to spatial equivalence when a sub-algebra is used to represent the space-time Clifford algebra.

Unit elements for $\mathbb{M} \cong M_4(C)$ are represented by:

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
S	L	M	N	V	D	E	F	iU	iX	iY	iZ	iT	iP	iQ	iR
e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}	e_{27}	e_{28}	e_{29}	e_{30}	e_{31}
iS	iL	iM	iN	iV	iD	iE	iF	U	X	Y	Z	T	P	Q	R

Unit elements for the trigintaduonion algebra, \mathbb{T} , are represented by:

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
σ_o	σ_ι	σ_j	σ_κ	λ_o	λ_ι	λ_j	λ_κ	μ_o	μ_ι	μ_j	μ_κ	ν_o	ν_ι	ν_j	ν_κ
e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}	e_{27}	e_{28}	e_{29}	e_{30}	e_{31}
α_o	α_ι	α_j	α_κ	β_o	β_ι	β_j	β_κ	γ_o	γ_ι	γ_j	γ_κ	δ_o	δ_ι	δ_j	δ_κ

Cayley tables for these algebras are shown in sections 8.1 and 9.3. They have been arranged so that, if the signs of products are ignored, they form the same latin square. This is referred to as “alignment” of the algebras. Note that the subscripts ι, j, κ identify orientation with respect to iX, iY, iZ for the alignment.

An algebra labeled \mathbb{U} is generated as the tensor product $\mathbb{T} \otimes \mathbb{M}$. For \mathbb{U} , all unit elements of its sub-algebra, \mathbb{T} , commute with all unit elements of its sub-algebra, \mathbb{M} . Unit elements of \mathbb{U} are labeled using combinations of the labels assigned to \mathbb{T} and \mathbb{M} , such as $\nu_\kappa iR$. The labels are listed in Section 10.

A further sub-algebra of \mathbb{U} , labeled \mathbb{W} , is identified which has a Cayley table, shown in section 3.4.2, which is aligned with those of \mathbb{T} and \mathbb{M} . It is a “resonant” subalgebra of \mathbb{U} , where “resonance” is defined as meeting a requirement for spatial equivalence for all sub-algebras when three unit elements of \mathbb{M} are used to represent unit spatial vector elements for a Clifford algebra.

Unit elements for \mathbb{W} are represented by:

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
$\sigma_o S$	$\sigma_\iota L$	$\sigma_j M$	$\sigma_\kappa N$	$\lambda_o V$	$\lambda_\iota D$	$\lambda_j E$	$\lambda_\kappa F$	$\mu_o iU$	$\mu_\iota iX$	$\mu_j iY$	$\mu_\kappa iZ$	$\nu_o iT$	$\nu_\iota iP$	$\nu_j iQ$	$\nu_\kappa iR$
e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}	e_{27}	e_{28}	e_{29}	e_{30}	e_{31}
$\alpha_o iS$	$\alpha_\iota iL$	$\alpha_j iM$	$\alpha_\kappa iN$	$\beta_o iV$	$\beta_\iota iD$	$\beta_j iE$	$\beta_\kappa iF$	$\gamma_o U$	$\gamma_\iota X$	$\gamma_j Y$	$\gamma_\kappa Z$	$\delta_o T$	$\delta_\iota P$	$\delta_j Q$	$\delta_\kappa R$

A label, \mathbb{D} is used for algebras isomorphic to the algebra of real 4×4 diagonal matrices.

3. Algebraic structures

3.1. Structure of $\mathbb{M} \cong M_4(C)$

The structure of \mathbb{M} is well known.

3.1.1. Embedded group and automorphism group. \mathbb{M} contains an embedded group, the Dirac matrix group, of order 64, generated by its 32 basis elements. The automorphism group for the Dirac matrix group, as determined using the Loops package for GAP4, has 42 conjugacy classes, and its structure description is:
 $((C2 \times C2 \times C2 \times C2) : A6) : (C2 \times C2)$

3.1.2. Sub-groups of order 32. The Dirac matrix group has 31 sub-groups of order 32, which can be sorted into 3 classes, as shown in table 10 in section 7.

3.1.3. Sub-groups of order 16. The Dirac matrix group has 155 sub-groups of order 16. There are five types of these sub-groups which differ in the signature of their unit components or in their commutation properties. Once unit matrices for \mathbb{M} are assigned to represent complexified unit multivector elements for a space-time Clifford algebra, further distinctions are found which identify sub-groups based on their relationship to space-like and time-like vectors. These sub-groups have been labeled as shown in table 2 in section 7.

3.1.4. Sub-groups of order 8. The Dirac matrix group has 155 sub-groups of order 8. There are 80 non-abelian sub-groups. There are 60 abelian sub-groups which exclude the unit imaginary, and 15 abelian sub-groups which include the unit imaginary. Once unit matrices for \mathbb{M} are assigned to represent complexified unit multivector elements for a space-time Clifford algebra, further distinctions are found which identify sub-groups based on their relationship to space-like and time-like vectors. These sub-groups are shown in tables 4-6 in section 7.

3.1.5. Graded algebras. \mathbb{M} can be used to represent multivectors for graded polar vector algebras such as the Clifford algebras $Cl_4(C)$, $Cl_{0,5}(R)$, $Cl_{2,3}(R)$, $Cl_{1,4}(R)$.

3.2. Structure of \mathbb{T}

The structure of the trigintaduonion algebra, \mathbb{T} , has been described by Cawagas et al[3], but that description does not detail the differing ways in which lower order subalgebras participate in sedenion-type subalgebras. These details are shown in tables 3-9 in section 7.

3.2.1. Embedded loop and automorphism group. \mathbb{T} contains an embedded loop T_L of order 64 generated by its 32 basis elements. The automorphism group for T_L , as determined using the Loops package[1] for GAP4[2], has 42 conjugacy classes, and its structure description is:

$$C_2 \times C_2 \times ((C_2 \times C_2 \times C_2) . PSL(3,2))$$

3.2.2. Sub-loops of order 32. T_L has 31 sedenion-type subloops of \mathbb{T} of order 32, falling into four isomorphism classes, which Cawagas et al. designated $S_L, S_L^\alpha, S_L^\beta, S_L^\gamma$, as shown in table 10 in section 7.

3.2.3. Sub-loops of order 16. T_L has 155 octonion-type sub-loops of order 16, falling into two isomorphism classes: octonion loops which Cawagas et al. designated O_L and quasi-octonion loops which they designated \tilde{O}_L . These octonion-type sub-loops of T_L participate in sedenion-type subloops of T_L in a variety of ways, as shown in table 3 in section 7.

3.2.4. Sub-loops of order 8. T_L has 155 quaternionic subloops of order 8, falling into one isomorphism class which Cawagas et al. designated Q_8 . These quaternionic subloops of T_L participate in sedenion-type subloops of T_L in a variety of ways, as shown in tables 4-6 in section 7.

3.2.5. Graded algebras. It is postulated that \mathbb{T} can be used to represent graded multivectors for axial vector algebras, which can be associated with graded multivectors for polar vector algebras represented by Clifford algebras such as $Cl_4(C)$, $Cl_{0,5}(R)$, $Cl_{2,3}(R)$, $Cl_{1,4}(R)$.

3.3. Structure of $\mathbb{U} \cong \mathbb{M} \otimes \mathbb{T}$

3.3.1. Embedded loop and automorphism group. \mathbb{U} contains an embedded loop U_L of order 2048 generated by its 1024 basis elements. The automorphism group for U_L , as determined using the Loops package for GAP4, is:

$C2xC2x(((C2xC2xC2xC2):A6):(C2xC2))x((C2xC2xC2).PSL(3,2))).$

3.3.2. Associative subalgebras of \mathbb{U} . \mathbb{M} is associative. \mathbb{T} is di-associative, pairs of its imaginary unit elements generate sub-algebras isomorphic to \mathbb{H} . As noted by Cawagas et al, there are 155 of these sub-algebras. So, for \mathbb{U} , there are 155 associative sub-algebras isomorphic to $\mathbb{H} \otimes \mathbb{M}$.

3.3.3. Aligned sub-algebras of \mathbb{U} . The Cayley tables for \mathbb{M} and \mathbb{T} , as presented in tables 12 and 13 in sections 8 and 9, have been arranged so that, if the signs of products are ignored, they form the same latin square. Referring to this as “alignment”, there are many possible ways of aligning unit elements of the two algebras. For sub-algebras of \mathbb{M} and \mathbb{T} , paired combinations of unit elements from this alignment generate “aligned” sub-algebras of \mathbb{U} .

3.3.4. Resonant subalgebras of \mathbb{U} . For the alignment of table 12 with 13, unit elements of aligned sub-algebras are listed in tables 4-10 in section 7. It can be seen that all \mathbb{M} subgroups that are related by spatial rotation are aligned with \mathbb{T} subloops which have related patterns of participation in sedenion type subloops. This property is designated as a “resonance”. A sub-algebra generated using paired products for a resonant alignment, such as \mathbb{W} , is designated a “resonant” sub-algebra.

3.3.5. Partition of \mathbb{U} . For \mathbb{U} , all unit elements from \mathbb{W} together with the unit imaginary element from \mathbb{M} all commute with all unit elements from a subalgebra of \mathbb{M} isomorphic to $Cl_{1,3}(R)$. If \mathbb{U} is used as a basis for a total space for a manifold with fiber bundles, this suggests the choice of a $Cl_{1,3}(R)$ base manifold. For a complexified unit element of \mathbb{W} such as $\mu_\nu X + \mu_\nu iX$, the partial derivative with respect to a coordinate, x , associated with the unit vector, would be a complex function of the associated unit element, μ_ν , of \mathbb{T} . This suggests association of $\mathbb{C} \otimes \mathbb{W}$ with fiber bundles, as covariant derivatives of solder forms for tangent vectors define torsion on tangent frame bundles[4].

3.4. Structure of \mathbb{W} and $\mathbb{C} \otimes \mathbb{W}$

Each unit element of \mathbb{W} is a product of a unit element of \mathbb{M} with a unit element of \mathbb{T} . As all imaginary unit elements of \mathbb{T} square to -1 and anti-commute, unit elements of \mathbb{W} other than the identity have opposite signature and opposite commutation properties to the corresponding unit elements of \mathbb{M} . This relates the Lie bracket of products for \mathbb{W} to the Jordan brace of products for \mathbb{M} and vice-versa, a form of supersymmetry.

3.4.1. Embedded loop and automorphism group. \mathbb{W} contains an embedded loop W_L of order 64 generated by its 32 basis elements. The automorphism group for W_L , as determined using the Loops package for GAP4, has 40 conjugacy classes, and its structure description is:

$$(C2) \times (C2) \times (C2) \times (S4).$$

The automorphism group of its complexification has 80 conjugacy classes and its structure description is:

$$(C2) \times (C2) \times (C2) \times (C2) \times (S4).$$

3.4.2. Cayley table for \mathbb{W} basis elements. The Cayley table for \mathbb{W} basis elements (excluding negative elements) is shown in table 1.

TABLE 1. Cayley table for \mathbb{W} basis elements

Ref.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31			
σ_S	σ_L	σ_M	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	
σ_L	σ_S	σ_T	σ_M	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
σ_M	σ_S	σ_T	σ_L	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
σ_N	σ_S	σ_T	σ_L	σ_M	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
σ_V	σ_S	σ_T	σ_L	σ_M	σ_N	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
σ_D	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
σ_E	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_D	σ_F	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
σ_F	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_D	σ_E	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
$\sigma_{d,T}$	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
$\sigma_{d,X}$	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,Y}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
$\sigma_{d,Y}$	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,Z}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
$\sigma_{d,Z}$	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,Q}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
$\sigma_{d,Q}$	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,R}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Z}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$
$\sigma_{d,R}$	σ_S	σ_T	σ_L	σ_M	σ_N	σ_V	σ_D	σ_E	σ_F	$\sigma_{d,T}$	$\sigma_{d,Z}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$	$\sigma_{d,T}$	$\sigma_{d,X}$	$\sigma_{d,Y}$	$\sigma_{d,Q}$	$\sigma_{d,R}$	$\sigma_{d,S}$	$\sigma_{d,N}$	$\sigma_{d,V}$	$\sigma_{d,D}$	$\sigma_{d,E}$	$\sigma_{d,F}$		

3.4.3. Sub-algebras of \mathbb{W} and $\mathbb{C} \otimes \mathbb{W}$. The Cayley tables of \mathbb{M} , \mathbb{T} and \mathbb{W} have been configured in a resonant alignment, their sub-algebras are also aligned, and are configured for spatial equivalence when \mathbb{M} is used to represent $Cl_{1,3}(R) \otimes \mathbb{C}$. The sub-algebras of \mathbb{W} are in one-to-one correspondence with sub-algebras of \mathbb{M} and \mathbb{T} .

The 155 sub-algebras of \mathbb{W} with four unit elements, all of which are associative, are listed in tables 4-6 in section 7. 80 of them relate to non-abelian sub-algebras of \mathbb{M} and 75 to abelian sub-algebras of \mathbb{M} . The sub-algebras of \mathbb{W} with eight unit elements, none of which are associative, are listed in tables 7-9 in section 7.

Analysis of the 155 sub-algebras of \mathbb{W} with 8 unit elements reveals that 15 of them are isomorphic to the split octonions. The other 140 are not power associative. Its 15 sub-algebras isomorphic to the split octonions generate, when complexified, 15 sub-algebras of $\mathbb{C} \otimes \mathbb{W}$ isomorphic to $\mathbb{C} \otimes \mathbb{O}$. Kohl Furey has postulated that minimal left ideals of a $Cl_6(C)$ algebra extracted from $\mathbb{C} \otimes \mathbb{O}$ correspond to one family of fundamental particles[5][6], and refers to others who have advocated the existence of a connection between non-associative algebras and particle theory[7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25]. $\mathbb{U} \cong [\mathbb{C} \otimes \mathbb{W} \otimes Cl_{1,3}(R)]$ has, as sub-algebras, the algebras on which all of these approaches are based, suggesting that it has the potential to provide a basis for the standard model of fundamental particles.

It is postulated that these 15 $\mathbb{C} \otimes \mathbb{O}$ sub-algebras, correspond to three spatial orientations for five families of particles - three families of standard model fermions and two families of dark matter particles. These 15 sub-algebras have, as sub-algebras, all the sub-algebras generated by complexification of the 75 four element subalgebras of \mathbb{W} related to abelian sub-algebras of \mathbb{M} .

The other 80 sub-algebras of \mathbb{W} with four unit elements, when complexified, generate sub-algebras isomorphic to $\mathbb{C} \otimes \mathbb{D}$, the algebra of 4×4 complex diagonal matrices. These are commuting and associative. It is postulated that they are associated with vector bosons.

It is postulated that the sub-algebra of \mathbb{W} with unit elements $[\sigma_o S, \sigma_o iS, \alpha_o S, \alpha_o iS]$, a complex doublet, is associated with the Higgs mechanism[26].

3.4.4. Grading distinctions. For sub-algebras of $\mathbb{C} \otimes \mathbb{W}$ with fewer than 32 unit elements, there are distinctions between otherwise isomorphic sub-algebras arising from the patterns of participation of quaternionic and octonion-type sub-loops of T_L in $S_L, S_L^\alpha, S_L^\beta, S_L^\gamma$ sub-loops of T_L . Once \mathbb{M} is assigned to represent $Cl_{1,3}(R) \otimes \mathbb{C}$, further distinctions arise between otherwise isomorphic sub-algebras of $\mathbb{C} \otimes \mathbb{W}$. It is postulated that these distinctions generate the complexity of the standard model.

3.4.5. Matrix representation of $\mathbb{C} \otimes \mathbb{W}$. The notation used to generate \mathbb{T} features dis-association operators, symbols that define the products of unit elements of \mathbb{T} using a modified form of the usual Moufang loop construction for octonions, as setout in table 14 in section 9.2.

A general element of $\mathbb{C} \otimes \mathbb{W}$ can be represented using a complex 4×4 matrix with entries with dis-association operators and using coefficients a_-, b_-i, c_-, d_-i with subscripts which are lower case versions of the associated unit element from \mathbb{M} .

$$\left(\begin{array}{cccc} +a_s + b_s i + c_s \sigma_o + d_s i \alpha_o & +a_y + b_y i + c_y \lambda_o + d_y i \beta_o & +a_m + b_m i + c_m \sigma_j + d_m i \alpha_j & +a_e + b_e i + c_e \lambda_j + d_e i \beta_j \\ +a_q + b_q i + c_q \delta_j + d_q i \nu_j & +a_y + b_y i + c_y \gamma_j + d_y i \mu_j & +a_t + b_t i + c_t \delta_o + d_t i \nu_o & +a_u + b_u i + c_u \gamma_o + d_u i \mu_o \\ +a_d + b_d i + c_d \lambda_i + d_d i \beta_i & +a_l + b_l i + c_l \sigma_i + d_l i \alpha_i & +a_f + b_f i + c_f \lambda_\kappa + d_f i \beta_\kappa & +a_n + b_n i + c_n \sigma_\kappa + d_n i \alpha_\kappa \\ +a_z + b_z i + c_z \gamma_\kappa + d_z i \mu_\kappa & +a_r + b_r i + c_r \delta_\kappa + d_r i \nu_\kappa & +a_x + b_x i + c_x \gamma_i + d_x i \mu_i & +a_p + b_p i + c_p \delta_i + d_p i \nu_i \\ \\ -a_v - b_v i - c_v \lambda_o - d_v i \beta_o & +a_s + b_s i + c_s \sigma_o + d_s i \alpha_o & -a_e - b_e i - c_e \lambda_j - d_e i \beta_j & +a_m + b_m i + c_m \sigma_j + d_m i \alpha_j \\ +a_y + b_y i + c_y \gamma_j + d_y i \mu_j & -a_q - b_q i - c_q \delta_j - d_q i \nu_j & +a_u + b_u i + c_u \gamma_o + d_u i \mu_o & -a_t - b_t i - c_t \delta_o - d_t i \nu_o \\ -a_l - b_l i - c_l \sigma_i - d_l i \alpha_i & +a_d + b_d i + c_d \lambda_i + d_d i \beta_i & -a_n - b_n i - c_n \sigma_\kappa - d_n i \alpha_\kappa & +a_f + b_f i + c_f \lambda_\kappa + d_f i \beta_\kappa \\ +a_r + b_r i + c_r \delta_\kappa + d_r i \nu_\kappa & -a_z - b_z i - c_z \gamma_\kappa - d_z i \mu_\kappa & +a_p + b_p i + c_p \delta_i + d_p i \nu_i & -a_x - b_x i - c_x \gamma_i - d_x i \mu_i \\ \\ -a_m - b_m i - c_m \sigma_j - d_m i \alpha_j & -a_e - b_e i - c_e \lambda_j - d_e i \beta_j & +a_s + b_s i + c_s \sigma_o + d_s i \alpha_o & +a_v + b_v i + c_v \lambda_o + d_v i \beta_o \\ -a_t - b_t i - c_t \delta_o - d_t i \nu_o & -a_u - b_u i - c_u \gamma_o - d_u i \mu_o & +a_q + b_q i + c_q \delta_j + d_q i \nu_j & +a_y + b_y i + c_y \gamma_j + d_y i \mu_j \\ +a_f + b_f i + c_f \lambda_\kappa + d_f i \beta_\kappa & +a_n + b_n i + c_n \sigma_\kappa + d_n i \alpha_\kappa & -a_d - b_d i - c_d \lambda_i - d_d i \beta_i & -a_l - b_l i - c_l \sigma_i - d_l i \alpha_i \\ +a_x + b_x i + c_x \gamma_i + d_x i \mu_i & +a_p + b_p i + c_p \delta_i + d_p i \nu_i & -a_z - b_z i - c_z \gamma_\kappa - d_z i \mu_\kappa & -a_r - b_r i - c_r \delta_\kappa - d_r i \nu_\kappa \end{array} \right)$$

4. Numbers of physical dimensions

The quest for a theory of everything has generated models with various physical dimensionalities - five dimensions for the original Kaluza-Klein theory[27], ten for string theories[28], eleven for M-theory[29]. For nature to be embedded in dimensionalities such as four, five, ten or eleven, whether real or complex, appears arbitrary.

We observe three spatial dimensions and one temporal dimension, which sets a minimum for the number of physical dimensions. The original Kaluza-Klein theory added a further physical dimension and extended general relativity unifying it with classical electromagnetism, suggesting the existence of five physical dimensions.

The Big Bang comprises a transition from a singularity to a manifold expanding through time. The singularity can be regarded as a one dimensional amplitude for zero dimensionality. If that was an unstable configuration, so that a transition to finite amplitudes for more than one dimension was favoured, this raises the question - why a transition to three, four, five, ten or eleven dimensions?

If physical n-space is regarded as composed of an assembly of distorted n-spherical quanta, and if the degree of distortion required is related to the densest possible packing ratio, those densest possible packing ratios vary with dimensionality (n). Suppose that a singularity corresponds to stacking those quanta in an n-needle, one “above” the other. That stack also has a packing ratio.

The packing ratios for assemblies of spheres for continua of different dimensionalities are:

The area of a 2 dimensional circle is $\pi R^2 = 3.142R^2$.
The volume of a 3 dimensional 3-ball is $4\pi/3.R^3 = 4.189R^3$.
The volume of a 4 dimensional 4-ball is $\pi^2/2.R^4 = 4.935R^4$.
The volume of a 5 dimensional 5-ball is $8\pi^2/15.R^5 = 5.264R^5$.
The volume of a 6 dimensional 6-ball is $\pi^3/6.R^6 = 5.168R^6$.
The volume of a 7 dimensional 7-ball is $16\pi^3/105.R^7 = 4.725R^7$.
The volume of an 8 dimensional 8-ball is $\pi^4/24.R^8 = 4.059R^8$.
The volume of a 9 dimensional 9-ball is $32\pi^4/945.R^9 = 3.299R^9$.
The volume of a 10 dimensional 10-ball is $\pi^5/120.R^{10} = 2.550R^{10}$.

For an assembly of n-balls stacked into an n-needle of unit quantum radius, the packing fractions are:

For a stack of 2 dimensional unit circles on edge: $(\pi R^2)/(2R \times 2R) \rightarrow 78.6\%$
For a 3-needle of unit 3-balls: $(4\pi^3/3.R^3)/(\pi R^2 \times 2R) \rightarrow 66.6\%$
For a 4-needle of unit 4-balls: $(\pi^2/2.R^4)/(4\pi^3/3.R^3 \times 2R) \rightarrow 58.9\%$
For a 5-needle of unit 5-balls: $(8\pi^2/15.R^5)/((\pi^2/2.R^4) \times 2R) \rightarrow 53.3\%$
For a 6-needle of unit 6-balls: $(\pi^3/6.R^6)/(8\pi^2/15.R^5 \times 2R) \rightarrow 49.1\%$
For a 7-needle of unit 7-balls: $(16\pi^3/105.R^7)/(\pi^3/6.R^6 \times 2R) \rightarrow 45.7\%$
For an 8-needle of unit 8-balls: $(\pi^4/24.R^8)/(16\pi^3/105.R^7 \times 2R) \rightarrow 45.9\%$
For a 9-needle of unit 9-balls: $(32\pi^4/945.R^9)/(\pi^4/24.R^8 \times 2R) \rightarrow 34.9\%$
For a 10-needle of unit 10-balls: $(\pi^5/120.R^{10})/(32\pi^4/945.R^9 \times 2R) \rightarrow 38.6\%$

For euclidean n-space the densest packing fractions, as listed by Cohn and Elkies[30], are:

For 2 dimensional 2-balls (circles) of equal radius: 91%
For 3 dimensional 3-balls of equal radius: 74%
For 4 dimensional 4-balls of equal radius: in the range 61.7 to 64.8%
For 5 dimensional 5-balls of equal radius: in the range 46.5 to 52.5%
For 6 dimensional 6-balls of equal radius: in the range 37.3 to 41.8%
For 7 dimensional 7-balls of equal radius: in the range: 29.5 to 32.8%
For 8 dimensional 8-balls of equal radius: 25.4%
For 9 dimensional 9-balls of equal radius: in the range: 14.6 to 19.5%
For 10 dimensional 10-balls of equal radius: in the range: 10.0 to 14.9%

For n-balls arranged in a n-disc, that is an euclidean n-space extended in $n - 1$ dimensions and limited to unit quantum diameter in the nth dimension, the densest packing fractions are product of the densest packing fractions for the euclidean $(n - 1)$ space and the packing fraction for the n-needle of the same dimensionality. For instance, for three dimensions spheres would be packed into the disc in the densest packing of parallel cylinders in a plane.

For euclidean space the densest packing fractions for n-discs are:

For 2 dimensional 2-balls (circles) of equal radius: 71%

For 3 dimensional 3-balls of equal radius: 49%

For 4 dimensional 4-balls of equal radius: in the range 36 to 38%

For 5 dimensional 5-balls of equal radius: in the range 25 to 28%

For 6 dimensional 6-balls of equal radius: in the range 18 to 21%

For 7 dimensional 7-balls of equal radius: in the range: 13 to 15%

For 8 dimensional 8-balls of equal radius: 11.7%

For 9 dimensional 9-balls of equal radius: in the range: 5 to 7%

For 10 dimensional 10-balls of equal radius: in the range: 4 to 6%

In dimensionalities lower than 5 an euclidean n-space has a higher densest possible packing fraction than that of an n-needle and that of an n-disc. This suggests the hypothesis that, for a four dimensional manifold, a singularity would be unstable, tending to expand into a 4 sphere. If that expansion were to overshoot, becoming disc like, there would be a tendency for it to contract again.

However, space is not necessarily euclidean. Packing fractions for hyperbolic space are higher than for euclidean space, but are difficult to calculate. For 3 dimensions and 4 dimensions, densest packing fractions have been calculated as:

For 3 dimensional 3-balls of equal radius: 85.3%

For 4 dimensional 4-balls of equal radius: 71.6%

Compared to euclidean n-space, these figures are 15% and 13% higher respectively. Extrapolating to higher dimensions, for dimensionalities higher than 5, euclidean n-space would still have a lower packing fraction than an n-needle. However, in 5 dimensions, it is possible that for hyperbolic 5-space there could be a denser packing than for a 5-needle.

This analysis suggests that expansion of a singularity into four dimensions for euclidean space or five dimensions for hyperbolic space could be favoured.

5. Types of dimensions

The original Kaluza-Klein theory[27] accounted for electro-magnetism by introducing an additional dimension. This dimension is not observed, suggesting that it would differ from the observed spatial dimensions. Time is observed, but also differs from spatial dimensions. \mathbb{M} is isomorphic to the Clifford algebra $Cl_{0,5}(R)$ and to $Cl_{1,3}(R) \otimes \mathbb{C}$. $Cl_{0,5}(R)$ is generated using five polar vector unit elements with negative signature. It is postulated that four dimensions corresponds to the dimensions of space and imaginary time and the fifth dimension to the extra dimension for 5D Kaluza-Klein theory, and that conventional time is emergent. This suggests the concept of reality as a 3-dimensional wavefront distorted into a fourth dimension propagating in a fifth dimension.

String and M-theories[28][29] also postulate additional dimensions. \mathbb{U} is of the same order as $Cl_{0,10}(R)$, but has a “natural” partition into $Cl_{1,3}(R) \otimes \mathbb{C} \otimes \mathbb{W}$, suggesting that its use in string/M theories in the place of $Cl_{0,10}(R)$ may generate a description of reality. However, as \mathbb{U} may constitute a representation of a combination of a manifold with 5 polar vectors and 5 axial vectors, this suggests that reality may be embedded in five-dimensional space with torsion, rather than 10-dimensional space. General relativity[31] is usually formulated using the assumption that affine connection has a vanishing torsion tensor, but non-vanishing torsion has been proposed for Einstein-Cartan-Sciama-Kibble and other theories. In an overview[32], Tejinder Singh comments:

“Thus on the one hand we have the torsion-dominated limit, which are the Dirac equations, and on the other hand we have the gravity dominated limit, which are the Einstein equations. In the former case, gravity is absent (Minkowski space-time) and matter behaviour is quantum. In the latter case matter behaviour is classical, and gravity dominates over torsion. Thus we may conclude that there must be a more general underlying theory in which the torsion-free part and the torsion part of the spin-connection are both present, and to which GTR and quantum theory are both approximations.”

The number of degrees of freedom for torsion for a given dimensionality are limited. For a manifold of dimension $d = 5$ with a maximally symmetric submanifold of dimension $n = 4$, there are up to $1 + 4 + 6 = 11$ allowed torsion components which are in general functions of the fifth coordinate[33]. The Einstein field equations (for four dimensions) have 10 degrees of freedom, four of which are unphysical. This suggests a correlation between the 4 + 6 allowed torsion components and the degrees of freedom for physical space-time with imaginary time substituted for time to make the four dimensions symmetric.

6. The Higgs mechanism

A grand unification algebra should provide a basis for the Higgs mechanism[26]. The mexican hat potential is unusual. This section describes algebraic elements that could account for that potential.

The Higgs mechanism acts on a complex doublet and involves scalar fields. For \mathbb{U} a scalar subalgebra can be assembled as the product:

$$[\sigma_o S, \sigma_o iS, \alpha_o S, \alpha_o iS] \otimes [\sigma_o S, \sigma_o T, \sigma_o V, \sigma_o U] \otimes [\sigma_o S, \lambda_o S, \mu_o S, \nu_o S].$$

$[\sigma_o S, \sigma_o T, \sigma_o V, \sigma_o U] \otimes [\sigma_o S, \lambda_o S, \mu_o S, \nu_o S]$ is isomorphic to $\mathbb{H} \otimes \mathbb{H}$ and to $M_4(R)$.

Its unit elements can be assigned unit matrices from table 1 as follows:

$$\begin{aligned} [\sigma_o S] &= [S], [\sigma_o T, \sigma_o V, \sigma_o U] = [TVU], [\lambda_o S, \mu_o S, \nu_o S] = [LMN] \\ [\lambda_o T, \mu_o T, \nu_o T] &= [PQR], [\lambda_o V, \mu_o V, \nu_o V] = [DEF], [\lambda_o U, \mu_o U, \nu_o U] = [XYZ] \end{aligned}$$

Subalgebras of $M_4(R)$ for which the scalar component is associated with a mexican hat potential can be found by considering unitary abelian subgroups of $M_4(R)$. Unitary abelian subgroups of $M_4(R)$ can be represented by diagonal 4×4 matrices.

$$\begin{bmatrix} e^{i\theta_1} & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 \\ 0 & 0 & 0 & e^{i\theta_4} \end{bmatrix}$$

where $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$, allowing it to be rewritten:

$$\begin{bmatrix} e^{ia} & 0 & 0 & 0 \\ 0 & e^{ib} & 0 & 0 \\ 0 & 0 & e^{ic} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of two elements of this type with parameters a, b, c and a', b', c' has parameters $a + a', b + b', c + c'$. A subgroup of the Heisenberg group $H(5)$ shares this property:

$$\begin{bmatrix} 1 & a & b & c + ab \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has determinant = 1, and the commuting products of the form:

$$\begin{bmatrix} 1 & a + a' & b + b' & c + c' + (a + a') \times (b + b') \\ 0 & 1 & 0 & b + b' \\ 0 & 0 & 1 & a + a' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix can be written in terms of unit elements of $M_4(R)$ as:

$$[S] + a/2[V + Y] + b/2[M + F] + (c + ab)/4[E + U + N + P].$$

There are other combinations of unit elements of $M_4(C)$ with similar properties. To find these combinations, it is useful to arrange the matrices in an array with anticommuting basis matrices and the identity in each row/column, forming a 6×6 array:

$$\begin{bmatrix} S & V & T & X & Y & Z \\ V & S & U & P & Q & R \\ T & U & S & D & E & F \\ X & P & D & S & N & M \\ Y & Q & E & N & S & L \\ Z & R & F & M & L & S \end{bmatrix}$$

Interchanging rows and matching columns preserves commutation/anticommutation relationships and group properties with respect to position in the array. For example, rows and columns 1 and 2 can be interchanged to make the array:

$$\begin{bmatrix} S & V & U & P & Q & R \\ V & S & T & X & Y & Z \\ U & T & S & D & E & F \\ P & X & D & S & N & M \\ Q & Y & E & N & S & L \\ R & Z & F & M & L & S \end{bmatrix}$$

Inspecting this array to assign unit matrices for an equivalent H5 subgroup group, they would be:

$$[S] + a/2[V + Q] + b/2[M + F] + (c + ab)/4[E + N + T + X]$$

This combination has the same properties. Interchanging rows and columns 1 and 2 has not changed the signatures of the matrices allocated to each position.

If a further interchange is made that does affect the signatures, e.g. interchanging rows and columns 1 and 4, to generate:

$$\begin{bmatrix} S & Q & U & P & V & R \\ Q & S & E & N & Y & L \\ U & E & S & D & T & F \\ P & N & D & S & X & M \\ V & Y & E & N & S & L \\ R & L & F & M & Z & S \end{bmatrix}$$

For the combination:

$$[S] + a/2[Y + Q] + b/2[M + F] + (c + ab)/4[P + U + T + X]$$

The determinant is no longer 1. To make this combination generate a unitary matrix, a factor has to be applied to $[S]$. That factor is $\sqrt{(\pm 1 \pm 2(a/2)^2)}$, provided that the factor is real and not imaginary.

For the resulting matrix, there are four plus/minus permutations, for which the possible matrices for $[S]$ are:

$$\begin{bmatrix} \sqrt{(1+a^2/2)} & 0 & 0 & 0 \\ 0 & \sqrt{(1+a^2/2)} & 0 & 0 \\ 0 & 0 & \sqrt{(1+a^2/2)} & 0 \\ 0 & 0 & 0 & \sqrt{(1+a^2/2)} \end{bmatrix}$$

Which always has real entries, and determinant = $1 + a^2 + a^4/4$

$$\begin{bmatrix} \sqrt{(-1-a^2/2)} & 0 & 0 & 0 \\ 0 & \sqrt{(-1-a^2/2)} & 0 & 0 \\ 0 & 0 & \sqrt{(-1-a^2/2)} & 0 \\ 0 & 0 & 0 & \sqrt{(-1-a^2/2)} \end{bmatrix}$$

Which never has real entries, and determinant = $1 + a^2 + a^4/4$

$$\begin{bmatrix} \sqrt{(1-a^2/2)} & 0 & 0 & 0 \\ 0 & \sqrt{(1-a^2/2)} & 0 & 0 \\ 0 & 0 & \sqrt{(1-a^2/2)} & 0 \\ 0 & 0 & 0 & \sqrt{(1-a^2/2)} \end{bmatrix}$$

Which has real entries for $a^2/2 \leq 1$, and determinant = $1 - a^2 + a^4/4$

$$\begin{bmatrix} \sqrt{(-1+a^2/2)} & 0 & 0 & 0 \\ 0 & \sqrt{(-1+a^2/2)} & 0 & 0 \\ 0 & 0 & \sqrt{(-1+a^2/2)} & 0 \\ 0 & 0 & 0 & \sqrt{(-1+a^2/2)} \end{bmatrix}$$

Which has real entries for $a^2/2 \geq 1$, and determinant = $1 - a^2 + a^4/4$

The function $f(a) = 1 - a^2 + a^4/4$ has the form of a mexican hat potential.

For the assignment of unit elements of \mathbb{U} to matrices:

$$[\sigma_o S] = [S], [\sigma_o T, \sigma_o V, \sigma_o U] = [TVU], [\lambda_o S, \mu_o S, \nu_o S] = [LMN] \\ [\lambda_o T, \mu_o T, \nu_o T] = [PQR], [\lambda_o V, \mu_o V, \nu_o V] = [DEF], [\lambda_o U, \mu_o U, \nu_o U] = [XYZ]$$

The group represented by a plus/minus choice for:

$$\sqrt{(\pm 1 \pm a^2/2)[S] + a/2[Y+Q] + b/2[M+F] + (c+ab)/4[P+U+T+X]}$$

is isomorphic to that for the same plus/minus choice for:

$$\sqrt{(\pm 1 \pm a^2/2)[\sigma_o S] + a/2[\mu_o U + \mu_o T] + b/2[\mu_o S + \nu_o V] + (c+ab)/4[\lambda_o T + \sigma_o U + \sigma_o T + \lambda_o U]}$$

which can be rearranged into:

$$\sqrt{(\pm 1 \pm a^2/2)[\sigma_o S] + (U+T) \times [a/2\mu_o + (c+ab)/4\sigma + (c+ab)/4\lambda] + b/2[\mu_o S + \nu_o V]}$$

for which $[TVU]$ symmetry is broken.

7. Subalgebra tables

TABLE 2. Classification of sub-algebras with 8 unit elements of $\mathbb{W} \cong M_4(C)$ with respect to unit elements of a $Cl_{1,3}$ multivector

Type 1 subgroups having [+ - - + + -] signature						
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	Subtype
1a	SLMNDEFV	1a	SLiM <i>i</i> N <i>i</i> EiFV	1a	SLiM <i>i</i> N <i>i</i> DEiFV	1a
1b	SLMNPQRT	1b	SLiM <i>i</i> N <i>i</i> P <i>i</i> QiRT	1b	SLiM <i>i</i> N <i>i</i> P <i>i</i> QiRT	1b
1c	SLMNXYZU	1c	SLiM <i>i</i> N <i>i</i> X <i>i</i> Y <i>i</i> ZU	1c	SLiM <i>i</i> N <i>i</i> X <i>i</i> Y <i>i</i> ZU	1c
1d	SVTUDPXL	1d	SVTiUDPiX _L	1d	SVTiUDPiX _L	1d
1d	SVTUEQYM	1d	SVTiUiE <i>i</i> QYiM	1d	SVTiUiE <i>i</i> QYiM	1d
1d	SVTUFRZN	1d	SVTiUiF <i>i</i> R <i>i</i> ZN	1d	SVTiUiF <i>i</i> R <i>i</i> ZN	1d
1e	SLEiFURQIX	1e	SLE <i>i</i> F <i>i</i> URQIX	1e	SLEiFURQIX	1e
1e	SMiFDiUPRiY	1e	SMFD <i>i</i> UiP <i>i</i> RY	1e	SMiFDUP <i>i</i> RY	1e
1e	SNiDEiUQP <i>i</i> Z	1e	SNDE <i>i</i> UiQ <i>i</i> P <i>i</i> Z	1e	SiNiDEUQ <i>i</i> P <i>i</i> Z	1e
1f	SLiQ <i>i</i> RiVZYiD	1f	SLQR <i>i</i> ViZiYiD	1f	SLiQ <i>i</i> RVZiYiD	1f
1f	SMiRiPiVXZiE	1f	SMR <i>i</i> P <i>i</i> ViX <i>i</i> ZE	1f	SiMiRPVXiZE	1f
1f	SNiP <i>i</i> QVYiXF	1f	SNP <i>i</i> Q <i>i</i> VYiXiF	1f	SiNiP <i>i</i> QVYiXiF	1f
1g	SLiY <i>i</i> ZiTFe <i>i</i> P	1g	SLY <i>i</i> ZiT <i>i</i> Fe <i>i</i> P	1g	SiLiY <i>i</i> ZTFe <i>i</i> P	1g
1g	SMiZiXiTDFiQ	1g	SMZX <i>i</i> T <i>i</i> D <i>i</i> FiQ	1g	SiMiZXT <i>i</i> DiFiQ	1g
1g	SNiXY <i>i</i> YTEDiR	1g	SNXY <i>i</i> T <i>i</i> E <i>i</i> DiR	1g	SiNiXY <i>i</i> TDiER	1g
Type 2 subgroups having [+ - - - +] signature						
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	
2a	SLMN <i>i</i> DiE <i>i</i> FV					
2b	SLMN <i>i</i> P <i>i</i> QiR <i>i</i> T					
2c	SLMN <i>i</i> X <i>i</i> Y <i>i</i> ZU					
2d	SVTU <i>i</i> DiPiX _L					
2d	SVTU <i>i</i> E <i>i</i> QYiM					
2d	SVTU <i>i</i> F <i>i</i> R <i>i</i> ZN					
2e	SLiE <i>i</i> F <i>i</i> UiR <i>i</i> QX					
Type 3 subgroups having [+ - + + - + +] signature						
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	
3a	SLiM <i>i</i> N <i>i</i> DEFV	3a	SiLM <i>i</i> N <i>i</i> DEFV	3a	SiLiM <i>i</i> N <i>i</i> DEFV	
3b	SLiM <i>i</i> N <i>i</i> P <i>i</i> Q <i>i</i> R <i>i</i> T	3b	SiLM <i>i</i> N <i>i</i> P <i>i</i> Q <i>i</i> R <i>i</i> T	3b	SiLiM <i>i</i> N <i>i</i> P <i>i</i> Q <i>i</i> R <i>i</i> T	
3c	SLiM <i>i</i> N <i>i</i> X <i>i</i> Y <i>i</i> ZU	3c	SiLM <i>i</i> N <i>i</i> X <i>i</i> Y <i>i</i> ZU	3c	SiLiM <i>i</i> N <i>i</i> X <i>i</i> Y <i>i</i> ZU	
3d	SVTiUiDP <i>i</i> X _L	3d	SiVTiUDPiX _L	3d	SiViTUDPiX _L	
3d	SVTiUiE <i>i</i> QYiM	3d	SVTiUiE <i>i</i> QYiM	3d	SiViTUE <i>i</i> QYiM	
3d	SVTiUiF <i>i</i> R <i>i</i> ZN	3d	SVTiUiF <i>i</i> R <i>i</i> ZN	3d	SiViTUFR <i>i</i> ZN	
3e	SLEFU <i>i</i> RQX	3e	SiLiE <i>i</i> F <i>i</i> UiR <i>i</i> QX	3e	SiLEiFURQX	
3e	SMFDUPR <i>i</i>	3e	SiMiFD <i>i</i> UiPR <i>i</i>	3e	SiMFIDUP <i>i</i> RY	
3e	SNDEUQP <i>i</i> Z	3e	SiNiDE <i>i</i> UiQ <i>i</i> P <i>i</i> Z	3e	SiNDiE <i>i</i> UiQ <i>i</i> P <i>i</i> Z	
3f	SLiQ <i>i</i> RVZYD	3f	SiLiQ <i>i</i> R <i>i</i> ViZYD	3f	SiLiQ <i>i</i> RiVZYD	
3f	SMRPVXZE	3f	SiMiRP <i>i</i> ViXZE	3f	SiMiRP <i>i</i> ViXZE	
3f	SNP <i>i</i> QVYiXF	3f	SiNiP <i>i</i> QVYiXF	3f	SiNP <i>i</i> QVYiXF	
3g	SLYZTEFP	3g	SiLiY <i>i</i> ZiT <i>i</i> EF <i>i</i> P	3g	SiLiY <i>i</i> ZiT <i>i</i> EF <i>i</i> P	
3g	SMZXTFDQ	3g	SiMiZ <i>i</i> XiTiFDQ	3g	SiMiZXiTiFDQ	
3g	SNXYTDER	3g	SiNiXY <i>i</i> T <i>i</i> DER	3g	SiNiXY <i>i</i> T <i>i</i> DER	
Type 4 subgroups having [+ + + - - -] signature						
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	
4a	SIVDiLV <i>i</i> DL <i>i</i> S	4a	SiVE <i>i</i> M <i>i</i> ViE <i>i</i> S	4a	SIVF <i>i</i> INV <i>i</i> FN <i>i</i> S	
4b	SiT <i>i</i> P <i>i</i> T <i>i</i> PL <i>i</i> S	4b	SiT <i>i</i> Q <i>i</i> M <i>i</i> T <i>i</i> QM <i>i</i> S	4b	SiT <i>i</i> R <i>i</i> NT <i>i</i> TR <i>i</i> N <i>i</i> S	
4c	SXi <i>i</i> UiL <i>i</i> XiUL <i>i</i> S	4c	SiY <i>i</i> Ui <i>i</i> Y <i>i</i> UM <i>i</i> S	4c	SZ <i>i</i> Ui <i>i</i> N <i>i</i> ZUN <i>i</i> S	
4d	SXER <i>i</i> XiE <i>i</i> R <i>i</i> S	4d	SYP <i>i</i> Fi <i>i</i> P <i>i</i> F <i>i</i> S	4d	SZDQ <i>i</i> Z <i>i</i> DiQ <i>i</i> S	
4d	SXQ <i>i</i> F <i>i</i> QiFiS	4d	SYDR <i>i</i> Y <i>i</i> DiR <i>i</i> S	4d	SZPE <i>i</i> Z <i>i</i> P <i>i</i> E <i>i</i> S	
Type 5 subgroups having [+ - - + + -] signature						
Subtype	Unit elements	Subtype	Unit elements	Subtype	Unit elements	
5a	SLMN <i>i</i> LiMiN <i>i</i> S					
5b	SLiE <i>i</i> FILEFiS	5b	SMiFiDiMFD <i>i</i> S	5b	SNiDiE <i>i</i> NDE <i>i</i> S	
5c	SLiQ <i>i</i> RiL <i>i</i> Q <i>i</i> R <i>i</i> S	5c	SMiR <i>i</i> P <i>i</i> MR <i>i</i> P <i>i</i> S	5c	SiNiP <i>i</i> Q <i>i</i> N <i>i</i> P <i>i</i> Q <i>i</i> S	
5d	SLiY <i>i</i> ZiLYZiS	5d	SMiZ <i>i</i> XiM <i>i</i> ZXiS	5d	SiNiXY <i>i</i> NY <i>i</i> S	
5e	SVTU <i>i</i> ViTiUiS					
5f	SViPiX <i>i</i> VP <i>i</i> X <i>i</i> S	5f	SViQiY <i>i</i> V <i>i</i> Q <i>i</i> Y <i>i</i> S	5f	SViR <i>i</i> Z <i>i</i> VR <i>i</i> Z <i>i</i> S	
5g	ST <i>i</i> X <i>i</i> D <i>i</i> TX <i>i</i> DiS	5g	ST <i>i</i> Y <i>i</i> E <i>i</i> TY <i>i</i> E <i>i</i> S	5g	ST <i>i</i> Z <i>i</i> F <i>i</i> T <i>i</i> Z <i>i</i> F <i>i</i> S	
5h	SUi <i>i</i> D <i>i</i> P <i>i</i> UD <i>i</i> S	5h	SUi <i>i</i> E <i>i</i> Q <i>i</i> UE <i>i</i> Q <i>i</i> S	5h	SUi <i>i</i> F <i>i</i> R <i>i</i> U <i>i</i> FR <i>i</i> S	

TABLE 3. Classification of sub-algebras with 8 unit elements of \mathbb{T} with respect to participation in sedenion-type sub-algebras

TABLE 4. Unit elements for aligned \mathbb{M} and \mathbb{T} sub-algebras with 4 unit elements and complexified aligned $\mathbb{C} \otimes \mathbb{W}$ sub-algebras

Unit elements of \mathbb{M}		Unit elements of \mathbb{T} and sedenion-type loop participation				Unit elements of $\mathbb{C} \otimes \mathbb{W}$		
Sub-group	Type	Sub-loop	Type	S_L	S_L^α	S_L^β	Complexified aligned sub-loop	
SLMN	Non-abelian	$\sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\kappa$	Non-Abelian	S_0	$S_8^{\alpha 1}$	$S_8^{\beta 1}$	$\sigma_\alpha S \sigma_\beta L \sigma_\gamma M \sigma_\kappa N \sigma_\alpha iS \sigma_\beta iL \sigma_\gamma iM \sigma_\kappa iN$	
SLIMN	Non-abelian	$\sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\kappa$	Non-Abelian	-	$S_8^{\alpha 2} S_8^{\beta 2}$	$S_8^{\alpha 2} S_8^{\beta 2}$	$\sigma_\alpha S \sigma_\beta L \sigma_\gamma M \sigma_\kappa N \sigma_\alpha iS \sigma_\beta iL \sigma_\gamma jM \sigma_\kappa N$	
SLIMN	Non-abelian	$\sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\kappa$	Non-Abelian	-	$S_8^{\alpha 3} S_6^{\alpha 3}$	$S_{10}^{\alpha 3} S_{13}^{\beta 3}$	$\sigma_\alpha S \sigma_\beta L \sigma_\gamma M \sigma_\kappa N \sigma_\alpha iS \sigma_\beta iL \sigma_\gamma jM \sigma_\kappa N$	
SLIMN	Non-abelian	$\sigma_\alpha \sigma_\beta \sigma_\gamma \sigma_\kappa$	Non-Abelian	-	$S_8^{\alpha 4} S_8^{\beta 4}$	$S_{11}^{\alpha 4} S_{14}^{\beta 4}$	$\sigma_\alpha S \sigma_\beta L \sigma_\gamma M \sigma_\kappa N \sigma_\alpha iS \sigma_\beta iL \sigma_\gamma jM \sigma_\kappa N$	
SLEF	Non-abelian	$\sigma_\alpha \sigma_\beta \lambda_j \lambda_k$	Non-Abelian	S_0	$S_8^{\alpha 5}$	$S_{10}^{\alpha 5} S_{16}^{\beta 5} S_{27}^{\gamma 5} S_{28}^{\delta 5}$	$\sigma_\alpha S \sigma_\beta L \lambda_j E \lambda_k F \sigma_\alpha iS \sigma_\beta iL \lambda_j iE \lambda_k iF$	
SLIEF	Non-abelian	$\sigma_\alpha \sigma_\beta \lambda_j \lambda_k$	Non-Abelian	-	$S_8^{\alpha 6} S_8^{\beta 6}$	$S_{16}^{\alpha 6} S_{22}^{\beta 6} S_{28}^{\gamma 6}$	$\sigma_\alpha S \sigma_\beta L \beta_j E \beta_k F \sigma_\alpha iS \sigma_\beta iL \beta_j E \beta_k F$	
SLIEF	Non-abelian	$\sigma_\alpha \sigma_\beta \lambda_j \lambda_k$	Non-Abelian	-	$S_8^{\alpha 7} S_6^{\alpha 7}$	$S_{16}^{\alpha 7} S_{22}^{\beta 7} S_{28}^{\gamma 7}$	$\sigma_\alpha S \sigma_\beta iL \beta_j E \lambda_k F \sigma_\alpha iS \alpha_k L \beta_j E \lambda_k iF$	
SILEF	Non-abelian	$\sigma_\alpha \sigma_\beta \lambda_j \lambda_k$	Non-Abelian	-	$S_8^{\alpha 8} S_8^{\beta 8}$	$S_{16}^{\alpha 8} S_{22}^{\beta 8} S_{28}^{\gamma 8}$	$\sigma_\alpha S \sigma_\beta iL \lambda_j E \beta_k F \sigma_\alpha iS \sigma_\beta iL \lambda_j E \beta_k F$	
SMFD	Non-abelian	$\sigma_\alpha \sigma_\beta \lambda_k \lambda_l$	Non-Abelian	S_0	$S_8^{\alpha 9}$	$S_{13}^{\beta 9}$	$\sigma_\alpha S \sigma_\beta M \lambda_k F \lambda_l D \sigma_\alpha iS \sigma_\beta iM \lambda_k iF \lambda_l iD$	
SMFID	Non-abelian	$\sigma_\alpha \sigma_\beta \kappa_k \beta_i$	Non-Abelian	-	$S_8^{\alpha 10} S_6^{\alpha 10}$	$S_{16}^{\alpha 10} S_{23}^{\beta 10} S_{29}^{\gamma 10}$	$\sigma_\alpha S \sigma_\beta M \beta_k F \beta_i D \sigma_\alpha iS \sigma_\beta iM \beta_k R F \beta_i D$	
SIMFID	Non-abelian	$\sigma_\alpha \sigma_\beta \kappa_k \lambda_i$	Non-Abelian	-	$S_8^{\alpha 11} S_8^{\beta 11}$	$S_{16}^{\alpha 11} S_{23}^{\beta 11} S_{29}^{\gamma 11}$	$\sigma_\alpha S \sigma_\beta iM \beta_k iF \lambda_i D \sigma_\alpha iS \alpha_k \beta_i M \beta_k F \lambda_i D$	
SIMFID	Non-abelian	$\sigma_\alpha \sigma_\beta \lambda_k \beta_i$	Non-Abelian	-	$S_8^{\alpha 12} S_8^{\beta 12}$	$S_{16}^{\alpha 12} S_{23}^{\beta 12} S_{29}^{\gamma 12}$	$\sigma_\alpha S \sigma_\beta iM \lambda_k F \beta_i D \sigma_\alpha iS \alpha_k M \lambda_k iF \beta_i D$	
SNDE	Non-abelian	$\sigma_\alpha \sigma_\kappa \lambda_i \lambda_j$	Non-Abelian	S_0	$S_8^{\alpha 13}$	$S_{16}^{\alpha 13} S_{17}^{\beta 13} S_{24}^{\gamma 13} S_{30}^{\delta 13}$	$\sigma_\alpha S \sigma_\kappa N \lambda_i D \lambda_j E \sigma_\alpha iS \sigma_\kappa iN \lambda_i iD \lambda_j iE$	
SNIDH	Non-abelian	$\sigma_\alpha \sigma_\kappa \lambda_i \beta_k$	Non-Abelian	-	$S_8^{\alpha 14} S_4^{\beta 14}$	$S_{16}^{\alpha 14} S_{24}^{\beta 14} S_{30}^{\gamma 14}$	$\sigma_\alpha S \sigma_\kappa N \beta_i D \beta_k F \sigma_\alpha iS \sigma_\kappa iN \beta_i D \beta_k F$	
SINIDE	Non-abelian	$\sigma_\alpha \sigma_\kappa \beta_i \lambda_j$	Non-Abelian	-	$S_8^{\alpha 15} S_6^{\alpha 15}$	$S_{16}^{\alpha 15} S_{23}^{\beta 15} S_{30}^{\gamma 15}$	$\sigma_\alpha S \sigma_\kappa iN \beta_i D \lambda_j E \sigma_\alpha iS \alpha_k N \beta_i D \lambda_j iE$	
SINIDE	Non-abelian	$\sigma_\alpha \sigma_\kappa \lambda_i \beta_j$	Non-Abelian	-	$S_8^{\alpha 16} S_6^{\alpha 16}$	$S_{16}^{\alpha 16} S_{24}^{\beta 16} S_{30}^{\gamma 16}$	$\sigma_\alpha S \sigma_\kappa iN \lambda_i D \beta_j E \sigma_\alpha iS \alpha_k N \lambda_i iD \beta_j E$	
SILYZ	Non-abelian	$\sigma_\alpha \sigma_\beta \lambda_k \lambda_l$	Non-Abelian	S_0	$S_8^{\alpha 17} S_8^{\beta 17}$	$S_{16}^{\alpha 17} S_{25}^{\beta 17} S_{29}^{\gamma 17}$	$\sigma_\alpha S \sigma_\beta L \gamma_j Y \gamma_k Z \sigma_\alpha iS \sigma_\beta iL \gamma_j iY \gamma_k iZ$	
SILIYZ	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_j \mu_k$	Non-Abelian	-	$S_8^{\alpha 18}$	$S_{16}^{\alpha 18} S_{17}^{\beta 18} S_{25}^{\gamma 18}$	$\sigma_\alpha S \sigma_\beta L \mu_j Y \mu_k iZ \sigma_\alpha iS \alpha_k L \mu_j Y \mu_k iZ$	
SILIYZ	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_j \gamma_k$	Non-Abelian	-	$S_8^{\alpha 19} S_6^{\alpha 19}$	$S_{16}^{\alpha 19} S_{17}^{\beta 19} S_{28}^{\gamma 19}$	$\sigma_\alpha S \sigma_\beta iL \mu_j Y \gamma_k Z \sigma_\alpha iS \alpha_k L \mu_j Y \gamma_k iZ$	
SILYZ	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_j \gamma_k$	Non-Abelian	-	$S_8^{\alpha 20} S_6^{\alpha 20}$	$S_{16}^{\alpha 20} S_{17}^{\beta 20} S_{28}^{\gamma 20}$	$\sigma_\alpha S \sigma_\beta iL \lambda_j E \beta_k F \sigma_\alpha iS \sigma_\beta iL \lambda_j iE \beta_k F$	
SMZN	Non-abelian	$\sigma_\alpha \sigma_\beta \gamma_k \gamma_l$	Non-Abelian	S_0	$S_8^{\alpha 21} S_8^{\beta 21}$	$S_{16}^{\alpha 21} S_{25}^{\beta 21} S_{29}^{\gamma 21}$	$\sigma_\alpha S \sigma_\beta M \gamma_k Z \gamma_l X \sigma_\alpha iS \sigma_\beta iM \gamma_k iZ \gamma_l iX$	
SMIZIX	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_k \mu_i$	Non-Abelian	-	$S_8^{\alpha 22} S_8^{\beta 22}$	$S_{16}^{\alpha 22} S_{17}^{\beta 22} S_{26}^{\gamma 22}$	$\sigma_\alpha S \sigma_\beta M \mu_k Z \mu_i X \sigma_\alpha iS \sigma_\beta iM \mu_k Z \mu_i X$	
SMIZIX	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_k \gamma_i$	Non-Abelian	-	$S_8^{\alpha 23} S_8^{\beta 23}$	$S_{17}^{\alpha 23} S_{26}^{\beta 23} S_{29}^{\gamma 23}$	$\sigma_\alpha S \sigma_\beta iM \mu_k iZ \gamma_i X \sigma_\alpha iS \alpha_k \mu_k Z \gamma_i X$	
SMIZIX	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_k \gamma_i$	Non-Abelian	-	$S_8^{\alpha 24} S_8^{\beta 24}$	$S_{17}^{\alpha 24} S_{26}^{\beta 24} S_{29}^{\gamma 24}$	$\sigma_\alpha S \sigma_\beta iM \mu_k iZ \gamma_i E \sigma_\alpha iS \alpha_k \mu_k Z \gamma_i E$	
SIMZIX	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_k \mu_i$	Non-Abelian	-	$S_8^{\alpha 25} S_8^{\beta 25}$	$S_{17}^{\alpha 25} S_{26}^{\beta 25} S_{29}^{\gamma 25}$	$\sigma_\alpha S \sigma_\beta iM \mu_k iZ \gamma_i E \sigma_\alpha iS \alpha_k \mu_k Z \gamma_i E$	
SIMZIX	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_k \mu_i$	Non-Abelian	-	$S_8^{\alpha 26} S_8^{\beta 26}$	$S_{17}^{\alpha 26} S_{26}^{\beta 26} S_{29}^{\gamma 26}$	$\sigma_\alpha S \sigma_\beta iM \mu_k iZ \gamma_i E \sigma_\alpha iS \alpha_k \mu_k Z \gamma_i E$	
SIMZIX	Non-abelian	$\sigma_\alpha \sigma_\beta \mu_k \mu_i$	Non-Abelian	-	$S_8^{\alpha 27} S_8^{\beta 27}$	$S_{17}^{\alpha 27} S_{26}^{\beta 27} S_{29}^{\gamma 27}$	$\sigma_\alpha S \sigma_\beta iM \mu_k iZ \gamma_i E \sigma_\alpha iS \alpha_k \mu_k Z \gamma_i E$	
SNXY	Non-abelian	$\sigma_\alpha \sigma_\kappa \gamma_k \gamma_j$	Non-Abelian	S_0	$S_8^{\alpha 28} S_8^{\beta 28}$	$S_{16}^{\alpha 28} S_{27}^{\beta 28}$	$\sigma_\alpha S \sigma_\kappa N \gamma_k X \gamma_j Y \sigma_\alpha iS \sigma_\kappa iN \gamma_k iX \gamma_j iY$	
SNIXY	Non-abelian	$\sigma_\alpha \sigma_\kappa \mu_i \mu_j$	Non-Abelian	-	$S_8^{\alpha 29} S_8^{\beta 29}$	$S_{16}^{\alpha 29} S_{17}^{\beta 29} S_{30}^{\gamma 29}$	$\sigma_\alpha S \sigma_\kappa N \mu_i L \mu_j iZ \sigma_\alpha iS \sigma_\kappa iN \mu_i X \mu_j Y$	
SINIXY	Non-abelian	$\sigma_\alpha \sigma_\kappa \mu_i \mu_j$	Non-Abelian	-	$S_8^{\alpha 30} S_8^{\beta 30}$	$S_{16}^{\alpha 30} S_{27}^{\beta 30} S_{30}^{\gamma 30}$	$\sigma_\alpha S \sigma_\kappa iN \mu_i X \mu_j Y \sigma_\alpha iS \sigma_\kappa iN \mu_i X \mu_j Y$	
SINIXY	Non-abelian	$\sigma_\alpha \sigma_\kappa \mu_i \mu_j$	Non-Abelian	-	$S_8^{\alpha 31} S_8^{\beta 31}$	$S_{16}^{\alpha 31} S_{27}^{\beta 31} S_{30}^{\gamma 31}$	$\sigma_\alpha S \sigma_\kappa iN \mu_i X \mu_j Y \sigma_\alpha iS \sigma_\kappa iN \mu_i X \mu_j Y$	
SLQR	Non-abelian	$\sigma_\alpha \sigma_\beta \delta_k \delta_l$	Non-Abelian	S_0	$S_8^{\alpha 32} S_8^{\beta 32}$	$S_{18}^{\alpha 32} S_{22}^{\beta 32} S_{25}^{\gamma 32}$	$\sigma_\alpha S \sigma_\beta L \delta_Q \delta_R \sigma_\alpha iS \sigma_\beta iL \delta_Q iQ \delta_R iR$	
SLQR	Non-abelian	$\sigma_\alpha \sigma_\beta \delta_k \delta_l$	Non-Abelian	-	$S_8^{\alpha 33}$	$S_{18}^{\alpha 33} S_{22}^{\beta 33} S_{25}^{\gamma 33}$	$\sigma_\alpha S \sigma_\beta L \nu_j Q \nu_k R \sigma_\alpha iS \sigma_\beta iL \nu_j Q \nu_k R$	
SILQIR	Non-abelian	$\sigma_\alpha \sigma_\beta \delta_k \delta_l$	Non-Abelian	-	$S_8^{\alpha 34} S_8^{\beta 34}$	$S_{18}^{\alpha 34} S_{22}^{\beta 34} S_{25}^{\gamma 34}$	$\sigma_\alpha S \sigma_\beta iL \delta_Q \nu_k R \sigma_\alpha iS \sigma_\beta iL \nu_j Q \delta_k \nu_R$	
SILQIR	Non-abelian	$\sigma_\alpha \sigma_\beta \delta_k \delta_l$	Non-Abelian	-	$S_8^{\alpha 35} S_8^{\beta 35}$	$S_{18}^{\alpha 35} S_{23}^{\beta 35} S_{24}^{\gamma 35}$	$\sigma_\alpha S \sigma_\beta M \delta_R R \sigma_\beta P \sigma_\alpha iS \sigma_\beta iM \delta_k iR \delta_i P$	
SMRP	Non-abelian	$\sigma_\alpha \sigma_\beta \delta_k \delta_l$	Non-Abelian	S_0	$S_8^{\alpha 36} S_8^{\beta 36}$	$S_{18}^{\alpha 36} S_{23}^{\beta 36} S_{26}^{\gamma 36}$	$\sigma_\alpha S \sigma_\beta M \nu_k R \nu_l P \sigma_\alpha iS \sigma_\beta iM \nu_k R \nu_l P$	
SMRIP	Non-abelian	$\sigma_\alpha \sigma_\beta \nu_k \nu_l$	Non-Abelian	-	$S_8^{\alpha 37} S_8^{\beta 37}$	$S_{18}^{\alpha 37} S_{23}^{\beta 37} S_{26}^{\gamma 37}$	$\sigma_\alpha S \sigma_\beta M \nu_k iR \nu_l P \sigma_\alpha iS \sigma_\beta iM \nu_k R \nu_l P$	
SMRIP	Non-abelian	$\sigma_\alpha \sigma_\beta \nu_k \nu_l$	Non-Abelian	-	$S_8^{\alpha 38} S_8^{\beta 38}$	$S_{18}^{\alpha 38} S_{23}^{\beta 38} S_{26}^{\gamma 38}$	$\sigma_\alpha S \sigma_\beta M \nu_k iR \nu_l P \sigma_\alpha iS \sigma_\beta iM \nu_k R \nu_l P$	
SMRIP	Non-abelian	$\sigma_\alpha \sigma_\beta \nu_k \nu_l$	Non-Abelian	-	$S_8^{\alpha 39} S_8^{\beta 39}$	$S_{18}^{\alpha 39} S_{23}^{\beta 39} S_{26}^{\gamma 39}$	$\sigma_\alpha S \sigma_\beta M \nu_k iR \nu_l P \sigma_\alpha iS \sigma_\beta iM \nu_k R \nu_l P$	
SMRIP	Non-abelian	$\sigma_\alpha \sigma_\beta \nu_k \nu_l$	Non-Abelian	-	$S_8^{\alpha 40} S_8^{\beta 40}$	$S_{18}^{\alpha 40} S_{23}^{\beta 40} S_{26}^{\gamma 40}$	$\sigma_\alpha S \sigma_\beta M \nu_k iR \nu_l P \sigma_\alpha iS \sigma_\beta iM \nu_k R \nu_l P$	
SVTU	Non-abelian	$\sigma_\alpha \lambda_\alpha \delta_\alpha \gamma_\alpha$	Non-Abelian	S_0	-	$S_{10}^{\alpha 10} S_{11}^{\beta 10}$	$S_{17}^{\alpha 10} S_{27}^{\beta 10} S_{27}^{\gamma 10}$	$\sigma_\alpha S \lambda_\alpha V \delta_\alpha T \gamma_\alpha U \sigma_\alpha iS \sigma_\beta iV \delta_\alpha iV \gamma_\alpha iU$
SVITIU	Non-abelian	$\sigma_\alpha \beta_\alpha \delta_\alpha \mu_\alpha$	Non-Abelian	-	$S_1^{\alpha 1} S_6^{\beta 1} S_6^{\gamma 1} S_6^{\delta 1}$	$S_{19}^{\alpha 1} S_{20}^{\beta 1} S_{21}^{\gamma 1} S_{21}^{\delta 1}$	$\sigma_\alpha S \beta_\alpha iV \delta_\alpha T \mu_\alpha iU \sigma_\alpha iS \beta_\alpha iV \delta_\alpha iT \mu_\alpha U$	
SVITIU	Non-abelian	$\sigma_\alpha \lambda_\alpha \nu_\alpha \mu_\alpha$	Non-Abelian	-	$S_2^{\alpha 2} S_3^{\beta 2} S_4^{\gamma 2} S_4^{\delta 2}$	$S_{19}^{\alpha 2} S_{20}^{\beta 2} S_{21}^{\gamma 2} S_{21}^{\delta 2}$	$\sigma_\alpha S \lambda_\alpha V \nu_\alpha iU \sigma_\alpha iS \nu_\alpha iV \mu_\alpha iU \nu_\alpha T \mu_\alpha U$	
SVITIU	Non-abelian	$\sigma_\alpha \beta_\alpha \nu_\alpha \gamma_\alpha$	Non-Abelian	-	$S_2^{\alpha 3} S_3^{\beta 3} S_4^{\gamma 3} S_4^{\delta 3}$	$S_{19}^{\alpha 3} S_{20}^{\beta 3} S_{21}^{\gamma 3} S_{21}^{\delta 3}$	$\sigma_\alpha S \beta_\alpha V \nu_\alpha iU \nu_\alpha iV \sigma_\alpha iS \beta_\alpha iV \nu_\alpha T \gamma_\alpha U$	
SUDP	Non-abelian	$\sigma_\alpha \gamma_\alpha \lambda_\alpha \delta_\alpha$	Non-Abelian	S_0	-	$S_{10}^{\alpha 4} S_{13}^{\beta 4} S_{14}^{\gamma 4} S_{14}^{\delta 4}$	$S_{17}^{\alpha 4} S_{23}^{\beta 4} S_{27}^{\gamma 4} S_{27}^{\delta 4}$	$\sigma_\alpha S \gamma_\alpha U \lambda_\alpha D \delta_\alpha P \sigma_\alpha iS \gamma_\alpha U \lambda_\alpha iD \delta_\alpha iP$
SUDP	Non-abelian	$\sigma_\alpha \mu_\alpha \beta_\alpha \delta_\alpha$	Non-Abelian	-	$S_8^{\alpha 5} S_3^{\beta 5} S_6^{\gamma 5} S_6^{\delta 5}$	$S_{19}^{\alpha 5} S_{23}^{\beta 5} S_{24}^{\gamma 5} S_{24}^{\delta 5}$	$\sigma_\alpha S \mu_\alpha U \beta_\alpha iD \delta_\alpha P \sigma_\alpha iS \mu_\alpha \beta_\alpha iD \delta_\alpha iP$	
SUDP	Non-abelian	$\sigma_\alpha \mu_\alpha \beta_\alpha \delta_\alpha$	Non-Abelian	-	$S_8^{\alpha 6} S_3^{\beta 6} S_6^{\gamma 6} S_6^{\delta 6}$	$S_{19}^{\alpha 6} S_{23}^{\beta 6} S_{24}^{\gamma 6} S_{24}^{\delta 6}$	$\sigma_\alpha S \mu_\alpha U \beta_\alpha iD \delta_\alpha P \sigma_\alpha iS \mu_\alpha \beta_\alpha iD \delta_\alpha iP$	
SUDP	Non-abelian	$\sigma_\alpha \mu_\alpha \beta_\alpha \delta_\alpha$	Non-Abelian	-	$S_8^{\alpha 7} S_3^{\beta 7} S_6^{\gamma 7} S_6^{\delta 7}$	$S_{19}^{\alpha 7} S_{23}^{\beta 7} S_{24}^{\gamma 7} S_{24}^{\delta 7}$	$\sigma_\alpha S \mu_\alpha U \beta_\alpha iD \delta_\alpha P \sigma_\alpha iS \mu_\alpha \beta_\alpha iD \delta_\alpha iP$	
SUDP	Non-abelian	$\sigma_\alpha \mu_\alpha \beta_\alpha \delta_\alpha$	Non-Abelian	-	$S_8^{\alpha 8} S_3^{\beta 8} S_6^{\gamma 8} S_6^{\delta 8}$	$S_{19}^{\alpha 8} S_{23}^{\beta 8} S_{24}^{\gamma 8} S_{24}^{\delta 8}$	$\sigma_\alpha S \mu_\alpha U \beta_\alpha iD \delta_\alpha P \sigma_\alpha iS \mu_\alpha \beta_\alpha iD \delta_\alpha iP$	
SUDP	Non-abelian	$\sigma_\alpha \mu_\alpha \beta_\alpha \delta_\alpha$	Non-Abelian	-	$S_8^{\alpha 9} S_3^{\beta 9} S_6^{\gamma 9} S_6^{\delta 9}$	$S_{19}^{\alpha 9} S_{23}^{\beta 9} S_{24}^{\gamma 9} S_{24}^{\delta 9}$	$\sigma_\alpha S \mu_\alpha U \beta_\alpha iD \delta_\alpha P \sigma_\alpha iS \mu_\alpha \beta_\alpha iD \delta_\alpha iP$	
SUDP	Non-abelian	$\sigma_\alpha \mu_\alpha \beta_\alpha \delta_\alpha$	Non-Abelian	-	$S_8^{\alpha 10} S_3^{\beta 10} S_6^{\gamma 10} S_6^{\delta 10}$	$S_{19}^{\alpha 10} S_{23}^{\beta 10} S_{24}^{\gamma 10} S_{24}^{\delta 10}$	$\sigma_\alpha S \mu_\alpha U \beta_\alpha iD \delta_\alpha P \sigma_\alpha iS \mu_\alpha \beta_\alpha iD \delta_\alpha iP$	
STXD	Non-abelian	$\sigma_\alpha \delta_\alpha \theta_\alpha \gamma_\alpha$	Non-Abelian	S_0	$S_6^{\alpha 11} S_7^{\beta 11}$	$S_{19}^{\alpha 11} S_{29}^{\beta 11} S_{30}^{\gamma 11}$	$\sigma_\alpha S \delta_\alpha T \gamma_\alpha X \lambda_i D \sigma_\alpha iS \delta_\alpha iT \gamma_\alpha iX \lambda_i D$	
STIXD	Non-abelian	$\sigma_\alpha \nu_\alpha \mu_\alpha \lambda_\alpha$	Non-Abelian	-	$S_2^{\alpha 12}$	$S_{19}^{\alpha 12} S_{14}^{\beta 12} S_{30}^{\gamma 12}$	$\sigma_\alpha S \nu_\alpha T \lambda_\alpha X \lambda_i D \sigma_\alpha iS \nu_\alpha iL \nu_\alpha X \lambda_i D$	
STIXD	Non-abelian	$\sigma_\alpha \delta_\alpha \mu_\alpha \beta_\alpha$	Non-Abelian	-	$S_8^{\alpha 13} S_7^{\beta 13}$	$S_{19}^{\alpha 13} S_{24}^{\beta 13} S_{24}^{\gamma 13}$	$\sigma_\alpha S \delta_\alpha T \mu_\alpha X \beta_\alpha D \sigma_\alpha iS \delta_\alpha iT \mu_\alpha X \beta_\alpha D$	
STIXD	Non-abelian	$\sigma_\alpha \nu_\alpha \theta_\alpha \beta_\alpha$	Non-Abelian	-	$S_8^{\alpha 14} S_7^{\beta 14}$	$S_{19}^{\alpha 14} S_{24}^{\beta 14} S_{24}^{\gamma 14}$	$\sigma_\alpha S \nu_\alpha T \theta_\alpha X \beta_\alpha D \sigma_\alpha iS \nu_\alpha iT \theta_\alpha X \beta_\alpha D$	
STYE	Non-abelian	$\sigma_\alpha \delta_\alpha \theta_\alpha \lambda_\alpha$	Non-Abelian	-	$S_8^{\alpha 15} S_7^{\beta 15}$	$S_{19}^{\alpha 15} S_{29}^{\beta 15} S_{30}^{\gamma 15}$	$\sigma_\alpha S \delta_\alpha T \gamma_\alpha X \beta_\alpha D \sigma_\alpha iS \delta_\alpha iT \gamma_\alpha iX \beta_\alpha D$	
STYE	Non-abelian	$\sigma_\alpha \nu_\alpha \theta_\alpha \lambda_\alpha$	Non-Abelian	-	$S_8^{\alpha 16} S_7^{\beta 16}$	$S_{19}^{\alpha 16} S_{29}^{\beta 16} S_{30}^{\gamma 16}$	$\sigma_\alpha S \nu_\alpha T \theta_\alpha X \beta_\alpha D \sigma_\alpha iS \nu_\alpha iT \theta_\alpha X \beta_\alpha D$	
STYE	Non-abelian	$\sigma_\alpha \delta_\alpha \theta_\alpha \lambda_\alpha$	Non-Abelian	-	$S_8^{\alpha 17} S_7^{\beta 17}$	$S_{19}^{\alpha 17} S_{29}^{\beta 17} S_{30}^{\gamma 17}$	$\sigma_\alpha S \delta_\alpha T \gamma_\alpha X \beta_\alpha D \sigma_\alpha iS \delta_\alpha iT \gamma_\alpha iX \beta_\alpha D$	
STYE	Non-abelian	$\sigma_\alpha \nu_\alpha \theta_\alpha \lambda_\alpha$	Non-Abelian	-</td				

TABLE 5. Unit elements for aligned \mathbb{M} and \mathbb{T} sub-algebras with 4 unit elements and complexified aligned $\mathbb{C} \otimes \mathbb{W}$ sub-algebras

Unit elements of \mathbb{M}		Unit elements of \mathbb{T} and sedenion-type loop participation				Unit elements of $\mathbb{C} \otimes \mathbb{W}$				
Sub-group	Type	Sub-loop	Type	S_L	S_L^α	S_L^β	S_L^γ	Complexified aligned sub-loop	Type	
SVDL	Abelian	$\sigma_o \lambda_o \lambda_i \sigma_e$	Non-Abelian	S_0	S_0^{α}	S_0^{β}	S_0^{γ}	$S_{15}^{\gamma} S_{19}^{\alpha} S_{29}^{\beta}$	$\sigma_o S \lambda_o V \lambda_i D \sigma_L \sigma_o iS \lambda_o iV \lambda_i j_1 E \sigma_j iL$	Non-Abelian
SViDIL	Abelian	$\sigma_o \beta_o \beta_e \sigma_i$	Non-Abelian	—	$S_1^{\alpha} S_6^{\beta}$	$S_2^{\beta} S_3^{\alpha}$	$S_3^{\gamma} S_{12}^{\beta}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{29}^{\beta}$	$\sigma_o S \beta_o iV \beta_i D \sigma_L \sigma_o iS \beta_o V \beta_i D \sigma_i iL$	Non-Abelian
SViDIL	Abelian	$\sigma_o \lambda_o \beta_e \sigma_i$	Non-Abelian	—	$S_3^{\alpha} S_4^{\beta}$	$S_{19}^{\beta} S_{11}^{\alpha}$	$S_{16}^{\gamma} S_{19}^{\beta} S_{25}^{\alpha}$	$S_{16}^{\gamma} S_{19}^{\beta} S_{25}^{\alpha}$	$\sigma_o S \lambda_o V \beta_i iL \sigma_o iL \sigma_o iS \beta_o V \beta_i D \alpha_i L$	Non-Abelian
SViDIL	Abelian	$\sigma_o \beta_o \lambda_i \sigma_e$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta}$	$S_{14}^{\beta} S_{14}^{\alpha}$	$S_{16}^{\gamma} S_{19}^{\beta} S_{25}^{\alpha}$	$\sigma_o S \beta_o iV \lambda_i D \alpha_i L \sigma_o iS \beta_o V \lambda_i D \alpha_i L$	Non-Abelian	
SVEM	Abelian	$\sigma_o \lambda_o \lambda_j \sigma_j$	Non-Abelian	S_0	S_0^{α}	S_0^{β}	S_0^{γ}	$S_{15}^{\gamma} S_{19}^{\alpha} S_{29}^{\beta} S_{26}^{\alpha}$	$\sigma_o S \lambda_o V \lambda_j E \sigma_j M \sigma_o iS \lambda_o iV \lambda_j j_1 E \sigma_j iM$	Non-Abelian
SViEIM	Abelian	$\sigma_o \beta_o \beta_j \sigma_j$	Non-Abelian	—	$S_1^{\alpha} S_6^{\beta}$	$S_8^{\beta} S_{13}^{\alpha}$	$S_{16}^{\gamma} S_{20}^{\beta} S_{26}^{\alpha}$	$\sigma_o S \beta_o iV \beta_j E \sigma_j M \sigma_o iS \beta_o V \beta_j E \sigma_j iM$	Non-Abelian	
SViEIM	Abelian	$\sigma_o \lambda_o \beta_j \sigma_j$	Non-Abelian	—	$S_2^{\alpha} S_4^{\beta}$	$S_6^{\beta} S_{11}^{\alpha}$	$S_{16}^{\gamma} S_{20}^{\beta} S_{26}^{\alpha}$	$\sigma_o S \lambda_o V \beta_j E \alpha_j iE \alpha_o iS \lambda_o iV \beta_j E \alpha_j M$	Non-Abelian	
SViEIM	Abelian	$\sigma_o \beta_o \lambda_j \sigma_j$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta}$	$S_{14}^{\beta} S_{14}^{\alpha}$	$S_{16}^{\gamma} S_{20}^{\beta} S_{26}^{\alpha}$	$\sigma_o S \beta_o iV \lambda_j E \alpha_j iM \sigma_o iS \beta_o V \lambda_j E \alpha_j M$	Non-Abelian	
SVFN	Abelian	$\sigma_o \lambda_o \lambda_k \sigma_n$	Non-Abelian	S_0	S_0^{α}	S_0^{β}	S_0^{γ}	$S_{15}^{\gamma} S_{19}^{\alpha} S_{27}^{\beta} S_{27}^{\alpha}$	$\sigma_o S \lambda_o V \lambda_k F \sigma_o N \sigma_o iS \lambda_o iV \lambda_k F \sigma_o iN$	Non-Abelian
SiViFN	Abelian	$\sigma_o \beta_o \beta_k \sigma_n$	Non-Abelian	—	$S_1^{\alpha} S_6^{\beta}$	$S_8^{\beta} S_{14}^{\alpha}$	$S_{16}^{\gamma} S_{21}^{\beta} S_{27}^{\alpha}$	$\sigma_o S \beta_o iV \beta_k F \sigma_o N \sigma_o iS \beta_o V \beta_k F \sigma_o iN$	Non-Abelian	
SiViFN	Abelian	$\sigma_o \lambda_o \beta_k \sigma_n$	Non-Abelian	—	$S_2^{\alpha} S_3^{\beta}$	$S_6^{\beta} S_{10}^{\alpha}$	$S_{16}^{\gamma} S_{21}^{\beta} S_{27}^{\alpha}$	$\sigma_o S \lambda_o V \beta_k F \alpha_o iN \sigma_o iS \beta_o V \lambda_k F \alpha_o N$	Non-Abelian	
SiViFN	Abelian	$\sigma_o \beta_o \alpha_k \sigma_n$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta}$	$S_{14}^{\beta} S_{14}^{\alpha}$	$S_{16}^{\gamma} S_{21}^{\beta} S_{27}^{\alpha}$	$\sigma_o S \beta_o iV \lambda_k F \alpha_o iN \sigma_o iS \beta_o V \lambda_k F \alpha_o N$	Non-Abelian	
STPL	Abelian	$\sigma_o \delta_o \delta_i \sigma_i$	Non-Abelian	S_0	$S_1^{\alpha} S_6^{\beta}$	S_9^{β}	$S_{15}^{\gamma} S_{19}^{\alpha} S_{28}^{\beta}$	$\sigma_o S \delta_o T \delta_i P \sigma_i L \sigma_o iS \delta_o iT \delta_i j_1 P \sigma_i iL$	Non-Abelian	
SiTiPL	Abelian	$\sigma_o \nu_o \nu_i \sigma_i$	Non-Abelian	—	S_2^{α}	$S_4^{\beta} S_{12}^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{28}^{\beta}$	$\sigma_o S \nu_o T \nu_i P \sigma_i L \sigma_o iS \nu_o V \nu_i P \sigma_i iL$	Non-Abelian	
STPiPL	Abelian	$\sigma_o \delta_o \nu_i \alpha_i$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta}$	$S_{10}^{\beta} S_{11}^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{28}^{\beta}$	$\sigma_o S \delta_o T \nu_i P \alpha_i L \sigma_o iS \delta_o iT \nu_i P \alpha_i L$	Non-Abelian	
STPiPL	Abelian	$\sigma_o \nu_o \delta_i \alpha_i$	Non-Abelian	—	$S_3^{\alpha} S_6^{\beta}$	$S_{13}^{\beta} S_{14}^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{28}^{\beta}$	$\sigma_o S \nu_o T \delta_i P \alpha_i L \sigma_o iS \nu_o T \delta_i P \alpha_i L$	Non-Abelian	
STQM	Abelian	$\sigma_o \delta_o \lambda_j \sigma_j$	Non-Abelian	S_0	$S_1^{\alpha} S_6^{\beta}$	S_{10}^{β}	$S_{15}^{\gamma} S_{19}^{\alpha} S_{28}^{\beta}$	$\sigma_o S \delta_o T \delta_j Q \sigma_j M \sigma_o iS \delta_o iT \delta_j Q \sigma_j iM$	Non-Abelian	
SiTiQM	Abelian	$\sigma_o \nu_o \nu_j \sigma_j$	Non-Abelian	—	S_3^{α}	$S_3^{\beta} S_{13}^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{29}^{\beta} S_{29}^{\alpha}$	$\sigma_o S \nu_i T \nu_j iQ \sigma_j M \sigma_o iS \nu_o T \nu_j Q \sigma_j M$	Non-Abelian	
SiTiQM	Abelian	$\sigma_o \nu_o \nu_j \alpha_j$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta}$	$S_{10}^{\beta} S_{11}^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{29}^{\beta} S_{29}^{\alpha}$	$\sigma_o S \delta_o T \nu_j iQ \alpha_j iM \sigma_o iS \delta_o iT \nu_j Q \alpha_j M$	Non-Abelian	
SiTiQM	Abelian	$\sigma_o \nu_o \delta_j \alpha_j$	Non-Abelian	—	$S_2^{\alpha} S_4^{\beta}$	$S_{13}^{\beta} S_{14}^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{29}^{\beta} S_{29}^{\alpha}$	$\sigma_o S \nu_o T \delta_j Q \alpha_j iM \sigma_o iS \nu_o T \delta_j Q \alpha_j M$	Non-Abelian	
STRN	Abelian	$\sigma_o \delta_o \delta_n \sigma_n$	Non-Abelian	S_0	$S_1^{\alpha} S_6^{\beta}$	S_{11}^{β}	$S_{15}^{\gamma} S_{21}^{\alpha} S_{29}^{\beta}$	$\sigma_o S \delta_o T \delta_n R \sigma_o N \sigma_o iS \delta_o iT \delta_n iF \sigma_o iN$	Non-Abelian	
SiTiRN	Abelian	$\sigma_o \nu_o \nu_k \sigma_n$	Non-Abelian	—	S_1^{α}	$S_6^{\beta} S_{14}^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{29}^{\beta} S_{29}^{\alpha}$	$\sigma_o S \nu_o T \nu_k iT \nu_n iP \sigma_o N \sigma_o iS \nu_o T \nu_k R \sigma_o iN$	Non-Abelian	
SiTiRN	Abelian	$\sigma_o \delta_o \nu_k \sigma_n$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta}$	$S_6^{\beta} S_{10}^{\alpha}$	$S_{15}^{\gamma} S_{21}^{\alpha} S_{29}^{\beta}$	$\sigma_o S \delta_o T \nu_k R \alpha_k iN \sigma_o iS \delta_o iT \nu_k R \alpha_k N$	Non-Abelian	
SiTiRN	Abelian	$\sigma_o \nu_o \delta_k \sigma_n$	Non-Abelian	—	$S_2^{\alpha} S_3^{\beta}$	$S_{15}^{\beta} S_{19}^{\alpha}$	$S_{15}^{\gamma} S_{21}^{\alpha} S_{29}^{\beta}$	$\sigma_o S \nu_o T \delta_k R \alpha_k iN \sigma_o iS \nu_o T \delta_k R \alpha_k N$	Non-Abelian	
SXUL	Abelian	$\sigma_o \gamma_o \gamma_o \sigma_o$	Non-Abelian	—	$S_8^{\alpha} S_8^{\beta} S_8^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{29}^{\beta} S_{29}^{\alpha}$	$\sigma_o S \gamma_o X \gamma_o U \sigma_o L \sigma_o iS \gamma_o iX \gamma_o iU \sigma_o iL$	$\sigma_o S \gamma_o X \gamma_o U \sigma_o L \sigma_o iS \gamma_o iX \gamma_o iU \sigma_o iL$	Non-Abelian	
SIXiUL	Abelian	$\sigma_o \mu_i \mu_o \sigma_i$	Non-Abelian	—	$S_3^{\alpha} S_2^{\beta} S_6^{\alpha}$	$S_{15}^{\gamma} S_{17}^{\alpha} S_{19}^{\beta} S_{22}^{\alpha}$	$\sigma_o S \mu_i X \mu_o U \sigma_i L \sigma_o iS \mu_o X \mu_o U \sigma_i L$	$\sigma_o S \mu_i X \mu_o U \sigma_i L \sigma_o iS \mu_o X \mu_o U \sigma_i L$	Non-Abelian	
SXIiUL	Abelian	$\sigma_o \tau_i \mu_o \alpha_i$	Non-Abelian	—	$S_3^{\alpha} S_4^{\beta} S_6^{\alpha} S_7^{\alpha}$	$S_{15}^{\gamma} S_{17}^{\alpha} S_{19}^{\beta} S_{22}^{\alpha}$	$\sigma_o S \gamma_i X \mu_o iL \sigma_o iS \gamma_i X \mu_o iL$	$\sigma_o S \gamma_i X \mu_o iL \sigma_o iS \gamma_i X \mu_o iL$	Non-Abelian	
SIXiUL	Abelian	$\sigma_o \mu_o \gamma_o \alpha_i$	Non-Abelian	—	$S_{10}^{\alpha} S_7^{\beta} S_{13}^{\alpha} S_{14}^{\beta}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{20}^{\beta} S_{29}^{\alpha}$	$\sigma_o S \mu_i X \gamma_o iL \sigma_o iS \mu_i X \gamma_o iU \alpha_i L$	$\sigma_o S \mu_i X \gamma_o iL \sigma_o iS \mu_i X \gamma_o iU \alpha_i L$	Non-Abelian	
SYUM	Abelian	$\sigma_o \gamma_o \gamma_o \sigma_o$	Non-Abelian	—	$S_8^{\alpha} S_8^{\beta} S_8^{\alpha}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{23}^{\beta} S_{23}^{\alpha}$	$\sigma_o S \gamma_o Y \gamma_o U \sigma_o M \sigma_o iS \gamma_o iY \gamma_o iU \sigma_o iM$	$\sigma_o S \gamma_o Y \gamma_o U \sigma_o M \sigma_o iS \gamma_o iY \gamma_o iU \sigma_o iM$	Non-Abelian	
SiYiUM	Abelian	$\sigma_o \mu_j \mu_o \sigma_j$	Non-Abelian	—	$S_3^{\alpha} S_3^{\beta} S_6^{\alpha} S_6^{\beta}$	$S_{15}^{\gamma} S_{17}^{\alpha} S_{20}^{\beta} S_{23}^{\alpha}$	$\sigma_o S \mu_j Y \mu_o U \sigma_j M \sigma_o iS \mu_j Y \mu_o U \sigma_j iM$	$\sigma_o S \mu_j Y \mu_o U \sigma_j M \sigma_o iS \mu_j Y \mu_o U \sigma_j iM$	Non-Abelian	
SiYiUM	Abelian	$\sigma_o \tau_j \mu_o \alpha_j$	Non-Abelian	—	$S_2^{\alpha} S_4^{\beta} S_6^{\alpha} S_7^{\beta}$	$S_{15}^{\gamma} S_{17}^{\alpha} S_{20}^{\beta} S_{23}^{\alpha}$	$\sigma_o S \tau_j Y \mu_o U \alpha_j iM \sigma_o iS \tau_j Y \mu_o U \alpha_j M$	$\sigma_o S \tau_j Y \mu_o U \alpha_j iM \sigma_o iS \tau_j Y \mu_o U \alpha_j M$	Non-Abelian	
SiYiUM	Abelian	$\sigma_o \mu_j \gamma_o \alpha_j$	Non-Abelian	—	$S_9^{\alpha} S_9^{\beta} S_9^{\alpha} S_9^{\beta}$	$S_{15}^{\gamma} S_{17}^{\alpha} S_{20}^{\beta} S_{23}^{\alpha}$	$\sigma_o S \mu_j Y \mu_o U \alpha_j iM \sigma_o iS \mu_j Y \mu_o U \alpha_j M$	$\sigma_o S \mu_j Y \mu_o U \alpha_j iM \sigma_o iS \mu_j Y \mu_o U \alpha_j M$	Non-Abelian	
SZUN	Abelian	$\sigma_o \gamma_o \gamma_o \sigma_o$	Non-Abelian	S_0	$S_8^{\alpha} S_8^{\beta} S_8^{\alpha}$	$S_{15}^{\gamma} S_{11}^{\alpha} S_{14}^{\beta} S_{14}^{\alpha}$	$\sigma_o S \gamma_o Z \gamma_o U \sigma_o N \sigma_o iS \gamma_o iZ \gamma_o u \sigma_o iN$	$\sigma_o S \gamma_o Z \gamma_o U \sigma_o N \sigma_o iS \gamma_o iZ \gamma_o u \sigma_o iN$	Non-Abelian	
SZiJUN	Abelian	$\sigma_o \mu_o \mu_o \sigma_o$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta} S_6^{\alpha}$	$S_{15}^{\gamma} S_{17}^{\alpha} S_{21}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \mu_o Z \mu_o U \sigma_o N \sigma_o iS \mu_o Z \mu_o U \sigma_o iN$	$\sigma_o S \mu_o Z \mu_o U \sigma_o N \sigma_o iS \mu_o Z \mu_o U \sigma_o iN$	Non-Abelian	
SZiJUN	Abelian	$\sigma_o \gamma_o \mu_o \alpha_o$	Non-Abelian	—	$S_2^{\alpha} S_3^{\beta} S_3^{\alpha} S_6^{\beta}$	$S_{15}^{\gamma} S_{17}^{\alpha} S_{21}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \gamma_o Z \mu_o U \sigma_o N \sigma_o iS \gamma_o iZ \mu_o U \sigma_o iN$	$\sigma_o S \gamma_o Z \mu_o U \sigma_o N \sigma_o iS \gamma_o iZ \mu_o U \sigma_o iN$	Non-Abelian	
SZiJUN	Abelian	$\sigma_o \mu_o \gamma_o \alpha_o$	Non-Abelian	—	$S_9^{\alpha} S_9^{\beta} S_{12}^{\alpha} S_{13}^{\beta}$	$S_{15}^{\gamma} S_{17}^{\alpha} S_{21}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \mu_o Z \mu_o U \sigma_o N \sigma_o iS \gamma_o iZ \mu_o U \sigma_o iN$	$\sigma_o S \mu_o Z \mu_o U \sigma_o N \sigma_o iS \gamma_o iZ \mu_o U \sigma_o iN$	Non-Abelian	
SZiJUN	Abelian	$\sigma_o \mu_o \lambda_o \sigma_o$	Non-Abelian	—	$S_9^{\alpha} S_9^{\beta} S_{12}^{\alpha} S_{13}^{\beta}$	$S_{15}^{\gamma} S_{21}^{\alpha} S_{24}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \mu_o Z \lambda_o U \sigma_o N \sigma_o iS \mu_o Z \lambda_o U \sigma_o iN$	$\sigma_o S \mu_o Z \lambda_o U \sigma_o N \sigma_o iS \mu_o Z \lambda_o U \sigma_o iN$	Non-Abelian	
SXER	Abelian	$\sigma_o \gamma_o \lambda_j \delta_k$	Non-Abelian	S_0	$S_3^{\alpha} S_6^{\beta}$	S_{12}^{β}	$S_{22}^{\gamma} S_{26}^{\alpha} S_{27}^{\beta}$	$\sigma_o S \gamma_o X \lambda_j E \delta_k R \sigma_o iS \gamma_o iX \lambda_j iE \delta_k iR$	Non-Abelian	
SXiEIR	Abelian	$\sigma_o \mu_i \lambda_j \nu_k$	Non-Abelian	—	S_5^{α}	$S_{15}^{\gamma} S_{19}^{\alpha} S_{24}^{\beta} S_{22}^{\alpha}$	$\sigma_o S \mu_i X \lambda_j \nu_k R \sigma_o iS \mu_i X \lambda_j iE \nu_k R$	$\sigma_o S \mu_i X \lambda_j \nu_k R \sigma_o iS \mu_i X \lambda_j iE \nu_k R$	Non-Abelian	
SXiEIR	Abelian	$\sigma_o \gamma_i \beta_j \nu_k$	Non-Abelian	—	$S_2^{\alpha} S_6^{\beta}$	$S_{22}^{\gamma} S_{26}^{\alpha} S_{26}^{\beta}$	$\sigma_o S \gamma_i X \beta_j \nu_k R \sigma_o iS \mu_i X \beta_j iE \nu_k R$	$\sigma_o S \gamma_i X \beta_j \nu_k R \sigma_o iS \mu_i X \beta_j iE \nu_k R$	Non-Abelian	
SXiEIR	Abelian	$\sigma_o \mu_i \beta_j \delta_k$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta}$	$S_{22}^{\gamma} S_{26}^{\alpha} S_{26}^{\beta}$	$\sigma_o S \mu_i X \beta_j \delta_k R \sigma_o iS \mu_i X \beta_j iE \nu_k R$	$\sigma_o S \mu_i X \beta_j \delta_k R \sigma_o iS \mu_i X \beta_j iE \nu_k R$	Non-Abelian	
SXiEIR	Abelian	$\sigma_o \tau_j \delta_i \lambda_k$	Non-Abelian	—	$S_3^{\alpha} S_6^{\beta}$	$S_{23}^{\gamma} S_{27}^{\alpha} S_{29}^{\beta}$	$\sigma_o S \tau_j Y \delta_i \nu_k R \sigma_o iS \tau_j Y \delta_i iE \nu_k R$	$\sigma_o S \tau_j Y \delta_i \nu_k R \sigma_o iS \tau_j Y \delta_i iE \nu_k R$	Non-Abelian	
SXQF	Abelian	$\sigma_o \gamma_i \beta_j \lambda_k$	Non-Abelian	S_0	$S_4^{\alpha} S_6^{\beta}$	S_{12}^{β}	$S_{22}^{\gamma} S_{27}^{\alpha} S_{29}^{\beta}$	$\sigma_o S \gamma_i X \beta_j \lambda_k R \sigma_o iS \mu_i X \beta_j iE \nu_k R$	Non-Abelian	
SXiQF	Abelian	$\sigma_o \gamma_i \nu_k \beta_e$	Non-Abelian	—	$S_5^{\alpha} S_5^{\beta}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{24}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \gamma_i X \nu_k iQ \beta_e R \sigma_o iS \gamma_i iZ \beta_e iQ$	$\sigma_o S \gamma_i X \nu_k iQ \beta_e R \sigma_o iS \gamma_i iZ \beta_e iQ$	Non-Abelian	
SXiQF	Abelian	$\sigma_o \mu_i \delta_j \beta_e$	Non-Abelian	—	$S_1^{\alpha} S_2^{\beta}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{24}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \mu_i X \delta_j iQ \beta_e R \sigma_o iS \mu_i X \delta_j iQ$	$\sigma_o S \mu_i X \delta_j iQ \beta_e R \sigma_o iS \mu_i X \delta_j iQ$	Non-Abelian	
SYPP	Abelian	$\sigma_o \tau_j \delta_i \lambda_k$	Non-Abelian	S_0	$S_4^{\alpha} S_6^{\beta}$	S_{13}^{β}	$S_{23}^{\gamma} S_{27}^{\alpha} S_{29}^{\beta}$	$\sigma_o S \tau_j Y \delta_i \nu_k R \sigma_o iS \tau_j Y \delta_i iE \nu_k R$	Non-Abelian	
SiYiPF	Abelian	$\sigma_o \mu_i \nu_k \lambda_k$	Non-Abelian	—	$S_6^{\alpha} S_6^{\beta}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{24}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \mu_i Y \nu_k iP \lambda_k R \sigma_o iS \mu_i Y \nu_k iP \lambda_k R$	$\sigma_o S \mu_i Y \nu_k iP \lambda_k R \sigma_o iS \mu_i Y \nu_k iP \lambda_k R$	Non-Abelian	
SiYiPF	Abelian	$\sigma_o \gamma_i \nu_k \beta_e$	Non-Abelian	—	$S_2^{\alpha} S_2^{\beta}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{24}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \gamma_i Y \nu_k iP \beta_e R \sigma_o iS \gamma_i Y \nu_k iP \beta_e R$	$\sigma_o S \gamma_i Y \nu_k iP \beta_e R \sigma_o iS \gamma_i Y \nu_k iP \beta_e R$	Non-Abelian	
SiYiPF	Abelian	$\sigma_o \mu_j \nu_k \beta_e$	Non-Abelian	—	$S_1^{\alpha} S_3^{\beta}$	S_{14}^{β}	$S_{23}^{\gamma} S_{27}^{\alpha} S_{28}^{\beta}$	$\sigma_o S \mu_j Y \nu_k iP \beta_e iF \sigma_o iS \mu_j Y \nu_k iP \beta_e iF$	Non-Abelian	
SYDR	Abelian	$\sigma_o \gamma_j \lambda_i \delta_k$	Non-Abelian	S_0	$S_2^{\alpha} S_2^{\beta}$	S_{13}^{β}	$S_{23}^{\gamma} S_{27}^{\alpha} S_{28}^{\beta}$	$\sigma_o S \gamma_j Y \lambda_i D \delta_k R \sigma_o iS \gamma_j Y \lambda_i D \delta_k iR$	Non-Abelian	
SiYDiD	Abelian	$\sigma_o \mu_j \lambda_i \nu_k$	Non-Abelian	—	S_6^{α}	$S_{15}^{\gamma} S_{19}^{\alpha} S_{24}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \mu_j Y \lambda_i D \nu_k R \sigma_o iS \mu_j Y \lambda_i D \nu_k R$	$\sigma_o S \mu_j Y \lambda_i D \nu_k R \sigma_o iS \mu_j Y \lambda_i D \nu_k R$	Non-Abelian	
SiYDiD	Abelian	$\sigma_o \mu_j \beta_i \nu_k$	Non-Abelian	—	$S_2^{\alpha} S_2^{\beta}$	$S_{15}^{\gamma} S_{19}^{\alpha} S_{24}^{\beta} S_{24}^{\alpha}$	$\sigma_o S \mu_j Y \beta_i D \delta_k Q \sigma_o iS \mu_j Z \beta_i D \delta_k iQ$	$\sigma_o S \mu_j Y \beta_i D \delta_k Q \sigma_o iS \mu_j Z \beta_i D \delta_k iQ$	Non-Abelian	
SiYDiD</										

TABLE 6. Unit elements for aligned \mathbb{M} and \mathbb{T} sub-algebras with 4 unit elements and complexified aligned $\mathbb{C} \otimes \mathbb{W}$ sub-algebras

Unit elements of \mathbb{M}		Unit elements of \mathbb{T} and sedenion-type loop participation					Unit elements of $\mathbb{C} \otimes \mathbb{W}$		
Sub-group	Type	Sub-loop	Type	S_L	S_L^α	S_L^β	S_L^γ	Complexified aligned sub-loop	Type
SLSiL	Abelian	$\sigma_o \sigma_e \sigma_o \alpha_i$	Non-Abelian	-	-	-	$S_{16}^{\gamma} S_{19}^{\gamma} S_{18}^{\gamma} S_{22}^{\gamma} S_{25}^{\gamma} S_{28}^{\gamma}$	$\sigma_o S \sigma_e L \alpha_o i S \alpha_i L \sigma_o i S \sigma_j i M \alpha_o S \alpha_j M$	Non-Abelian
SMiSiM	Abelian	$\sigma_o \sigma_j \alpha_o \alpha_j$	Non-Abelian	-	-	-	$S_{16}^{\gamma} S_{19}^{\gamma} S_{18}^{\gamma} S_{20}^{\gamma} S_{23}^{\gamma} S_{26}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \sigma_j M \alpha_o i S \alpha_j i M \sigma_o i S \sigma_j i M \alpha_o S \alpha_j M$	Non-Abelian
SNiSiN	Abelian	$\sigma_o \sigma_\kappa \alpha_o \alpha_\kappa$	Non-Abelian	-	-	-	$S_{16}^{\gamma} S_{19}^{\gamma} S_{18}^{\gamma} S_{23}^{\gamma} S_{24}^{\gamma} S_{27}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \sigma_\kappa N \alpha_o i S \sigma_\kappa i N \sigma_o i S \sigma_\kappa i N \alpha_o S \alpha_\kappa N$	Non-Abelian
SViSiV	Abelian	$\sigma_o \lambda_o \alpha_o \beta_o$	Non-Abelian	-	-	-	$S_{16}^{\gamma} S_{19}^{\gamma} S_{20}^{\gamma} S_{21}^{\gamma} S_{25}^{\gamma} S_{26}^{\gamma} S_{27}^{\gamma}$	$\sigma_o S \lambda_o V \alpha_o i S \beta_o V \sigma_o i S \lambda_o V \alpha_o S \beta_o V$	Non-Abelian
STiSiT	Abelian	$\sigma_o \delta_o \alpha_o \nu_o$	Non-Abelian	-	-	-	$S_{18}^{\gamma} S_{19}^{\gamma} S_{20}^{\gamma} S_{21}^{\gamma} S_{25}^{\gamma} S_{26}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \delta_o T \alpha_o i S \nu_o i T \sigma_o i S \delta_o i T \alpha_o S \nu_o T$	Non-Abelian
SUiSiU	Abelian	$\sigma_o \gamma_o \alpha_o \mu_o$	Non-Abelian	-	-	-	$S_{17}^{\gamma} S_{19}^{\gamma} S_{20}^{\gamma} S_{21}^{\gamma} S_{22}^{\gamma} S_{23}^{\gamma} S_{24}^{\gamma}$	$\sigma_o S \gamma_o U \alpha_o i S \mu_o i U \sigma_o i S \gamma_o i U \alpha_o S \mu_o U$	Non-Abelian
SXiSiX	Abelian	$\sigma_o \gamma_i \alpha_o \mu_i$	Non-Abelian	-	-	-	$S_{17}^{\gamma} S_{19}^{\gamma} S_{22}^{\gamma} S_{26}^{\gamma} S_{27}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \gamma_i X \alpha_o i S \mu_i X \sigma_o i S \gamma_i X \alpha_o \mu_i X$	Non-Abelian
SYiSiY	Abelian	$\sigma_o \gamma_j \alpha_o \mu_j$	Non-Abelian	-	-	-	$S_{17}^{\gamma} S_{20}^{\gamma} S_{23}^{\gamma} S_{24}^{\gamma} S_{25}^{\gamma} S_{28}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \gamma_j Y \alpha_o i S \mu_j Y \sigma_o i S \gamma_j i Y \alpha_o S \mu_j Y$	Non-Abelian
SZiSiZ	Abelian	$\sigma_o \gamma_k \alpha_o \mu_k$	Non-Abelian	-	-	-	$S_{17}^{\gamma} S_{20}^{\gamma} S_{23}^{\gamma} S_{24}^{\gamma} S_{25}^{\gamma} S_{27}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \gamma_k Z \alpha_o i S \mu_k Z \sigma_o i S \gamma_k i Z \alpha_o S \mu_k Z$	Non-Abelian
SPiSiP	Abelian	$\sigma_o \delta_i \alpha_o \nu_i$	Non-Abelian	-	-	-	$S_{18}^{\gamma} S_{19}^{\gamma} S_{23}^{\gamma} S_{24}^{\gamma} S_{26}^{\gamma} S_{27}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \delta_i P \alpha_o i S \nu_i i P \sigma_o i S \delta_i i P \alpha_o S \nu_i P$	Non-Abelian
SQiSiQ	Abelian	$\sigma_o \delta_j \alpha_o \nu_j$	Non-Abelian	-	-	-	$S_{18}^{\gamma} S_{20}^{\gamma} S_{24}^{\gamma} S_{25}^{\gamma} S_{27}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \delta_j Q \alpha_o i S \nu_j i Q \sigma_o i S \delta_j i Q \alpha_o S \nu_j Q$	Non-Abelian
SRiSiR	Abelian	$\sigma_o \delta_k \alpha_o \nu_k$	Non-Abelian	-	-	-	$S_{18}^{\gamma} S_{20}^{\gamma} S_{22}^{\gamma} S_{23}^{\gamma} S_{25}^{\gamma} S_{26}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \delta_k R \alpha_o i S \nu_k i R \sigma_o i S \delta_k i R \alpha_o S \nu_k R$	Non-Abelian
SDiSiD	Abelian	$\sigma_o \lambda_i \alpha_o \beta_i$	Non-Abelian	-	-	-	$S_{16}^{\gamma} S_{19}^{\gamma} S_{23}^{\gamma} S_{24}^{\gamma} S_{25}^{\gamma} S_{29}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \lambda_i D \alpha_o i S \beta_i i D \sigma_o i S \lambda_i i D \alpha_o S \beta_i D$	Non-Abelian
SEiSiE	Abelian	$\sigma_o \lambda_j \alpha_o \mu_j$	Non-Abelian	-	-	-	$S_{16}^{\gamma} S_{20}^{\gamma} S_{22}^{\gamma} S_{24}^{\gamma} S_{26}^{\gamma} S_{28}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \lambda_j E \alpha_o i S \beta_j i E \sigma_o i S \lambda_j i E \alpha_o S \beta_j E$	Non-Abelian
SFiSiF	Abelian	$\sigma_o \lambda_k \alpha_o \beta_k$	Non-Abelian	-	-	-	$S_{16}^{\gamma} S_{21}^{\gamma} S_{22}^{\gamma} S_{23}^{\gamma} S_{27}^{\gamma} S_{28}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \lambda_k F \alpha_o i S \beta_k i F \sigma_o i S \lambda_k i F \alpha_o S \beta_k F$	Non-Abelian

TABLE 7. Unit elements for aligned \mathbb{M} , \mathbb{T} and \mathbb{W} sub-algebras with 8 unit elements

Unit elements of \mathbb{M}		Unit elements of \mathbb{T} and sedenion-type loop participation					Unit elements of $\mathbb{C} \otimes \mathbb{W}$		
Sub-group	Type	Sub-loop	Type	S_L	S_L^α	S_L^β	S_L^γ	Complexified aligned sub-loop	Type
SViDiLiViDLiS	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \beta_o \lambda_i \alpha_i \lambda_o \beta_i \sigma_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{18}^{\gamma} S_{19}^{\gamma} S_{28}^{\gamma}$	$\sigma_o S \beta_o V \lambda_i D \alpha_i L \lambda_o V \beta_i i D \sigma_i L \alpha_o i$	Split Octonion
SiTiPiLTiPLiS	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \nu_o \delta_i \alpha_i \delta_o \nu_i \sigma_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{16}^{\gamma} S_{19}^{\gamma} S_{25}^{\gamma}$	$\sigma_o S \nu_o i T \delta_i P \alpha_i L \delta_o T \nu_i i P \sigma_i L \alpha_o i$	Split Octonion
SXiUiLiXuLiS	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \gamma_i \mu_o \alpha_i \mu_o \gamma_i \sigma_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{17}^{\gamma} S_{19}^{\gamma} S_{22}^{\gamma}$	$\sigma_o S \gamma_i X \mu_o i U \alpha_i L \mu_i i S \gamma_i U \sigma_i L \alpha_o i$	Split Octonion
SXeRiXiErIeRiS	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \gamma_i \lambda_i \delta_i \delta_o \nu_i \beta_i \nu_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{22}^{\gamma} S_{27}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \gamma_i X \lambda_i E \delta_o R \beta_i i E \nu_i X \beta_i j E \nu_o X \beta_i R \alpha_o i$	Split Octonion
SXQFiXiQiFiS	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \gamma_i \delta_j \lambda_k \mu_i \nu_j \beta_k \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{22}^{\gamma} S_{26}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \gamma_i X \delta_Q \lambda_F \mu_i F \nu_j F \nu_j i Q \beta_k i F \alpha_o i$	Split Octonion
SiveimViEMiS	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \beta_o \lambda_j \alpha_j \lambda_o \beta_j \sigma_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{18}^{\gamma} S_{20}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \beta_o i V \lambda_j E \alpha_j L \lambda_o V \beta_j i E \sigma_j M \alpha_o i$	Split Octonion
SiTiQMTiQMiS	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \nu_o \delta_j \alpha_j \delta_o \nu_j \sigma_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{16}^{\gamma} S_{20}^{\gamma} S_{26}^{\gamma}$	$\sigma_o S \nu_o i T \delta_j Q \alpha_j M \delta_o T \nu_j i Q \sigma_j M \alpha_o i$	Split Octonion
SyiuimYumiS	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \gamma_j \mu_o \alpha_j \mu_j \gamma_o \sigma_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{17}^{\gamma} S_{20}^{\gamma} S_{23}^{\gamma}$	$\sigma_o S \gamma_j Y \mu_o i U \alpha_j i M \mu_j i Y \gamma_o U \sigma_j M \alpha_o i$	Split Octonion
SyppiYipififis	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \gamma_j \delta_i \lambda_k \mu_j \nu_i \beta_k \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{23}^{\gamma} S_{27}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \gamma_j Y \delta_i P \lambda_F \mu_j F \nu_i P \beta_k i F \alpha_o i$	Split Octonion
Sydrividiridris	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \gamma_j \lambda_i \delta_k \mu_j \beta_i \nu_k \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{23}^{\gamma} S_{27}^{\gamma} S_{28}^{\gamma}$	$\sigma_o S \gamma_j Y \lambda_i D \delta_k R \mu_j i Y \beta_i D \nu_k i R \alpha_o i$	Split Octonion
SivfinviFnis	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \beta_o \lambda_o \alpha_o \lambda_o \beta_o \sigma_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{18}^{\gamma} S_{21}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \beta_o i V \lambda_o R \alpha_o i N \lambda_o V \beta_o i F \sigma_o N \alpha_o i$	Split Octonion
Sitrinrnis	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \nu_o \delta_o \alpha_o \delta_o \nu_o \sigma_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{16}^{\gamma} S_{21}^{\gamma} S_{27}^{\gamma}$	$\sigma_o S \nu_o i T \delta_o R \alpha_o i N \delta_o T \nu_o i R \sigma_o N \alpha_o i$	Split Octonion
Szruinuzinis	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \nu_o \mu_o \alpha_o \mu_o \gamma_o \sigma_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{17}^{\gamma} S_{21}^{\gamma} S_{24}^{\gamma}$	$\sigma_o S \gamma_o Z \mu_o i U \alpha_o i N \mu_o Z \gamma_o U \sigma_o N \alpha_o i$	Split Octonion
Szdqizidiqis	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \gamma_i \lambda_i \delta_i \mu_i \beta_i \nu_i \beta_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{24}^{\gamma} S_{26}^{\gamma} S_{28}^{\gamma}$	$\sigma_o S \gamma_i Z \lambda_i D \delta_i Q \mu_o Z \beta_i D \nu_i i Q \alpha_o i$	Split Octonion
Szpeizipeis	$\mathbb{D} \otimes \mathbb{C}$	$\sigma_o \gamma_k \delta_i \lambda_j \mu_k \nu_k \beta_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{24}^{\gamma} S_{25}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \gamma_k Z \delta_i P \lambda_j E \mu_k i Z \nu_i i P \beta_j i E \alpha_o i$	Split Octonion
SlmniLiminiS	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_i \sigma_j \sigma_k \alpha_i \alpha_j \alpha_k \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{16}^{\gamma} S_{17}^{\gamma} S_{18}^{\gamma}$	$\sigma_o S \sigma_i L \sigma_j M \sigma_k N \alpha_i i L \alpha_j i M \alpha_k i N \alpha_o i$	Distributive
Sliefilefies	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_i \sigma_j \beta_k \alpha_i \lambda_j \lambda_k \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{18}^{\gamma} S_{22}^{\gamma} S_{25}^{\gamma}$	$\sigma_o S \sigma_i L \beta_j i E \beta_k i F \alpha_i L \lambda_j E \lambda_k F \alpha_o i$	Distributive
SliQriLQRis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_i \nu_k \nu_n \alpha_i \delta_j \delta_k \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{16}^{\gamma} S_{22}^{\gamma} S_{28}^{\gamma}$	$\sigma_o S \sigma_i L \nu_j i Q \nu_k i R \alpha_i L \delta_j Q \delta_k R \alpha_o i$	Distributive
SlyiZiLyZis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_i \mu_j \mu_k \mu_i \alpha_i \gamma_j \gamma_k \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{17}^{\gamma} S_{22}^{\gamma} S_{28}^{\gamma}$	$\sigma_o S \sigma_i L \mu_j i Z \mu_k i Z \alpha_i L \gamma_j Y \gamma_k Z \alpha_o i$	Distributive
Smifidimpidis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_j \beta_k \beta_i \alpha_j \delta_i \lambda_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{18}^{\gamma} S_{23}^{\gamma} S_{26}^{\gamma}$	$\sigma_o S \sigma_j M \beta_k i F \beta_i D \alpha_j i M \lambda_k F \lambda_i D \alpha_o i$	Distributive
Smripimpris	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_j \mu_k \nu_i \alpha_j \delta_i \delta_k \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{16}^{\gamma} S_{23}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \sigma_j M \nu_k i R \nu_i i P \alpha_j i M \delta_k R \delta_i P \alpha_o i$	Distributive
Smiziximzxis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_j \mu_k \mu_i \alpha_j \gamma_k \gamma_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{17}^{\gamma} S_{26}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \sigma_j M \mu_k Z \mu_i X \alpha_j i M \gamma_k Z \gamma_i X \alpha_o i$	Distributive
Snidihendeis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_k \beta_i \beta_j \alpha_i \lambda_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{18}^{\gamma} S_{24}^{\gamma} S_{27}^{\gamma}$	$\sigma_o S \sigma_k N \beta_i D \beta_j i E \alpha_i K \alpha_i N \lambda_i D \lambda_j E \alpha_o i$	Distributive
SnipiQinpQis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_k \nu_i \nu_j \alpha_k \delta_j \delta_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{16}^{\gamma} S_{24}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \sigma_k N \nu_i i P \nu_j i Q \alpha_i N \delta_i P \delta_j Q \alpha_o i$	Distributive
Snixiyinxysi	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_k \mu_j \mu_j \alpha_k \gamma_i \gamma_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{17}^{\gamma} S_{27}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \sigma_k N \mu_i X \mu_j i Y \alpha_i N \gamma_i X \gamma_j Y \alpha_o i$	Distributive
Svtuivitiuis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \delta_o \gamma_o \beta_o \nu_o \mu_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{19}^{\gamma} S_{20}^{\gamma} S_{21}^{\gamma}$	$\sigma_o S \lambda_o V \delta_o T \gamma_o U \beta_o i V \nu_o i T \mu_o i U \alpha_o i$	Distributive
Svipixvpxis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_i \mu_i \beta_o \delta_i \gamma_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{19}^{\gamma} S_{29}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \lambda_o V \nu_i i P \mu_i X \beta_o i V \delta_i P \gamma_i X \alpha_o i$	Distributive
Stixiditxidis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \delta_o \mu_i \beta_i \nu_o \gamma_i \lambda_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{19}^{\gamma} S_{27}^{\gamma} S_{27}^{\gamma}$	$\sigma_o S \delta_o T \mu_i i S \beta_i D \nu_o i T \gamma_i X \lambda_i D \alpha_o i$	Distributive
Suidipudipis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \gamma_i \beta_i \nu_i \mu_o \lambda_i \delta_i \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{19}^{\gamma} S_{29}^{\gamma} S_{24}^{\gamma}$	$\sigma_o S \gamma_o U \beta_i i D \nu_i i P \mu_o i U \lambda_i D \delta_i P \alpha_o i$	Distributive
Sviqiyivqyis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_j \mu_j \beta_o \delta_j \gamma_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{20}^{\gamma} S_{25}^{\gamma} S_{30}^{\gamma}$	$\sigma_o S \lambda_o V \nu_j i Q \mu_j i Y \beta_o i V \delta_j Q \gamma_j Y \alpha_o i$	Distributive
Stiyietyeis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \delta_o \mu_j \beta_o \nu_o \gamma_j \lambda_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{20}^{\gamma} S_{25}^{\gamma} S_{27}^{\gamma}$	$\sigma_o S \delta_o T \mu_j i D \beta_j E \nu_o i T \gamma_j Y \lambda_j E \alpha_o i$	Distributive
Suheiqueqis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \gamma_o \beta_j \nu_j \mu_o \lambda_j \delta_j \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{20}^{\gamma} S_{22}^{\gamma} S_{24}^{\gamma}$	$\sigma_o S \gamma_o U \beta_j i E \nu_j i Q \mu_o i U \lambda_j E \delta_j Q \alpha_o i$	Distributive
Svirizivrzis	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \mu_o \beta_o \delta_o \gamma_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{21}^{\gamma} S_{25}^{\gamma} S_{29}^{\gamma}$	$\sigma_o S \lambda_o V \nu_o i R \mu_o i U \lambda_o Z \beta_o R \gamma_o Z \alpha_o i$	Distributive
Stizifitzfris	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \delta_o \mu_o \beta_o \nu_o \gamma_o \lambda_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{21}^{\gamma} S_{25}^{\gamma} S_{23}^{\gamma}$	$\sigma_o S \delta_o T \mu_o i Z \beta_o i R \nu_o i U \gamma_o Z \lambda_o F \alpha_o i$	Distributive
Sofifirufuris	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \gamma_o \beta_o \nu_o \mu_o \delta_o \delta_o \alpha_o$	\mathbb{O}_0^c	-	-	-	$S_{21}^{\gamma} S_{22}^{\gamma} S_{23}^{\gamma}$	$\sigma_o S \gamma_o U \beta_o i F \nu_o i R \mu_o i U \lambda_o F \delta_o R \alpha_o i$	Distributive

TABLE 8. Unit elements for aligned \mathbb{M} , \mathbb{T} and \mathbb{W} sub-algebras with 8 unit elements

Unit elements of \mathbb{M}		Unit elements of \mathbb{T} and sedentron-type loop participation				Unit elements of $\mathbb{C} \otimes \mathbb{W}$	
Sub-group	Type	Sub-loop	Type	S_L	S_L^α	S_L^β	S_{18}^7
SLMNDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \sigma_K \lambda_i \lambda_j \lambda_K \lambda_o$	$\tilde{\sigma}_0^c$	S_0	S_1^α	S_{18}^7	$\sigma_o S \sigma_e L \sigma_j M \sigma_K N \lambda_i D \lambda_j E \lambda_K F \lambda_o$
SLIMINIDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \alpha_K \lambda_i \beta_j \beta_K \lambda_o$	$\tilde{\sigma}_0^f$	—	S_9^α	S_{18}^7	$\sigma_o S \sigma_e L \alpha_j i M \alpha_i N \lambda_i D \beta_j E \beta_i F \lambda_o$
SLIMINIDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \alpha_K \beta_i \lambda_j \beta_K \lambda_o$	$\tilde{\sigma}_0^f$	—	S_6^α	S_{10}^β	$\sigma_o S \sigma_e i L \sigma_j M \alpha_i N \beta_i t D \lambda_j E \beta_i F \lambda_o$
SLIMINIDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \alpha_j \sigma_K \beta_i \lambda_j \lambda_K \lambda_o$	$\tilde{\sigma}_0^f$	—	S_7^α	S_{18}^7	$\sigma_o S \alpha_i L \alpha_j i M \sigma_K N \beta_i t D \beta_j E \lambda_K F \lambda_o$
SLIMINIDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \alpha_j \sigma_K \beta_i \beta_j \lambda_K \lambda_o$	$\tilde{\sigma}_0^j$	—	S_8^β	S_{18}^7	$\sigma_o S \alpha_i L \alpha_j i M \sigma_K N \beta_i t D \beta_j E \beta_o i F \lambda_o$
SLIMINIDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \alpha_K \beta_i \beta_j \beta_K \beta_o$	$\tilde{\sigma}_0^j$	—	S_8^β	$S_{15}^7 S_{18}^7$	$\sigma_o S \sigma_e L \sigma_j M \sigma_K N \beta_i t D \beta_j E \beta_o i F \beta_o i$
SLIMINIDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \alpha_K \beta_i \beta_j \lambda_K \beta_o$	$\tilde{\sigma}_0^j$	—	S_2^α	S_{12}^β	$\sigma_o S \sigma_e L \alpha_j i M \alpha_i N \beta_i t D \lambda_j E \lambda_K F \beta_o i$
SLIMINIDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \beta_i \lambda_j \lambda_K \beta_o$	$\tilde{\sigma}_0^f$	—	S_3^α	S_{13}^β	$\sigma_o S \alpha_i L \sigma_j M \alpha_i N \lambda_i D \beta_j E \lambda_K F \beta_o i$
SLIMINIDEFV	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \alpha_j \sigma_K \beta_i \lambda_j \beta_K \beta_o$	$\tilde{\sigma}_0^f$	—	S_4^α	S_{14}^β	$\sigma_o S \alpha_i L \alpha_j i M \sigma_K N \lambda_i D \lambda_j E \beta_K i F \beta_o i$
SLMNPNQRT	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \alpha_j \sigma_K \delta_i \delta_j \delta_K \delta_o$	$\tilde{\sigma}_0^a$	S_0	—	—	$S_{15}^7 S_{18}^7$
SLIMINPQIRT	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \alpha_K \delta_i \nu_j \nu_K \delta_o$	$\tilde{\sigma}_0^f$	—	S_2^α	S_{16}^β	$\sigma_o S \sigma_e L \alpha_j i M \alpha_i N \delta_i P \nu_j i Q \nu_K i R \delta_o$
SLIMINPQIRT	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \nu_i \delta_j \nu_K \delta_o$	$\tilde{\sigma}_0^f$	—	S_3^α	S_{16}^β	$\sigma_o S \alpha_i L \sigma_j M \alpha_i N \nu_i i P \delta_j Q \nu_K i R \delta_o$
SLIMINPQIRT	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \alpha_j \sigma_K \nu_i \nu_j \delta_K \delta_o$	$\tilde{\sigma}_0^f$	—	S_4^α	S_{11}^β	$\sigma_o S \alpha_i L \alpha_j i M \sigma_K N \nu_i i P \nu_j i Q \delta_K R \delta_o$
SLIMINPQIRIT	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \sigma_K \nu_i \nu_j \nu_K \nu_o$	$\tilde{\sigma}_0^f$	—	S_1^α	S_8^β	$\sigma_o S \sigma_e L \sigma_j M \sigma_K N \nu_i i P \nu_j i Q \nu_K i R \nu_o i$
SLIMINPQIRIT	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \sigma_K \delta_i \nu_j \delta_K \nu_o$	$\tilde{\sigma}_0^f$	—	S_5^α	S_{16}^β	$\sigma_o S \sigma_e \alpha_i L \alpha_j i M \alpha_i N \nu_i i P \delta_Q \delta_K R \nu_o i$
SLIMINPQIRIT	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \sigma_K \delta_i \nu_j \delta_K \nu_o$	$\tilde{\sigma}_0^f$	—	S_6^α	S_6^β	$\sigma_o S \alpha_i L \sigma_j M \alpha_i N \delta_i P \nu_j i Q \delta_K R \nu_o i$
SLIMINPQIRIT	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \nu_i \delta_j \nu_K \nu_o$	$\tilde{\sigma}_0^f$	—	S_7^α	S_{14}^β	$\sigma_o S \alpha_i L \alpha_j i M \sigma_K N \nu_i i P \nu_j i Q \nu_K i R \nu_o i$
SLIMINXYZU	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \sigma_j \sigma_K \gamma_i \gamma_j \gamma_K \gamma_o$	$\tilde{\sigma}_0^g$	S_0	—	S_8^β	S_{17}^7
SLIMINXYZU	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \sigma_e \alpha_j \alpha_K \gamma_i \mu_j \mu_K \gamma_o$	$\tilde{\sigma}_0^g$	—	$S_6^g S_{12}^\beta$	S_{17}^7	$\sigma_o S \sigma_e L \alpha_j i M \alpha_i N \gamma_i X \mu_j Y \mu_i Z \gamma_o$
SLIMINXYZU	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \mu_i \nu_j \mu_K \gamma_o$	$\tilde{\sigma}_0^i$	—	$S_{10}^g S_{13}^\beta$	S_{17}^7	$\sigma_o S \sigma_e L \alpha_j i M \alpha_i N \mu_i X \gamma_j Y \mu_i Z \gamma_o$
SLIMINXYZU	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \mu_i \nu_j \mu_K \gamma_o$	$\tilde{\sigma}_0^i$	—	$S_{11}^g S_{14}^\beta$	S_{17}^7	$\sigma_o S \alpha_i L \alpha_j i M \sigma_K N \mu_i X \mu_j Y \gamma_K Z \gamma_o$
SLIMINXYZU	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \mu_i \nu_j \mu_K \mu_o$	$\tilde{\sigma}_0^b$	—	S_1^α	$S_{15}^7 S_{17}^7$	$\sigma_o S \sigma_e L \sigma_j M \sigma_K N \mu_i X \mu_j Y \mu_i Z \gamma_o$
SLIMINXYZU	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \mu_i \nu_j \mu_K \mu_o$	$\tilde{\sigma}_0^h$	—	$S_2^\alpha S_5^\beta$	S_{17}^7	$\sigma_o S \alpha_i L \sigma_j M \alpha_i N \mu_i X \gamma_j Y \mu_i Z \gamma_o$
SLIMINXYZU	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \mu_i \nu_j \mu_K \mu_o$	$\tilde{\sigma}_0^h$	—	$S_3^\alpha S_6^\beta$	S_{17}^7	$\sigma_o S \alpha_i L \sigma_j M \alpha_i N \mu_i X \gamma_j Y \gamma_K Z \mu_o i$
SLIMINXYZU	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \alpha_i \sigma_j \alpha_K \gamma_i \mu_j \mu_K \mu_o$	$\tilde{\sigma}_0^h$	—	$S_4^\alpha S_7^\beta$	S_{17}^7	$\sigma_o S \alpha_i L \alpha_j i M \sigma_K N \gamma_i X \gamma_j Y \gamma_K Z \mu_o i$
SVTUDPXL	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \delta_o \gamma_o \lambda_i \delta_i \gamma_L \sigma_L$	$\tilde{\sigma}_0^g$	S_0	—	S_9^β	S_{19}^7
SVTUDPXL	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \mu_o \lambda_i \nu_i \mu_i \sigma_i$	$\tilde{\sigma}_0^d$	—	$S_1^\alpha S_6^\beta$	S_{19}^7	$\sigma_o S \lambda_o V \nu_o T \mu_o i U \lambda_i D \nu_i P \mu_i X \sigma_i$
SVTUDPXL	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \delta_o \mu_o \beta_i \nu_i \mu_i \sigma_i$	$\tilde{\sigma}_0^k$	—	S_2^α	$S_{15}^7 S_{19}^7$	$\sigma_o S \beta_o i V \delta_o T \mu_o i U \beta_i D \delta_i P \mu_i i X \sigma_i$
SVTUDPXL	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \beta_i \nu_i \gamma_L \sigma_L$	$\tilde{\sigma}_0^i$	—	$S_4^g S_{12}^\beta$	S_{19}^7	$\sigma_o S \beta_o i V \nu_o i U \beta_i D \nu_i P \gamma_i X \sigma_i$
SVTUDPPIXL	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \delta_o \gamma_o \beta_i \nu_i \mu_i \alpha_i$	$\tilde{\sigma}_0^i$	—	$S_{10}^g S_{11}^\beta$	S_{19}^7	$\sigma_o S \lambda_o V \delta_o T \gamma_o U \beta_i D \nu_i P \mu_i X \alpha_i$
SVTUDPPIXL	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \mu_o \beta_i \nu_i \delta_i \gamma_i \alpha_i$	$\tilde{\sigma}_0^h$	—	$S_6^g S_{12}^\beta$	S_{19}^7	$\sigma_o S \lambda_o V \delta_o T \mu_o i U \beta_i D \nu_i P \gamma_i X \alpha_i$
SVTUDPPIXL	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \mu_o \lambda_i \nu_i \gamma_i \alpha_i$	$\tilde{\sigma}_0^h$	—	$S_7^g S_{13}^\beta$	S_{19}^7	$\sigma_o S \beta_o i V \delta_o T \mu_o i U \beta_i D \nu_i P \gamma_i X \alpha_i$
SVTUDPPIXL	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_i \delta_i \mu_i \alpha_i$	$\tilde{\sigma}_0^i$	—	$S_8^g S_4^\beta$	S_{19}^7	$\sigma_o S \beta_o i V \nu_o i T \lambda_i D \delta_i P \mu_i i X \alpha_i$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \mu_o \lambda_j \delta_j \gamma_j \sigma_j$	$\tilde{\sigma}_0^g$	S_0	—	S_{10}^β	S_{20}^7
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \mu_o \lambda_j \delta_j \gamma_j \sigma_j$	$\tilde{\sigma}_0^d$	—	$S_1^\alpha S_6^\beta$	S_{20}^7	$\sigma_o S \lambda_o V \nu_o T \mu_o i U \lambda_j D \delta_j Q \mu_j Y \sigma_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \mu_o \lambda_j \delta_j \gamma_j \sigma_j$	$\tilde{\sigma}_0^k$	—	$S_5^\alpha S_{20}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \delta_o T \mu_o i U \beta_j E \delta_j Q \mu_j Y \sigma_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \beta_j \nu_j \mu_j \sigma_j$	$\tilde{\sigma}_0^i$	—	$S_8^g S_{13}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \nu_j i Q \gamma_j Y \sigma_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \gamma_o \beta_j \nu_j \mu_j \alpha_j$	$\tilde{\sigma}_0^i$	—	$S_9^g S_{11}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \nu_j i Q \mu_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \gamma_o \beta_j \nu_j \mu_j \alpha_j$	$\tilde{\sigma}_0^h$	—	$S_5^g S_7^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \mu_o i U \beta_j E \delta_j Q \gamma_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \delta_o \lambda_j \nu_j \gamma_j \alpha_j$	$\tilde{\sigma}_0^h$	—	$S_2^\alpha S_4^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \delta_o T \mu_o i U \beta_j E \nu_j i Q \gamma_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \delta_o \lambda_j \nu_j \gamma_j \alpha_j$	$\tilde{\sigma}_0^i$	—	$S_{12}^g S_{14}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \mu_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^g$	—	$S_1^\alpha S_6^\beta$	S_{20}^7	$\sigma_o S \lambda_o V \delta_o T \gamma_o U \lambda_j D \delta_j P \mu_i X \alpha_i$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^d$	—	$S_2^\alpha S_{13}^\beta$	S_{20}^7	$\sigma_o S \lambda_o V \delta_o T \mu_o i U \beta_j E \delta_j P \gamma_j X \alpha_i$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^k$	—	$S_3^\alpha S_{13}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \delta_o T \mu_o i U \beta_j E \nu_j i P \gamma_j X \alpha_i$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^i$	—	$S_4^\alpha S_{14}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \gamma_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^h$	—	$S_5^\alpha S_{21}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \mu_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^h$	—	$S_6^\alpha S_{20}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \mu_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^i$	—	$S_7^\alpha S_{21}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \mu_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^i$	—	$S_8^\alpha S_{21}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \mu_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^h$	—	$S_9^\alpha S_{21}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \mu_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^h$	—	$S_{10}^\alpha S_{21}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \mu_j Y \alpha_j$
SVTUEQYM	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_j \delta_j \nu_j \sigma_j$	$\tilde{\sigma}_0^i$	—	$S_{12}^g S_{14}^\beta$	S_{20}^7	$\sigma_o S \beta_o i V \nu_o T \beta_j E \delta_j Q \mu_j Y \alpha_j$
SVTUFIRZIN	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \delta_o \gamma_o \beta_o \nu_o \mu_o \alpha_o$	$\tilde{\sigma}_0^i$	—	$S_9^g S_{10}^\beta$	S_{21}^7	$\sigma_o S \lambda_o V \delta_o T \gamma_o U \beta_o i F \nu_o i R \mu_o i Z \alpha_o i$
SVTUFIRZIN	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \lambda_o \nu_o \mu_o \beta_o \delta_o \gamma_o \alpha_o$	$\tilde{\sigma}_0^h$	—	$S_5^g S_6^\beta$	S_{21}^7	$\sigma_o S \lambda_o V \nu_o T \beta_o i F \delta_o R \gamma_o Z \alpha_o i$
SVTUFIRZIN	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \delta_o \mu_o \lambda_o \nu_o \gamma_o \alpha_o$	$\tilde{\sigma}_0^h$	—	$S_2^g S_3^\beta$	S_{21}^7	$\sigma_o S \beta_o V \delta_o T \mu_o i U \beta_o i F \delta_o R \gamma_o Z \alpha_o i$
SVTUFIRZIN	$\mathbb{H} \otimes \mathbb{C}$	$\sigma_o \beta_o \nu_o \gamma_o \lambda_o \delta_o \nu_o \mu_o \alpha_o$	$\tilde{\sigma}_0^i$	—	$S_{12}^g S_{13}^\beta$	S_{21}^7	$\sigma_o S \beta_o V \nu_o T \gamma_o U \lambda_o F \delta_o R \mu_o i Z \alpha_o i$

TABLE 9. Unit elements for aligned \mathbb{M} , \mathbb{T} and \mathbb{W} sub-algebras with 8 unit elements

TABLE 10. Unit elements for aligned \mathbb{M} , \mathbb{T} and \mathbb{W} sub-algebras with 16 unit elements

Ref.	M sub-group	Ref.	T sub-loop	W sub-loop
	Unit elements	Ref.	Unit elements	Unit elements
E ₀	$SLMNUXYZ$ $VDEFPTPQR$	S ₀	$\sigma_o \sigma_i \sigma_j \sigma_k \gamma_o \gamma_i \gamma_j \gamma_k$ $\lambda_o \alpha_L \lambda_j \alpha_K \delta_o \delta_i \delta_j \delta_K$	$\sigma_o S \sigma_i L \sigma_j M \sigma_k N \gamma_o U \gamma_i X \gamma_j Y \gamma_k Z$ $\lambda_o V \lambda_t D \lambda_j E \lambda_K F \delta_o T \delta_i P \delta_j Q \delta_K R$
E ₁	$SLMNViDiEiF$ $iUiXiYiZTPQR$	S ₁	$\sigma_o \sigma_i \sigma_j \sigma_k \beta_o \beta_i \beta_j \beta_K$ $\mu_o \mu_i \mu_j \mu_k \delta_o \delta_i \delta_j \delta_K$	$\sigma_o S \sigma_i L \sigma_j M \sigma_k N \beta_o V \beta_i iD \beta_j iE \beta_K iF$ $\mu_o iU \mu_i X \mu_j Y \mu_k iZ \delta_o T \delta_i P \delta_j Q \delta_K R$
E ₂	$SLUXViDiTiP$ $iMiNiYiZEFPQR$	S ₂	$\sigma_o \sigma_i \gamma_o \gamma_i \beta_o \beta_i \nu_o \nu_i$ $\alpha_j \alpha_K \mu_j \mu_k \lambda_j \lambda_K \delta_o \delta_K$	$\sigma_o S \sigma_i L \gamma_o U \gamma_i X \beta_o iV \beta_i iD \nu_o iT \nu_i iP$ $\alpha_j iM \alpha_k iN \mu_j Y \mu_k iZ \lambda_j E \lambda_K F \delta_j Q \delta_K R$
E ₃	$SMUYiViEiTQ$ $iLiNiXiZDFPR$	S ₃	$\sigma_o \sigma_j \gamma_o \gamma_j \beta_o \beta_j \nu_o \nu_j$ $\alpha_i \alpha_K \mu_i \mu_k \lambda_i \lambda_K \delta_o \delta_K$	$\sigma_o S \sigma_j M \gamma_o U \gamma_j Y \beta_o iV \beta_j iE \nu_o iT \nu_j iQ$ $\alpha_i iL \alpha_k iN \mu_i X \mu_k iZ \lambda_t D \lambda_K F \delta_t P \delta_K R$
E ₄	$SNUViFiTiR$ $iLiMiXiYDEPQ$	S ₄	$\sigma_o \sigma_k \gamma_o \gamma_j \beta_o \beta_k \nu_o \nu_K$ $\alpha_i \alpha_j \mu_i \mu_j \lambda_i \lambda_j \delta_t \delta_J$	$\sigma_o S \sigma_k N \gamma_o U \gamma_Z \beta_o iV \beta_k iF \nu_o iT \nu_K iR$ $\alpha_i iL \alpha_j iM \mu_i X \mu_j Y \lambda_t D \lambda_J E \delta_t P \delta_J Q$
E ₅	$SLYZiViDiQiR$ $iMiNiUiXEFPT$	S ₅	$\sigma_o \sigma_i \gamma_j \gamma_K \beta_o \beta_i \nu_j \nu_K$ $\alpha_j \alpha_K \mu_o \mu_l \lambda_j \lambda_K \delta_o \delta_L$	$\sigma_o S \sigma_i L \gamma_j Y \gamma_K Z \beta_o iV \beta_i iD \nu_j iQ \nu_K iR$ $\alpha_j iM \alpha_k iN \mu_o iU \mu_i X \lambda_j E \lambda_K F \delta_o T \delta_i P$
E ₆	$SMXZiViEiPiR$ $iLiNiUiYDFTQ$	S ₆	$\sigma_o \sigma_j \gamma_i \gamma_K \beta_o \beta_j \nu_i \nu_K$ $\alpha_i \alpha_K \mu_o \mu_j \lambda_i \lambda_j \delta_o \delta_J$	$\sigma_o S \sigma_j M \gamma_i X \gamma_K Z \beta_o iV \beta_j iE \nu_i iP \nu_K iR$ $\alpha_i iL \alpha_k iN \mu_o iU \mu_j Y \lambda_t D \lambda_K F \delta_o T \delta_J Q$
E ₇	$SNXYiViPiQ$ $iLiMiUiZDETR$	S ₇	$\sigma_o \sigma_k \gamma_U \gamma_j \beta_o \beta_K \nu_i \nu_j$ $\alpha_i \alpha_j \mu_o \mu_K \lambda_i \lambda_j \delta_o \delta_K$	$\sigma_o S \sigma_K N \gamma_i X \gamma_j Y \beta_o iV \beta_K iF \nu_i iP \nu_j iQ$ $\alpha_i iL \alpha_j iM \mu_o iU \mu_K iZ \lambda_t D \lambda_J E \delta_o T \delta_K R$
E ₈	$SLMNViDiPiQ$ $iUiXiYiZVDEF$	S ₈	$\sigma_o \sigma_i \sigma_K \sigma_o \nu_i \nu_j \nu_K$ $\mu_o \mu_i \mu_j \mu_k \lambda_o \lambda_j \lambda_K$	$\sigma_o S \sigma_i L \sigma_M \sigma_K N \nu_o T \nu_i iP \nu_j iQ \nu_K iR$ $\mu_o iU \mu_i X \mu_j Y \mu_k iZ \lambda_o V \lambda_t D \lambda_J E \lambda_K F$
E ₉	$SLUXViEiFiQiR$ $iMiNiYiZVDT$	S ₉	$\sigma_o \sigma_i \gamma_o \gamma_i \beta_o \beta_K \nu_i \nu_K$ $\alpha_j \alpha_K \mu_j \mu_K \lambda_o \lambda_i \delta_o \delta_L$	$\sigma_o S \sigma_i L \gamma_o U \gamma_i X \beta_i E \beta_K iF \nu_j iQ \nu_K iR$ $\alpha_j iM \alpha_k iN \mu_j Y \mu_K iZ \lambda_o V \lambda_t D \delta_o T \delta_L P$
E ₁₀	$SMUYiDiPiR$ $iLiNiXiZVETQ$	S ₁₀	$\sigma_o \sigma_j \gamma_o \gamma_j \beta_i \beta_K \nu_i \nu_K$ $\alpha_i \alpha_K \mu_i \mu_K \lambda_o \lambda_j \delta_o \delta_J$	$\sigma_o S \sigma_j M \gamma_o U \gamma_j Y \beta_i iD \beta_K iF \nu_i iP \nu_K iR$ $\alpha_i iL \alpha_k iN \mu_i X \mu_j Y \lambda_o V \lambda_J E \delta_o T \delta_J Q$
E ₁₁	$SNUZiDiEiPiQ$ $iLiMiXiYVFT$	S ₁₁	$\sigma_o \sigma_K \gamma_i \gamma_K \beta_i \beta_j \nu_i \nu_j$ $\alpha_i \alpha_j \mu_i \mu_j \lambda_o \lambda_K \delta_o \delta_K$	$\sigma_o S \sigma_K N \gamma_o U \gamma_K Z \beta_i iD \beta_j iE \nu_i iP \nu_j iQ$ $\alpha_i iL \alpha_j iM \mu_i X \mu_j Y \lambda_o V \lambda_K F \delta_o T \delta_K R$
E ₁₂	$SLYZiEiFiTiP$ $iMiNiUiXVDQR$	S ₁₂	$\sigma_o \sigma_i \gamma_j \gamma_K \beta_j \beta_K \nu_o \nu_i$ $\alpha_j \alpha_K \mu_o \mu_l \lambda_o \lambda_j \delta_o \delta_K$	$\sigma_o S \sigma_i L \gamma_j Y \gamma_K Z \beta_j iE \beta_K iF \nu_o iT \nu_i iP$ $\alpha_j iM \alpha_k iN \mu_o iU \mu_i X \lambda_o V \lambda_t D \delta_J Q \delta_K R$
E ₁₃	$SMXZiDiTiQ$ $iLiNiUiYVEPR$	S ₁₃	$\sigma_o \sigma_j \gamma_i \gamma_K \beta_i \beta_K \nu_o \nu_j$ $\alpha_i \alpha_K \mu_o \mu_j \lambda_o \lambda_j \delta_o \delta_K$	$\sigma_o S \sigma_j M \gamma_i X \gamma_K Z \beta_i iD \beta_K iF \nu_o iT \nu_j iQ$ $\alpha_i iL \alpha_k iN \mu_o iU \mu_j Y \lambda_o V \lambda_J E \delta_o P \delta_K R$
E ₁₄	$SNXYiDiEiT$ $iLiMiUiZVFPQ$	S ₁₄	$\sigma_o \sigma_K \gamma_i \gamma_j \beta_i \beta_j \nu_o \nu_K$ $\alpha_i \alpha_j \mu_o \mu_K \lambda_o \lambda_K \delta_o \delta_J$	$\sigma_o S \sigma_K N \gamma_i X \gamma_j Y \beta_i iD \beta_j iE \nu_o iT \nu_K iR$ $\alpha_i iL \alpha_j iM \mu_o iU \mu_K iZ \lambda_o V \lambda_K F \delta_o T \delta_J Q$
E ₁₅	$SLMNUXYZ$ $iViDiEiFiTiPiR$	S ₁₅	$\sigma_o \sigma_i \sigma_j \sigma_K \gamma_o \gamma_i \gamma_j \gamma_K$ $\beta_o \beta_i \beta_j \beta_K \nu_o \nu_i \nu_j \nu_K$	$\sigma_o S \sigma_i L \sigma_j M \sigma_K N \gamma_o U \gamma_i X \gamma_j Y \gamma_K Z$ $\beta_o iV \beta_i iD \beta_j iE \beta_K iF \nu_o iT \nu_i iP \nu_j iQ \nu_K iR$
E ₁₆	$SLMNUXYZ$ $iSiLiMiNiUiXiYiZ$	S ₁₆	$\sigma_o \sigma_i \sigma_K \sigma_o \gamma_i \gamma_j \gamma_K$ $\alpha_o \alpha_i \alpha_j \alpha_K \mu_o \mu_j \mu_K$	$\sigma_o S \sigma_i L \sigma_j M \sigma_o N \gamma_o U \gamma_i X \gamma_j Y \gamma_K Z$ $\alpha_o iS \alpha_i iL \alpha_j iM \alpha_K iN \mu_o iU \mu_i X \mu_j Y \mu_K iZ$
E ₁₇	$SLMNViDiEiF$ $iSiLiMiNVDEF$	S ₁₇	$\sigma_o \sigma_i \sigma_K \sigma_o \beta_o \beta_i \beta_j \beta_K$ $\alpha_o \alpha_i \alpha_j \alpha_K \lambda_o \lambda_j \lambda_K$	$\sigma_o S \sigma_i L \sigma_j M \sigma_K N \beta_o V \beta_i iD \beta_j iE \beta_K iF$ $\alpha_o iS \alpha_i iL \alpha_j iM \alpha_K iN \lambda_o V \lambda_t D \lambda_J E \lambda_K F$
E ₁₈	$SLMNViTiPiQ$ $iSiLiMiNTPQR$	S ₁₈	$\sigma_o \sigma_i \sigma_j \sigma_K \nu_o \nu_i \nu_j \nu_K$ $\alpha_o \alpha_i \alpha_j \alpha_K \delta_o \delta_i \delta_j \delta_K$	$\sigma_o S \sigma_i L \sigma_j M \sigma_K N \nu_o iT \nu_i iP \nu_j iQ \nu_K iR$ $\alpha_o iS \alpha_i iL \alpha_j iM \alpha_K iN \delta_o T \delta_i P \delta_J Q \delta_K R$
E ₁₉	$SLUXViDiTiP$ $iSiLiUiXVDT$	S ₁₉	$\sigma_o \sigma_i \gamma_o \gamma_j \beta_o \beta_i \nu_o \nu_i$ $\alpha_o \alpha_i \mu_o \mu_i \lambda_o \lambda_i \delta_o \delta_L$	$\sigma_o S \sigma_i L \gamma_o U \gamma_i X \beta_o iV \beta_i iD \nu_o iT \nu_i iP$ $\alpha_o iS \alpha_i iL \mu_o U \mu_i iZ \lambda_o V \lambda_t D \delta_o T \delta_i P$
E ₂₀	$SMUYiViTiQ$ $iSiMiUiYVETQ$	S ₂₀	$\sigma_o \sigma_j \gamma_o \gamma_j \beta_o \beta_i \nu_o \nu_j$ $\alpha_o \alpha_j \mu_o \mu_j \lambda_o \lambda_j \delta_o \delta_J$	$\sigma_o S \sigma_j M \gamma_o U \gamma_j Y \beta_o iV \beta_j iE \nu_o iT \nu_j iQ$ $\alpha_o iS \alpha_j iM \mu_o iU \mu_j Y \lambda_o V \lambda_J E \delta_o T \delta_J Q$
E ₂₁	$SNUZiViFiTiR$ $iSiNiUiZVFT$	S ₂₁	$\sigma_o \sigma_K \gamma_o \gamma_K \beta_o \beta_K \nu_o \nu_K$ $\alpha_o \alpha_K \mu_o \mu_K \lambda_o \lambda_K \delta_o \delta_K$	$\sigma_o S \sigma_K N \gamma_o U \gamma_K Z \beta_o iV \beta_K iF \nu_o iT \nu_K iR$ $\alpha_o iS \alpha_K iN \mu_o iU \mu_K iZ \lambda_o V \lambda_K F \delta_o T \delta_K R$
E ₂₂	$SLYZiViDiQiR$ $iSiLiUiZVDFQR$	S ₂₂	$\sigma_o \sigma_i \gamma_j \gamma_K \beta_o \beta_i \nu_j \nu_K$ $\alpha_o \alpha_i \mu_j \mu_K \lambda_o \lambda_j \delta_o \delta_K$	$\sigma_o S \sigma_i L \gamma_j Y \gamma_K Z \beta_o iV \beta_i iD \nu_j iQ \nu_i iR$ $\alpha_o iS \alpha_i iL \mu_j Y \mu_K iZ \lambda_o V \lambda_t D \delta_J Q \delta_K R$
E ₂₃	$SMXZiViEiPiR$ $iSiMiXiZVEPR$	S ₂₃	$\sigma_o \sigma_j \gamma_i \gamma_K \beta_o \beta_j \nu_i \nu_K$ $\alpha_o \alpha_j \mu_i \mu_K \lambda_o \lambda_j \delta_o \delta_K$	$\sigma_o S \sigma_j M \gamma_i X \gamma_K Z \beta_o iV \beta_j iE \nu_i iP \nu_K iR$ $\alpha_o iS \alpha_j iM \mu_i X \mu_K iZ \lambda_o V \lambda_J E \delta_t P \delta_K R$
E ₂₄	$SNXYiViFiPiQ$ $iSiNiXiYVFPQ$	S ₂₄	$\sigma_o \sigma_j \gamma_i \gamma_j \beta_o \beta_K \nu_i \nu_j$ $\alpha_o \alpha_K \mu_o \mu_Q \lambda_o \lambda_K \delta_o \delta_Q$	$\sigma_o S \sigma_K N \gamma_i X \gamma_j Y \beta_o iV \beta_K iF \nu_i iP \nu_j iQ$ $\alpha_o iS \alpha_K iN \mu_i X \mu_j Y \lambda_o V \lambda_K F \delta_o T \delta_Q R$
E ₂₅	$SLUXViEiFiQiR$ $iSiLiUiXEFQR$	S ₂₅	$\sigma_o \sigma_i \gamma_i \gamma_K \beta_j \beta_K \nu_i \nu_K$ $\alpha_o \alpha_i \mu_o \mu_j \lambda_o \lambda_K \delta_o \delta_K$	$\sigma_o S \sigma_i L \gamma_o U \gamma_i X \beta_i E \beta_K iF \nu_j iQ \nu_K iR$ $\alpha_o iS \alpha_i iL \mu_o U \mu_i X \lambda_j E \lambda_K F \delta_J Q \delta_K R$
E ₂₆	$SMUYiDiPiR$ $iSiMiUiYDFPR$	S ₂₆	$\sigma_o \sigma_j \gamma_i \gamma_j \beta_i \beta_K \nu_i \nu_K$ $\alpha_o \alpha_j \mu_o \mu_i \lambda_o \lambda_K \delta_o \delta_K$	$\sigma_o S \sigma_j M \gamma_o U \gamma_j Y \beta_i iD \beta_K iF \nu_i iP \nu_K iR$ $\alpha_o iS \alpha_j iM \mu_o U \mu_j Y \lambda_t D \lambda_K F \delta_t P \delta_K R$
E ₂₇	$SNUZiDiEiPiQ$ $iSiNiUiZDEPQ$	S ₂₇	$\sigma_o \sigma_K \gamma_o \gamma_K \beta_i \beta_j \nu_i \nu_j$ $\alpha_o \alpha_K \mu_o \mu_K \lambda_i \lambda_j \delta_o \delta_J$	$\sigma_o S \sigma_K N \gamma_o U \gamma_K Z \beta_i iD \beta_j iE \nu_i iP \nu_j iQ$ $\alpha_o iS \alpha_K iN \mu_o iU \mu_K iZ \lambda_t D \lambda_J E \delta_t P \delta_J Q$
E ₂₈	$SLYZiViFiTiP$ $iSiLiUiZEFTP$	S ₂₈	$\sigma_o \sigma_i \gamma_j \gamma_K \beta_j \beta_K \nu_o \nu_i$ $\alpha_o \alpha_i \mu_j \mu_K \lambda_j \lambda_K \delta_o \delta_L$	$\sigma_o S \sigma_i L \gamma_j Y \gamma_K Z \beta_j iE \beta_K iF \nu_o iT \nu_i iP$ $\alpha_o iS \alpha_i L \mu_j Y \mu_K iZ \lambda_J E \lambda_K F \delta_o T \delta_P$
E ₂₉	$SMXZiDiFiTiQ$ $iSiMiXiZDFTQ$	S ₂₉	$\sigma_o \sigma_j \gamma_i \gamma_K \beta_K \beta_K \nu_o \nu_j$ $\alpha_o \alpha_j \mu_i \mu_K \lambda_i \lambda_K \delta_o \delta_J$	$\sigma_o S \sigma_j M \gamma_i X \gamma_K Z \beta_i iD \beta_K iF \nu_o iT \nu_j iQ$ $\alpha_o iS \alpha_j iM \mu_i X \mu_K iZ \lambda_t D \lambda_K F \delta_o T \delta_J Q$
E ₃₀	$SNXYiDiEiT$ $iSiNiXiYDETR$	S ₃₀	$\sigma_o \sigma_K \gamma_i \gamma_j \beta_i \beta_j \nu_o \nu_K$ $\alpha_o \alpha_K \mu_i \mu_j \lambda_i \lambda_j \delta_o \delta_K$	$\sigma_o S \sigma_K N \gamma_i X \gamma_j Y \beta_i iD \beta_j iE \nu_o iT \nu_K iR$ $\alpha_o iS \alpha_K iN \mu_i X \mu_j Y \lambda_t D \lambda_J E \delta_o T \delta_K R$

8. Basis of the notation used for unit elements of $\mathbb{M} \cong M_4(C)$, and its Cayley table

Capital roman letters are used to label real matrix unit elements of $M_4(R)$, as shown in table 11. These labels are combined with i to represent imaginary counterparts. Their Cayley table is shown in table 12.

TABLE 11. Notation used to label 4×4 unit matrices

$$\begin{aligned}
S &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & L &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & M &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} & N &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \\
V &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} & D &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & E &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} & F &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
iU &= \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} & iX &= \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} & iY &= \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} & iZ &= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \\
iT &= \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} & iP &= \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} & iQ &= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} & iR &= \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix} \\
iS &= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} & iL &= \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} & iM &= \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} & iN &= \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \\
iV &= \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} & iD &= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} & iE &= \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} & iF &= \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \\
U &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} & X &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} & Y &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & Z &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & P &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} & Q &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} & R &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}
\end{aligned}$$

Note that some matrix labels have been changed from those used in a previous paper [34].

Observations of structure of a possible unification algebra

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 TABLE 12. Labels and Cayley table for $\mathbb{M} \cong M_4(C)$

Ref no.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31		
Label	S	L	M	N	V	D	E	F	U	iX	iY	iZ	iT	iP	iQ	iR	iS	iL	iM	iN	iV	iD	iE	iF	U	X	Y	Z	T	P	Q	R		
S	+S	+L	+M	+N	+V	+D	+E	+F	+U	+iX	+iY	+iZ	+iT	+iP	+iQ	+iR	+iS	+iL	+iM	+iN	+iV	+iD	+iE	+iF	+U	+X	+Y	+Z	+T	+P	+Q	+R		
L	+L	-S	+N	-M	-D	+V	-E	+F	+G	+iX	+iY	+iZ	+iT	+iP	+iT	+iQ	+iR	+iS	+iN	+iM	+iD	+iV	+iE	+iF	+U	-Z	-Y	-P	+T	-R	+Q			
M	+M	-N	-S	+L	+E	-F	-V	+D	+E	+iX	+iY	+iZ	+iT	+iR	+iT	+iP	+iM	-iN	-iS	-iL	-iE	-iF	-iG	-iH	+E	+X	-U	+Z	-Y	-P	+T	-R	-Q	
N	+N	+M	-L	-S	-F	-E	+D	+V	-iZ	-iY	+iX	+iU	+iV	+iQ	-iP	-iT	+iN	+iM	-iL	-iS	-iL	-iE	-iF	-iG	-iH	+D	+V	-Z	-Y	+U	+R	+Q	-P	
V	+V	-D	+E	-F	-E	-D	+V	+iZ	-iY	+iX	+iU	+iV	+iQ	-iP	-iT	+iN	+iM	-iL	-iS	-iL	-iE	-iF	-iG	-iH	+D	+V	-Z	-Y	+U	+R	+Q	-P		
D	+D	+V	+F	+E	+E	+S	+N	+M	+P	+T	+E	+Q	+G	+D	+U	+Z	+D	+D	+V	+E	+F	+G	+H	+I	+S	+N	+M	+P	+T	+E	+Q	+Z	+Y	
E	+E	+F	-V	-D	-E	-F	-V	+D	+E	+F	+G	+H	+I	+J	+K	+L	+M	+N	+O	+P	+Q	+R	+S	+T	+U	+V	+W	+X	+Y	+Z	-U	-X		
F	+F	-E	-D	-V	+N	-M	-L	-S	-F	-E	-D	-Q	-P	-T	-Z	-Y	-X	-W	-V	-U	-X	-Y	-Z	-U	-T	-S	-R	-Q	-P	-T	-Z	-Y	-X	-U
iU	+iU	+iX	-iY	-iZ	-iT	-iP	-iQ	+iR	+iS	+iL	-iM	-iN	-iV	-iD	-iV	-iN	-iM	-iL	-iS	-iR	-iU	-iW	-iX	-iY	-iZ	-iT	-iP	-iQ	-iR	-iU	-iV	-iW		
iX	+iX	-iU	-iZ	-iY	-iP	-iT	-iR	-iQ	+iL	-iM	-iN	-iV	-iD	-iE	-iF	-iG	-iH	-iI	-iJ	-iK	-iL	-iS	-iT	-iP	-iQ	-iR	-iU	-iV	-iW	-iX	-iY	-iZ		
iY	+iY	-iZ	-iU	-iX	-iQ	-iR	-iT	-iP	-iM	-iN	-iS	-iL	-iE	-iF	-iG	-iH	-iI	-iJ	-iK	-iL	-iS	-iT	-iP	-iQ	-iR	-iU	-iV	-iW	-iX	-iY	-iZ			
iZ	+iZ	-iY	-iX	-iU	-iV	-iQ	-iP	-iT	-iR	-iM	-iL	-iN	-iV	-iD	-iE	-iF	-iG	-iH	-iI	-iJ	-iK	-iL	-iS	-iT	-iP	-iQ	-iR	-iU	-iV	-iW	-iX	-iY	-iZ	
iT	+iT	-iT	-iP	-iQ	-iR	-iU	-iV	-iZ	-iY	-iX	-iL	-iM	-iF	-iG	-iH	-iI	-iJ	-iK	-iL	-iS	-iT	-iP	-iQ	-iR	-iU	-iV	-iW	-iX	-iY	-iZ	-iT			
iP	+iP	-iT	-iQ	-iR	-iU	-iV	-iZ	-iY	-iX	-iL	-iM	-iN	-iV	-iD	-iE	-iF	-iG	-iH	-iI	-iL	-iS	-iT	-iP	-iQ	-iR	-iU	-iV	-iW	-iX	-iY	-iZ	-iP		
iQ	+iQ	+R	+T	+P	+V	+Z	+U	+X	+E	+F	+G	+H	+I	+J	+K	+L	+M	+N	+O	+P	+Q	+R	+S	+T	+U	+V	+W	+X	+Y	+Z	+U	+V	+W	
iR	+iR	-iQ	+iP	-iT	-iZ	-iR	-iU	-iV	-iX	-iY	-iL	-iM	-iN	-iV	-iD	-iE	-iF	-iG	-iH	-iI	-iL	-iS	-iT	-iP	-iQ	-iR	-iU	-iV	-iW	-iX	-iY	-iZ		
iS	+iS	+L	+M	+N	+V	+W	+X	+Y	+Z	+U	+T	+R	+Q	+P	+O	+N	+M	+L	+S	+D	+E	+F	+G	+H	+I	+J	+K	+L	+M	+N	+O	+P		
iL	+iL	-iS	+iN	-iM	-iD	-iV	-iE	-iX	-iU	-iZ	-iY	-iP	-iT	-iR	-iQ	-iL	-iN	-iM	-iS	-iD	-iV	-iE	-iX	-iU	-iZ	-iY	-iP	-iT	-iR	-iQ	-iL			
iM	+iM	-iN	-iM	-iD	-iV	-iE	-iX	-iU	-iZ	-iY	-iP	-iT	-iR	-iQ	-iL	-iN	-iM	-iS	-iD	-iV	-iE	-iX	-iU	-iZ	-iY	-iP	-iT	-iR	-iQ	-iL	-iM			
iN	+iN	-iM	-iL	-iS	-iF	-iE	-iD	-iV	-iW	-iZ	-iY	-iX	-iU	-iT	-iR	-iQ	-iN	-iM	-iL	-iS	-iF	-iE	-iD	-iV	-iW	-iZ	-iY	-iX	-iU	-iT	-iR	-iQ		
iV	+iV	-iP	-iQ	-iR	-iU	-iV	-iZ	-iY	-iX	-iL	-iM	-iN	-iV	-iD	-iE	-iF	-iG	-iH	-iI	-iL	-iS	-iT	-iP	-iQ	-iR	-iU	-iV	-iW	-iX	-iY	-iZ	-iV		
iD	+iD	+V	+F	+E	+L	+S	+N	+M	+P	+T	+R	+Q	+O	+N	+U	+Z	+Y	+X	+W	+V	+U	+T	+S	+N	+M	+P	+T	+R	+Q	+O	+N	+U		
iE	+iE	+F	+E	+L	+S	+N	+M	+P	+T	+R	+Q	+O	+N	+U	+Z	+Y	+X	+W	+V	+U	+T	+S	+N	+M	+P	+T	+R	+Q	+O	+N	+U			
iF	+iF	-iE	-iD	+V	+N	+M	+L	+I	+S	+R	+Q	+P	+T	+Z	+Y	+X	+U	+F	+E	+D	+V	+N	+M	+L	+I	+S	+R	+Q	+P	+T	+Z	+Y	+X	+U
U	+U	+X	-Y	-Z	-T	-P	+Q	+R	-iS	-iL	+iM	+iN	+iV	+iD	-iE	-iF	+iU	+iX	-iY	-iZ	-iT	-iP	+iQ	-iR	-iS	-L	+M	+N	+V	+D	-E	-F		
X	+X	-U	-Z	-T	-P	-R	-Q	-R	-iS	-iL	-iM	-iN	-iV	-iD	-iE	-iF	-iG	-iH	-iI	-iL	-iS	-iT	-iP	-iQ	-iR	-iU	-iV	-iW	-iX	-iY	-iZ	-iP		
Y	+Y	-Z	-U	-X	-Q	-R	-T	-P	-iM	-iN	-iS	-iL	-iE	-iF	-iG	-iH	-iI	-iJ	-iL	-iS	-iT	-iP	-iM	-iN	-iS	-L	-E	+F	-V	+D	+W			
Z	+Z	+Y	+X	+U	+R	+Q	+P	+T	+E	+F	+G	+H	+I	+J	+K	+L	+M	+N	+O	+P	+Q	+R	+S	+T	+U	+V	+W	+X	+Y	+Z	+U	+V	+W	+D
T	+T	-P	-Q	-R	-U	-X	-Y	-Z	-iV	-iD	-iE	-iF	-iG	-iH	-iI	-iM	-iN	-iT	-iP	-iQ	-iR	-iS	-iL	-iM	-iN	-iT	-iP	-iQ	-iR	-iS	-L	+M	-N	
P	+P	+T	+R	-Q	-X	-U	-Z	-Y	-iD	-iV	-iW	-iX	-iS	-iN	-iM	-iL	-iO	-iP	-iQ	-iR	-iS	-iT	-iP	-iR	-iQ	-iL	-iM	-iN	-iO	-iP	-iQ	-iR	-iS	-N
Q	+Q	+R	+T	+P	+Y	+Z	+U	+X	+E	+F	+V	+W	+X	+D	+M	+N	+S	+I	+S	+R	+T	+P	+Q	+R	+U	+V	+W	+X	+Y	+Z	+U	+V	+W	+D
R	+R	-Q	+P	-T	-Z	+Y	-X	+U	+iF	-iE	+iD	-iV	-iN	+iM	-iL	+iS	+iR	-iQ	-iP	-iT	-iZ	+iY	-iX	+iU	+F	-E	+D	-V	-N	+M	-L	+S		

 TABLE 13. Labels and Cayley table for \mathbb{T} basis elements

Ref no.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	
Label	σ_o	σ_i	σ_j	σ_k	λ_o	λ_i	λ_j	λ_k	μ_o	μ_i	μ_j	μ_k	ν_o	ν_i	ν_j	ν_k	α_o	α_i	α_j	α_k	β_o	β_i	β_j	β_k	γ_o	γ_i	γ_j	γ_k	δ_o	δ_i	δ_j	δ_k	
σ_o	+ σ_o	+ σ_i	+ σ_j	+ σ_k	+ λ_o	+ λ_i	+ λ_j	+ λ_k	+ μ_o	+ μ_i	+ μ_j	+ μ_k	+ ν_o	+ ν_i	+ ν_j	+ ν_k	+ α_o	+ α_i	+ α_j	+ α_k	+ β_o	+ β_i	+ β_j	+ β_k	+ γ_o	+ γ_i	+ γ_j	+ γ_k	+ δ_o	+ δ_i	+ δ_j	+ δ_k	
σ_i	+ σ_i	- σ_o	+ σ_k	- σ_j	+ λ_i	- λ_o	+ λ_j	- λ_k	+ μ_i	- μ_o	+ μ_j	- μ_k	+ ν_i	- ν_o	+ ν_j	- ν_k	- α_o	- α_k	+ α_i	- α_j	+ β_i	- β_o	+ β_j	- β_k	+ γ_i	- γ_o	- γ_j	+ γ_k	+ δ_i	- δ_o	- δ_j	+ δ_k	
σ_j	+ σ_j	- σ_o	- σ_i	+ σ_k	+ λ_i	- λ_o	- λ_k	+ λ_j	+ μ_j	- μ_o	- μ_i	+ μ_k	+ ν_j	- ν_o	- ν_i	+ ν_k	- α_i	- α_o	- α_k	+ α_j	- β_i	- β_o	+ β_j	- β_k	+ γ_j	- γ_o	- γ_i	+ γ_k	+ δ_j	- δ_o	- δ_i	+ δ_k	
σ_k	+ σ_k	- σ_o	- σ_i	- σ_j	+ λ_o	- λ_i	- λ_j	+ λ_k	+ μ_k	- μ_i	- μ_j	+ μ_o	+ ν_k	- ν_i	- ν_j	+ ν_o	- α_i	- α_o	- α_j	+ α_k </													

9. Basis of notation used for unit elements of \mathbb{T} and its Cayley table

Conventional notation for unit elements of \mathbb{T} would use the numbers 0 to 31, standing alone or subscripted as e_0 to e_{31} . In this paper a different form of notation is used based on a modified Moufang loop construction for \mathbb{T} .

9.1. Moufang Loop construction for octonions

For Moufang loop construction of octonions, based on quaternion pairs, a dis-association operator, ω , is assigned to the second pair, and a product rule which generates octonions can be defined:

For quaternion pair $(p, \omega p') \times$ quaternion pair $(q, \omega q')$:

$$\begin{aligned} p.q &= (pq) \\ p.\omega q' &= \omega(p^{-1}q') \\ \omega p'.q &= \omega(qp') \\ \omega p'.\omega q' &= -(q'p'^{-1}) \end{aligned}$$

9.2. Modified Moufang Loop construction for trintadtaduonions

The Moufang loop construction for octonions uses one dis-association operator. An identity operator, σ , and a set of seven dis-association operators, $\lambda, \mu, \nu, \alpha, \beta, \gamma, \delta$, can be applied to quaternions to define trintadtaduonions, where products for two quaternions: p and q , are constructed in accordance with table 14.

TABLE 14. Multiplication procedures for non-associative components

	σq	λq	μq	νq	αq	βq	γq	δq
σp	$+ \sigma pq$	$+ \lambda qp$	$+ \mu qp$	$+ \nu qp^{-1}$	$+ \alpha qp$	$+ \beta qp^{-1}$	$+ \gamma qp^{-1}$	$+ \delta qp$
λp	$+ \lambda pq^{-1}$	$- \sigma q^{-1}p$	$+ \nu pq$	$- \mu p^{-1}q$	$+ \beta pq$	$- \alpha p^{-1}q$	$- \delta pq$	$+ \gamma p^{-1}q$
μp	$+ \mu pq^{-1}$	$- \nu qp$	$- \sigma q^{-1}p$	$+ \lambda qp^{-1}$	$+ \gamma pq$	$+ \delta qp$	$- \alpha p^{-1}q$	$- \beta qp^{-1}$
νp	$+ \nu pq$	$+ \mu q^{-1}p$	$- \lambda pq^{-1}$	$- \sigma p^{-1}q$	$+ \delta pq^{-1}$	$- \gamma q^{-1}p$	$+ \beta pq^{-1}$	$- \alpha q^{-1}p$
αp	$+ \alpha pq^{-1}$	$- \beta qp$	$- \gamma qp$	$- \delta qp^{-1}$	$- \sigma q^{-1}p$	$+ \lambda qp^{-1}$	$+ \mu qp^{-1}$	$+ \nu qp$
βp	$+ \beta pq$	$+ \alpha q^{-1}p$	$- \delta pq$	$+ \gamma p^{-1}q$	$- \lambda pq^{-1}$	$- \sigma p^{-1}q$	$- \nu pq$	$+ \mu p^{-1}q$
γp	$+ \gamma pq$	$+ \delta qp$	$+ \alpha q^{-1}p$	$- \beta qp^{-1}$	$- \mu pq^{-1}$	$+ \nu qp$	$- \sigma p^{-1}q$	$- \lambda qp^{-1}$
δp	$+ \delta pq^{-1}$	$- \gamma q^{-1}p$	$+ \beta pq^{-1}$	$+ \alpha p^{-1}q$	$- \nu pq$	$- \mu q^{-1}p$	$+ \lambda pq^{-1}$	$- \sigma q^{-1}p$

Unit trintadtaduonions have been labeled using: $\sigma, \lambda, \mu, \nu, \alpha, \beta, \gamma, \delta$, combined with subscripts: o, ι, j, κ . Each label denotes a unit quaternion and a dis-association operator defining a multiplication procedure. Their Cayley table is shown in table 13.

If, instead of being applied to the quaternions, these dis-association operators and multiplication procedures are applied to the reals, they generate the octonions. If applied to complex numbers, they generate the sedenions. However, when applied to the octonions, they do not generate the sexagintaquaternionions. It is possible to apply the dis-association operators to other algebras, such as Clifford algebras, generating different loops. This was described for $Cl_{1,4}(R)$ in a previous paper by the author[35].

10. Unit elements of \mathbb{U}

TABLE 15. Unit elements of \mathbb{U}

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