Foundations of Conic Conformal Geometric Algebra and Simplified Versors for Rotation, Translation and Scaling

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Abstract. This paper explains in algebraic detail how two-dimensional conics can be defined by the outer products of conformal geometric algebra (CGA) points in higher dimensions. These multivector expressions code all types of conics in arbitrary scale, location and orientation. Conformal geometric algebra of two-dimensional Euclidean geometry is fully embedded as an algebraic subset. With small model preserving modifications, it is possible to consistently define in conic CGA versors for rotation, translation and scaling, similar to [9], but simpler, especially for translations.

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1. Introduction

Two dimensional conics can be represented in extended conformal geometric algebras (CGA) in several ways. The method we study in algebraic detail was outlined by C. Perwass in Chapter 4.5 of [10]. Most recently double CGA (DCGA)[5] and quadric CGA (QCGA)[1] have been proposed[1]. We also compare with a recent modified approach provided in [9]. We first focus on the elegant algebraic core structure of conic conformal geometric algebra (CCGA) to be presented in some detail. While doing that, we introduce a small model preserving modification in the basic algebraic definitions, that is key to preserving the translation formulation of [10], but allows to consistently keep the full system of versors for rotations, translations and scaling.

The paper is structured as follows. Section 2 defines CCGA as an extension of CGA, and provides a detailed set of important algebraic relations computed for

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[1]For further literature see the references in [1].

CCGA. Section 3 shows how points and all two-dimensional CGA objects are defined and embedded in CCGA. Section 4 defines blade expressions for conics and axis aligned conics in CCGA. Section 5 explains how to compute intersections, and Section 6 formulates versors for rotations, translations and scaling. The paper ends with a conclusion in Section 7 followed by acknowledgments and references.

2. CCGA definition

This section introduces CCGA. We specify its basis vectors in a slightly modified way, and show important blade computations. We use the following notation: Lower-case bold letters denote basis blades and multivectors (vector or multivector $\mathbf{a}$). Italic lower-case letters refer to multivector components $(a_1, x, y^2, \cdots)$. For example, a_i is the ith coordinate of the (multi)vector $\mathbf{a}$. Constant scalars are denoted using lower-case default text font (constant radius r) or simply r. The superscript star used in $\mathbf{x}^*$ represents the dualization of the multivector $\mathbf{x}$. Finally, subscript ε on $\mathbf{x}_\varepsilon$ refers to the two-dimensional Euclidean vector associated to the vector $\mathbf{x}$ of CCGA.

Note that when used in geometric algebra the inner product, contractions and the outer product have priority over the full geometric product. For instance, $\mathbf{a} \wedge \mathbf{b} \mathbf{I} = (\mathbf{a} \wedge \mathbf{b})\mathbf{I}$.

2.1. CCGA basis and metric

The algebraic equations in this section can be either computed by hand, expanding all blades in terms of basis vectors, or they can be computed with software, like the Clifford Multivector Toolbox (for MATLAB)[11]. The CCGA $Cl(5,3)$ is defined over the 8-dimensional real vector space $\mathbb{R}^{5,3}$. The basis vectors of the space can be divided into three groups: $\{\mathbf{e}_1, \mathbf{e}_2\}$ (corresponding to the Euclidean vectors of $\mathbb{R}^2$), and six null vectors $\{\mathbf{e}_{o1}, \mathbf{e}_{o2}, \mathbf{e}_{o3}\}$, and $\{\mathbf{e}_{\infty 1}, \mathbf{e}_{\infty 2}, \mathbf{e}_{\infty 3}\}$. The inner products between them are defined in Table 1.

TABLE 1. Inner products between CCGA basis vectors.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{o1}$	$\mathbf{e}_{\infty 1}$	$\mathbf{e}_{o2}$	$\mathbf{e}_{\infty 2}$	$\mathbf{e}_{o3}$	$\mathbf{e}_{\infty 3}$
$\mathbf{e}_1$	1	0	·	·	·	·	·	·
$\mathbf{e}_2$	0	1	·	·	·	·	·	·
$\mathbf{e}_{o1}$	·	·	0	−1	·	·	·	·
$\mathbf{e}_{\infty 1}$	·	·	−1	0	·	·	·	·
$\mathbf{e}_{o2}$	·	·	·	·	0	−1	·	·
$\mathbf{e}_{\infty 2}$	·	·	·	·	−1	0	·	·
$\mathbf{e}_{o3}$	·	·	·	·	·	·	0	−1
$\mathbf{e}_{\infty 3}$	·	·	·	·	·	·	−1	0

For efficient computation, a diagonal metric matrix may often be useful. The algebra $Cl(5,3)$ generated by the Euclidean basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, and three basis vectors $\{\mathbf{e}_{+1}, \mathbf{e}_{+2}, \mathbf{e}_{+3}\}$ squaring to $+1$ along with three other basis vectors $\{\mathbf{e}_{-1}, \mathbf{e}_{-2}, \mathbf{e}_{-3}\}$ squaring to -1 would correspond to a diagonal metric matrix.

Relating the diagonal two-dimensional Minkowski plane basis $\{\mathbf{e}_{+i}, \mathbf{e}_{-i}\}$ to the null basis $\{\mathbf{e}_{oi}, \mathbf{e}_{\infty i}\}$, for $i = 1, 2$ or 3, is unique up to three non-zero parameters $\lambda_i \in \mathbb{R} \setminus \{0\}$,

$$\mathbf{e}_{\infty i} = \frac{\lambda_i}{\sqrt{2}}(\mathbf{e}_{+i} + \mathbf{e}_{-i}), \qquad \mathbf{e}_{oi} = \frac{1}{\lambda_i \sqrt{2}}(\mathbf{e}_{-i} - \mathbf{e}_{+i}). \tag{1}$$

The parameters[2] λ_i, $i = 1, 2, 3$, are usually in conformal geometric algebra of the plane, or in the modeling of conics in $Cl(5,3)$ by Perwass [10] or by Hrdina et al [9], simple all set to $\lambda_1 = \lambda_2 = \lambda_3 = \sqrt{2}$. In contrast to that, e.g., El Mir et al [4] have exploited this degree of freedom in conformal geometry for an elegant new formulation of viewpoint change representation. By reasons, that will become evident during the course of this paper, we decide to rather choose the symmetric values $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

The transformation from the diagonal metric basis to that of Table 1 can now be defined as follows: for $1 \leq i, j \leq 3$,

$$\mathbf{e}_{\infty i} = \frac{1}{\sqrt{2}}(\mathbf{e}_{+i} + \mathbf{e}_{-i}), \qquad \mathbf{e}_{oi} = \frac{1}{\sqrt{2}}(\mathbf{e}_{-i} - \mathbf{e}_{+i}). \tag{2}$$

We further define for later use another pair of null vectors

$$\mathbf{e}_{\infty} = \tfrac{1}{2}(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2}), \qquad \mathbf{e}_o = \mathbf{e}_{o1} + \mathbf{e}_{o2}. \tag{3}$$

Inner products lead to

$$\mathbf{e}_{\infty i} \cdot \mathbf{e}_{oi} = -1, \qquad \mathbf{e}_{\infty} \cdot \mathbf{e}_o = -1, \qquad \mathbf{e}_o^2 = \mathbf{e}_{\infty}^2 = 0, \tag{4}$$

$$\mathbf{e}_{\infty 1} \cdot \mathbf{e}_o = \mathbf{e}_{\infty 2} \cdot \mathbf{e}_o = -1, \qquad \mathbf{e}_{\infty 3} \cdot \mathbf{e}_o = 0, \qquad \mathbf{e}_{\infty i} \cdot \mathbf{e}_{\infty} = 0, \tag{5}$$

We further define the bivectors E_i, E, as

$$E_i = \mathbf{e}_{\infty i} \wedge \mathbf{e}_{oi} = \mathbf{e}_{+i} \mathbf{e}_{-i}, \qquad E = \mathbf{e}_{\infty} \wedge \mathbf{e}_o, \tag{6}$$

and obtain the following products

$$E_i^2 = 1, \qquad E_i E_j = E_j E_i, \tag{7}$$

$$\mathbf{e}_{oi} E_i = -E_i \mathbf{e}_{oi} = -\mathbf{e}_{oi}, \qquad \mathbf{e}_{\infty i} E_i = -E_i \mathbf{e}_{\infty i} = \mathbf{e}_{\infty i}, \tag{8}$$

$$\mathbf{e}_{oj} E_i \overset{i \neq i}{=} E_i \mathbf{e}_{oj}, \qquad \mathbf{e}_{\infty j} E_i \overset{i \neq i}{=} E_i \mathbf{e}_{\infty j}, \tag{9}$$

$$E^2 = 1, \quad \mathbf{e}_o E = -E \mathbf{e}_o = -\mathbf{e}_o, \quad \mathbf{e}_{\infty} E = -E \mathbf{e}_{\infty} = \mathbf{e}_{\infty}. \tag{10}$$

For brevity of notation and clarity, we also define the following blades:

$$\mathbf{I}_{\infty 12} = \mathbf{e}_{\infty 1} \mathbf{e}_{\infty 2}, \quad \mathbf{I}_{\infty} = \mathbf{I}_{\infty 12} \mathbf{e}_{\infty 3}, \quad \mathbf{I}_{o12} = \mathbf{e}_{o1} \mathbf{e}_{o2}, \quad \mathbf{I}_o = \mathbf{I}_{o12} \mathbf{e}_{o3}, \tag{11}$$

$$\mathbf{I}_{\infty o} = \mathbf{I}_{\infty} \wedge \mathbf{I}_o = -E_1 E_2 E_3, \tag{12}$$

$$\mathbf{e}_{\infty 12}^{\triangleright} = \mathbf{e}_{\infty 1} - \mathbf{e}_{\infty 2}, \quad \mathbf{I}_{\infty}^{\triangleright} = \mathbf{e}_{\infty 12}^{\triangleright} \mathbf{e}_{\infty 3}, \quad \mathbf{e}_{o12}^{\triangleright} = \mathbf{e}_{o1} - \mathbf{e}_{o2}, \quad \mathbf{I}_o^{\triangleright} = \mathbf{e}_{o12}^{\triangleright} \mathbf{e}_{o3}, \tag{13}$$

$$\mathbf{I}^{\triangleright} = \mathbf{I}_{\infty}^{\triangleright} \wedge \mathbf{I}_o^{\triangleright}. \tag{14}$$

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[2]The parameters λ_i, $i = 1, 2, 3$, parametrize a continuous set of horospheres [4].

We note that

$$\mathbf{I}_{\infty o}^2 = 1, \quad \mathbf{I}_o \mathbf{I}_{\infty o} = -\mathbf{I}_{\infty o}\mathbf{I}_o = \mathbf{I}_o, \quad \mathbf{I}_\infty \mathbf{I}_{\infty o} = -\mathbf{I}_{\infty o}\mathbf{I}_\infty = -\mathbf{I}_\infty, \tag{15}$$

$$(\mathbf{e}_{\infty 12}^\rhd \wedge \mathbf{e}_{o12}^\rhd)^{-1} = \tfrac{1}{4}\mathbf{e}_{\infty 12}^\rhd \wedge \mathbf{e}_{o12}^\rhd, \quad \mathbf{I}_{\infty 12} \wedge \mathbf{I}_{o12} = -E_1 E_2, \tag{16}$$

$$\mathbf{I}^\rhd = -\mathbf{e}_{\infty 12}^\rhd \wedge \mathbf{e}_{o12}^\rhd E_3, \quad (\mathbf{I}^\rhd)^2 = (\mathbf{e}_{\infty 12}^\rhd \wedge \mathbf{e}_{o12}^\rhd)^2 = 4, \quad (\mathbf{I}^\rhd)^{-1} = \tfrac{1}{4}\mathbf{I}^\rhd \tag{17}$$

$$\mathbf{I}_\infty^\rhd \cdot \mathbf{I}_o^\rhd = \mathbf{I}_o^\rhd \cdot \mathbf{I}_\infty^\rhd = \mathbf{I}_\infty^\rhd \rfloor \mathbf{I}_o^\rhd = \mathbf{I}_\infty^\rhd \lfloor \mathbf{I}_o^\rhd = -2. \tag{18}$$

Note, that in the last line, the product symbols $\rfloor$ and $\lfloor$ express the left- and right contraction, respectively. We have the following outer product relationships

$$\mathbf{I}_{\infty 12} = -\mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 12}^\rhd = -\mathbf{e}_{\infty 2} \wedge \mathbf{e}_{\infty 12}^\rhd = -\mathbf{e}_\infty \wedge \mathbf{e}_{\infty 12}^\rhd$$

$$= -\mathbf{e}_{\infty 1}\,\mathbf{e}_{\infty 12}^\rhd = -\mathbf{e}_{\infty 2}\,\mathbf{e}_{\infty 12}^\rhd = -\mathbf{e}_\infty\,\mathbf{e}_{\infty 12}^\rhd, \tag{19}$$

$$\mathbf{I}_{o12} = -\mathbf{e}_{o1} \wedge \mathbf{e}_{o12}^\rhd = -\mathbf{e}_{o2} \wedge \mathbf{e}_{o12}^\rhd = -\tfrac{1}{2}\mathbf{e}_o \wedge \mathbf{e}_{o12}^\rhd$$

$$= -\mathbf{e}_{o1}\,\mathbf{e}_{o12}^\rhd = -\mathbf{e}_{o2}\,\mathbf{e}_{o12}^\rhd = -\tfrac{1}{2}\mathbf{e}_o\,\mathbf{e}_{o12}^\rhd, \tag{20}$$

$$\mathbf{I}_{\infty 12} \wedge \mathbf{I}_{o12} = -E_1 \wedge \mathbf{e}_{\infty 12}^\rhd \wedge \mathbf{e}_{o12}^\rhd = -E_2 \wedge \mathbf{e}_{\infty 12}^\rhd \wedge \mathbf{e}_{o12}^\rhd \tag{21}$$

$$= -\tfrac{1}{2}E \wedge \mathbf{e}_{\infty 12}^\rhd \wedge \mathbf{e}_{o12}^\rhd = -\tfrac{1}{2}E\,\mathbf{e}_{\infty 12}^\rhd \wedge \mathbf{e}_{o12}^\rhd.$$

And we have the following inner products ($i = 1,2$)

$$\mathbf{e}_{o12}^\rhd = 2\mathbf{e}_\infty \cdot \mathbf{I}_{o12}, \quad \mathbf{I}_o^\rhd = 2\mathbf{e}_\infty \cdot \mathbf{I}_o, \quad \mathbf{e}_{\infty 12}^\rhd = \mathbf{e}_o \cdot \mathbf{I}_{\infty 12}, \quad \mathbf{I}_\infty^\rhd = \mathbf{e}_o \cdot \mathbf{I}_\infty, \tag{22}$$

$$(\mathbf{e}_{oi} \cdot \mathbf{I}_\infty) \cdot \mathbf{I}_o = \mathbf{e}_{oi}, \qquad (\mathbf{e}_{\infty i} \cdot \mathbf{I}_o) \cdot \mathbf{I}_\infty = \mathbf{e}_{\infty i}, \tag{23}$$

$$(\mathbf{e}_o \cdot \mathbf{I}_\infty) \cdot \mathbf{I}_o = \mathbf{e}_o, \qquad (\mathbf{e}_\infty \cdot \mathbf{I}_o) \cdot \mathbf{I}_\infty = \mathbf{e}_\infty \tag{24}$$

$$\mathbf{e}_\infty \cdot \mathbf{I}_{\infty o} = -\tfrac{1}{2}\mathbf{I}_\infty \wedge \mathbf{I}_o^\rhd, \qquad \mathbf{e}_o \cdot \mathbf{I}_{\infty o} = \mathbf{I}_\infty^\rhd \wedge \mathbf{I}_o, \tag{25}$$

$$\mathbf{e}_{\infty i} \cdot \mathbf{e}_{\infty 12}^\rhd = 0, \qquad \mathbf{e}_{\infty i} \cdot \mathbf{I}_\infty^\rhd = 0, \qquad \mathbf{e}_\infty \cdot \mathbf{e}_{\infty 12}^\rhd = 0, \qquad \mathbf{e}_\infty \cdot \mathbf{I}_\infty^\rhd = 0, \tag{26}$$

$$\mathbf{e}_{oi} \cdot \mathbf{e}_{o12}^\rhd = 0, \qquad \mathbf{e}_{oi} \cdot \mathbf{I}_o^\rhd = 0, \qquad \mathbf{e}_o \cdot \mathbf{e}_{o12}^\rhd = 0, \qquad \mathbf{e}_o \cdot \mathbf{I}_o^\rhd = 0, \tag{27}$$

$$\mathbf{e}_\infty \cdot \mathbf{e}_{o12}^\rhd = 0, \qquad \mathbf{e}_\infty \cdot \mathbf{I}_o^\rhd = 0, \qquad \mathbf{e}_o \cdot \mathbf{e}_{\infty 12}^\rhd = 0, \qquad \mathbf{e}_o \cdot \mathbf{I}_\infty^\rhd = 0, \tag{28}$$

$$\mathbf{e}_\infty \cdot \mathbf{I}^\rhd = 0, \qquad \mathbf{e}_o \cdot \mathbf{I}^\rhd = 0, \qquad E \cdot \mathbf{I}^\rhd = 0. \tag{29}$$

As consequence we obtain

$$\mathbf{I}_\infty = -\mathbf{e}_{\infty 1}\mathbf{I}_\infty^\rhd = -\mathbf{e}_{\infty 1} \wedge \mathbf{I}_\infty^\rhd = -\mathbf{e}_{\infty 2}\mathbf{I}_\infty^\rhd = -\mathbf{e}_{\infty 2} \wedge \mathbf{I}_\infty^\rhd$$

$$= -\mathbf{e}_\infty \mathbf{I}_\infty^\rhd = -\mathbf{e}_\infty \wedge \mathbf{I}_\infty^\rhd, \tag{30}$$

$$\mathbf{I}_\infty \wedge \mathbf{I}_o^\rhd = -\mathbf{e}_{\infty 1} \wedge \mathbf{I}^\rhd = -\mathbf{e}_{\infty 2} \wedge \mathbf{I}^\rhd = -\mathbf{e}_\infty \wedge \mathbf{I}^\rhd = -\mathbf{e}_\infty \mathbf{I}^\rhd = -\mathbf{I}^\rhd \mathbf{e}_\infty, \tag{31}$$

$$2\mathbf{I}_o = -\mathbf{e}_o \mathbf{I}_o^\rhd = -\mathbf{e}_o \wedge \mathbf{I}_o^\rhd, \qquad -2\mathbf{I}_\infty^\rhd \wedge \mathbf{I}_o = \mathbf{e}_o \mathbf{I}^\rhd = \mathbf{e}_o \wedge \mathbf{I}^\rhd = \mathbf{I}^\rhd \mathbf{e}_o, \tag{32}$$

$$2\mathbf{I}_{\infty o} = E\mathbf{I}^\rhd = E \wedge \mathbf{I}^\rhd = \mathbf{I}^\rhd E, \qquad \mathbf{I}^\rhd = 2E\mathbf{I}_{\infty o} = 2\mathbf{I}_{\infty o}E. \tag{33}$$

The last relationship means that the quadvector $\mathbf{I}^\rhd$ can be obtained from the 6-vector $\mathbf{I}_{\infty o}$ by left- or right division with bivector E, since $E^{-1} = E$. We can summarize for the multivector basis of the geometric algebra Minkowski plane spanned by $\{\mathbf{e}_o, \mathbf{e}_\infty\}$, and the quadvector $\mathbf{I}^\rhd$ the important set of relations

$$\{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \wedge \mathbf{I}^\rhd = \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}\mathbf{I}^\rhd = \mathbf{I}^\rhd \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}. \tag{34}$$

Similar relationships hold separately for products with the simple bivector $\mathbf{I}_\infty^\triangleright$, or with $\mathbf{I}_o^\triangleright$, respectively,

$$\{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \wedge \mathbf{I}_\infty^\triangleright = \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \mathbf{I}_\infty^\triangleright = \mathbf{I}_\infty^\triangleright \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}, \tag{35}$$

$$\{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \wedge \mathbf{I}_o^\triangleright = \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\} \mathbf{I}_o^\triangleright = \mathbf{I}_o^\triangleright \{1, \mathbf{e}_o, \mathbf{e}_\infty, E\}. \tag{36}$$

We define the two-dimensional Euclidean pseudo-scalar $\mathbf{I}_\varepsilon$ bivector in $\mathbb{R}^2$:

$$\mathbf{I}_\varepsilon = \mathbf{e}_1 \mathbf{e}_2, \qquad \mathbf{I}_\varepsilon^2 = -1, \qquad \mathbf{I}_\varepsilon^{-1} = -\mathbf{I}_\varepsilon, \tag{37}$$

and the conformal pseudo-scalar $\mathbf{I}_C$ by

$$\mathbf{I}_C = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_\infty \wedge \mathbf{e}_o = \mathbf{I}_\varepsilon E, \qquad \mathbf{I}_C^2 = -1, \qquad \mathbf{I}_C^{-1} = -\mathbf{I}_C. \tag{38}$$

The full pseudo-scalar $\mathbf{I}$ and its inverse $\mathbf{I}^{-1}$ (used for dualization) are:

$$\mathbf{I} = \mathbf{I}_\varepsilon \mathbf{I}_{\infty o} = \tfrac{1}{2} \mathbf{I}_C \mathbf{I}^\triangleright = \tfrac{1}{2} \mathbf{I}_\varepsilon E \mathbf{I}^\triangleright = -\mathbf{I}_\varepsilon E_1 E_2 E_3, \quad \mathbf{I}^2 = -1, \quad \mathbf{I}^{-1} = -\mathbf{I}. \tag{39}$$

The dual of a multivector indicates division by the pseudo-scalar, e.g., $\mathbf{a}^* = -\mathbf{a}\mathbf{I}$, $\mathbf{a} = \mathbf{a}^*\mathbf{I}$. From eq. (1.19) in [7], we have the useful duality between outer and inner products of non-scalar blades A, B in geometric algebra:

$$(A \wedge B)^* = A \cdot B^*, \quad A \wedge (B^*) = (A \cdot B)^* \iff A \wedge (B\mathbf{I}) = (A \cdot B)\mathbf{I}, \tag{40}$$

which indicates that

$$A \wedge B = 0 \iff A \cdot B^* = 0, \qquad A \cdot B = 0 \iff A \wedge B^* = 0. \tag{41}$$

Further useful duality relationships are

$$\mathbf{I}_{\infty o}^* = -\mathbf{I}_\varepsilon, \quad (\mathbf{I}_\infty \wedge \mathbf{I}_o^\triangleright)^* = 2\mathbf{I}_\varepsilon \mathbf{e}_\infty,$$

$$(\mathbf{I}_\varepsilon (\mathbf{e}_{oi} \cdot \mathbf{I}_\infty) \wedge \mathbf{I}_o)^* = \mathbf{e}_{oi}, \quad (\mathbf{I}_\varepsilon \mathbf{I}_\infty \wedge (\mathbf{e}_{\infty i} \cdot \mathbf{I}_o))^* = -\mathbf{e}_{\infty i},$$

$$(\mathbf{I}_\varepsilon (\mathbf{e}_o \cdot \mathbf{I}_\infty) \wedge \mathbf{I}_o)^* = \mathbf{e}_o, \quad (\mathbf{I}_\varepsilon \mathbf{I}_\infty \wedge (\mathbf{e}_\infty \cdot \mathbf{I}_o))^* = -\mathbf{e}_\infty. \tag{42}$$

3. CCGA objects

CCGA is an extension of CGA; thus the objects defined in CGA are also defined in CCGA. The following sections introduce the important definition of a general point in CCGA, and show next how all round and flat geometric objects (point pairs, flat points, circles, lines) of CGA can straightforwardly be embedded in CCGA.

3.1. Points, point pairs and lines in CCGA

The point $\mathbf{x}$ of CCGA corresponding to the Euclidean point $\mathbf{x}_\varepsilon = x\mathbf{e}_1 + y\mathbf{e}_2 \in \mathbb{R}^2$ is defined by adding four null vector components[3]

$$\mathbf{x} = \mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2 \mathbf{e}_{\infty 1} + y^2 \mathbf{e}_{\infty 2}) + xy\, \mathbf{e}_{\infty 3} + \mathbf{e}_o. \tag{43}$$

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[3]Note that our definition (43) is different from that given in [9], equation (2).

Note that the null vector $\mathbf{e}_{o3}$ is not present in the definition of the point. This is merely to keep the convenient properties of the CGA points, namely, the inner product between two points is identical with the squared distance between them. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points, their inner product is

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = (\mathbf{x}_{1\varepsilon} + \tfrac{1}{2}x_1^2\mathbf{e}_{\infty 1} + \tfrac{1}{2}y_1^2\mathbf{e}_{\infty 2} + x_1 y_1\mathbf{e}_{\infty 3} + \mathbf{e}_o)$$
$$\cdot (\mathbf{x}_{2\varepsilon} + \tfrac{1}{2}x_2^2\mathbf{e}_{\infty 1} + \tfrac{1}{2}y_2^2\mathbf{e}_{\infty 2} + x_2 y_2\mathbf{e}_{\infty 3} + \mathbf{e}_o). \tag{44}$$

from which together with Table 1, it follows that

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_{1\varepsilon} \cdot \mathbf{x}_{2\varepsilon} - \tfrac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2) \stackrel{(47)}{=} \mathbf{x}_{1C} \cdot \mathbf{x}_{2C} = -\tfrac{1}{2}(\mathbf{x}_{1\varepsilon} - \mathbf{x}_{2\varepsilon})^2. \tag{45}$$

We see that the inner product is equivalent to minus half of the squared Euclidean distance between $\mathbf{x}_1$ and $\mathbf{x}_2$.

In the remainder of the paper the following result will be useful, because it relates a point $\mathbf{x}$ in CCGA to the representation $\mathbf{x}_C$ it would have in the subalgebra CGA $\mathbb{R}^{3,1}$ with vector basis $\{\mathbf{e}_o, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_\infty\}$.

$$\begin{aligned}
\mathbf{x} \wedge \mathbf{I}_\infty^{\triangleright} &= \left(\mathbf{x}_\varepsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2}) + \mathbf{e}_o\right) \wedge \mathbf{I}_\infty^{\triangleright} \\
&= (\mathbf{x}_\varepsilon + \mathbf{e}_o) \wedge \mathbf{I}_\infty^{\triangleright} + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2}) \wedge \mathbf{I}_\infty^{\triangleright} \\
&\stackrel{(30)}{=} (\mathbf{x}_\varepsilon + \mathbf{e}_o) \wedge \mathbf{I}_\infty^{\triangleright} + \tfrac{1}{2}(x^2 + y^2)\mathbf{e}_\infty \wedge \mathbf{I}_\infty^{\triangleright} \\
&= (\mathbf{x}_\varepsilon + \mathbf{e}_o) \wedge \mathbf{I}_\infty^{\triangleright} + \tfrac{1}{2}\mathbf{x}_\varepsilon^2\mathbf{e}_\infty \wedge \mathbf{I}_\infty^{\triangleright} \\
&= (\mathbf{x}_\varepsilon + \tfrac{1}{2}\mathbf{x}_\varepsilon^2\mathbf{e}_\infty + \mathbf{e}_o) \wedge \mathbf{I}_\infty^{\triangleright} = \mathbf{x}_C \wedge \mathbf{I}_\infty^{\triangleright} = \mathbf{x}_C \mathbf{I}_\infty^{\triangleright},
\end{aligned} \tag{46}$$

where we have already dropped in the first line the cross term $xy\,\mathbf{e}_{\infty 3}$, because wedging with $\mathbf{e}_{\infty 3}$, by (11) a factor in $\mathbf{I}_\infty^{\triangleright}$, eliminates it. Moreover, the last identity is based on (35). Therefore, if a point in CCGA appears wedged with $\mathbf{I}_\infty^{\triangleright}$, we can replace it by the form[4]

$$\mathbf{x}_C = \mathbf{x}_\varepsilon + \tfrac{1}{2}\mathbf{x}_\varepsilon^2\mathbf{e}_\infty + \mathbf{e}_o \stackrel{(18)}{=} -\tfrac{1}{2}(\mathbf{x} \wedge \mathbf{I}_\infty^{\triangleright})\lfloor \mathbf{I}_o^{\triangleright}. \tag{47}$$

it would have in CGA. This in turn means, that we can embed in CCGA the known CGA representations of round and flat objects, by taking the outer products of between one and four points with $\mathbf{I}_\infty^{\triangleright}$, as shown in the following.

3.2. Round and flat objects of CGA in CCGA

With *round* objects, we mean points, point pairs and circles with uniform curvature. Similar to CGA, these can be defined by the outer product of one to three points with the simple bivector $\mathbf{I}_\infty^{\triangleright}$. Their center $\mathbf{c}_C$, radius r and Euclidean carrier D can be easily extracted. Alternatively, they can be directly constructed from their center $\mathbf{c}_C$, radius r and Euclidean carrier D.

Wedging any round object with the point at infinity $\mathbf{e}_\infty$, gives the corresponding *flat* object multivector. From it the orthogonal distance to the origin $\mathbf{c}_{\varepsilon\perp}$ and the Euclidean carrier D can easily be extracted.

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[4]We have introduced the lower index C for entities in the subalgebra, generated by $\{\mathbf{e}_o, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_\infty\}$, isomorphic to $Cl(3,1)$, the CGA of the two-dimensional Euclidean plane $\mathbb{R}^2$.

We now briefly review the CGA description of round and flat objects embedded in CCGA. The round objects are point, point pair and circle,

$$P = \mathbf{x} \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_C \, \mathbf{I}_\infty^\triangleright, \tag{48}$$

$$Pp = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1C} \wedge \mathbf{x}_{2C} \, \mathbf{I}_\infty^\triangleright, \tag{49}$$

$$Circle = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1C} \wedge \mathbf{x}_{2C} \wedge \mathbf{x}_{3C} \, \mathbf{I}_\infty^\triangleright, \tag{50}$$

The corresponding flat objects are flat point, line and the whole plane,

$$Flat\,p = P \wedge \mathbf{e}_\infty = \mathbf{x} \wedge \mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_C \wedge \mathbf{e}_\infty \, \mathbf{I}_\infty^\triangleright, \tag{51}$$

$$Line = Pp \wedge \mathbf{e}_\infty = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1C} \wedge \mathbf{x}_{2C} \wedge \mathbf{e}_\infty \, \mathbf{I}_\infty^\triangleright, \tag{52}$$

$$Plane = Circle \wedge \mathbf{e}_\infty = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{e}_\infty \wedge \mathbf{I}_\infty^\triangleright = \mathbf{x}_{1C} \wedge \mathbf{x}_{2C} \wedge \mathbf{x}_{3C} \wedge \mathbf{e}_\infty \, \mathbf{I}_\infty^\triangleright. \tag{53}$$

The above embeddings by means of the outer product with $\mathbf{I}_\infty^\triangleright$, allow to make use of standard CGA results found in [7]. All embedded round entities of point, point pair and circle have one common multivector form[5]

$$S = \left(D \wedge \mathbf{c}_\varepsilon + [\tfrac{1}{2}(\mathbf{c}_\varepsilon^2 + r^2)D - \mathbf{c}_\varepsilon\mathbf{c}_\varepsilon\rfloor D]\mathbf{e}_\infty + D\mathbf{e}_o + D\lfloor \mathbf{c}_\varepsilon E \right) \mathbf{I}_\infty^\triangleright = S_C \, \mathbf{I}_\infty^\triangleright,$$

$$S_C = -\tfrac{1}{2}S \lfloor \mathbf{I}_o^\triangleright. \tag{54}$$

The Euclidean carriers D are for each object

$$D = \begin{cases} 1, & \text{point } \mathbf{x} \\ \mathbf{d}_\varepsilon, & \text{point pair } Pp \\ \mathbf{I}_\varepsilon, & \text{circle } Circle, \end{cases} \tag{55}$$

where the unit point pair connection direction vector is $\mathbf{d}_\varepsilon = (\mathbf{x}_{1\varepsilon} - \mathbf{x}_{2\varepsilon})/2r$. The radius r of a round object and its center $\mathbf{c}_C$ are generally determined by

$$r^2 = \frac{S_C \widetilde{S_C}}{(S_C \wedge \mathbf{e}_\infty)(S_C \wedge \mathbf{e}_\infty)^\sim}, \qquad \mathbf{c}_C = S_C \, \mathbf{e}_\infty \, S_C. \tag{56}$$

where $\widetilde{S_C}$ indicates the *reverse* of S_C. By further wedging a circle S with the simple bivector $\mathbf{I}_o^\triangleright$ we obtain a 7-vector and its dual

$$S \wedge \mathbf{I}_o^\triangleright = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{I}_\infty^\triangleright \wedge \mathbf{I}_o^\triangleright = \mathbf{x}_{1C} \wedge \mathbf{x}_{2C} \wedge \mathbf{x}_{3C} \, \mathbf{I}^\triangleright = (\mathbf{c}_C - \tfrac{1}{2}r^2\mathbf{e}_\infty)\mathbf{I}, \tag{57}$$

where the dual vector $\mathbf{c}_C - \tfrac{1}{2}r^2\mathbf{e}_\infty$ is exactly the dual vector of the sphere S_C in CGA

$$S_C^* = \mathbf{c}_C - \tfrac{1}{2}r^2\mathbf{e}_\infty = S_C \mathbf{I}_C^{-1}. \tag{58}$$

All embedded flat entities of flat point, line and plane have one common multivector form

$$F = S \wedge \mathbf{e}_\infty = (D \wedge \mathbf{c}_\varepsilon\mathbf{e}_\infty - DE) \mathbf{I}_\infty^\triangleright = (D\mathbf{c}_{\varepsilon\perp}\mathbf{e}_\infty - DE) \mathbf{I}_\infty^\triangleright = F_C \, \mathbf{I}_\infty^\triangleright, \tag{59}$$

$$F_C = S_C \wedge \mathbf{e}_\infty = -\tfrac{1}{2}F\lfloor \mathbf{I}_o^\triangleright. \tag{60}$$

where the orthogonal Euclidean distance of the flat object from the origin is

$$\mathbf{c}_{\varepsilon\perp} = \begin{cases} \mathbf{x}_\varepsilon, & \text{finite-infinite point pair } Flat\,p \\ \mathbf{c}_{\varepsilon\perp}, & \text{line } Line \\ 0, & \text{plane } Plane, \end{cases} \tag{60}$$

---

[5]Note, that the left- and right contraction, respectively, are needed essentially.

The Euclidean carrier blade D, and the orthogonal Euclidean distance vector of F from the origin, can both be directly determined from the flat object multivector as

$$D = -F_C \lfloor E, \qquad \mathbf{c}_{\varepsilon\perp} = D^{-1}(F_C \wedge \mathbf{e}_o)\lfloor E. \tag{61}$$

By further wedging a line F with the simple bivector $\mathbf{I}_o^{\triangleright}$ we obtain a 7-vector and its dual

$$F \wedge \mathbf{I}_o^{\triangleright} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{e}_\infty \wedge \mathbf{I}_\infty^{\triangleright} \wedge \mathbf{I}_o^{\triangleright} = \mathbf{x}_{1C} \wedge \mathbf{x}_{2C} \wedge \mathbf{e}_\infty \mathbf{I}^{\triangleright} = 2|\mathbf{x}_{2\varepsilon} - \mathbf{x}_{1\varepsilon}|(\mathbf{n}_\varepsilon + d\mathbf{e}_\infty)\mathbf{I}, \tag{62}$$

where up to the factor $2|\mathbf{x}_{2\varepsilon} - \mathbf{x}_{1\varepsilon}|$ the dual vector $\mathbf{n}_\varepsilon + d\mathbf{e}_\infty$ is exactly the dual vector of the line F_C in CGA

$$F_C^* = F_C \mathbf{I}_C^{-1} = |\mathbf{x}_{2\varepsilon} - \mathbf{x}_{1\varepsilon}|(\mathbf{n}_\varepsilon + d\mathbf{e}_\infty) \propto \mathbf{n}_\varepsilon + d\mathbf{e}_\infty, \tag{63}$$

with Euclidean unit normal vector to the line $\mathbf{n}_\varepsilon = -(\mathbf{x}_{2\varepsilon} - \mathbf{x}_{1\varepsilon})\mathbf{I}_\varepsilon/|\mathbf{x}_{2\varepsilon} - \mathbf{x}_{1\varepsilon}|$ and shortest distance of the line F from the origin $d = -(\mathbf{x}_{1\varepsilon} \wedge \mathbf{x}_{2\varepsilon})\mathbf{I}_\varepsilon/|\mathbf{x}_{2\varepsilon} - \mathbf{x}_{1\varepsilon}|$.

Because by (45) the distance between two points is given by their inner product, the vector F_C^* like in CGA can also be found as the difference of two points $\mathbf{y}_1$ and $\mathbf{y}_2$, bisected by the same line F, i.e. $F_C^* \propto \mathbf{y}_2 - \mathbf{y}_1$.

4. Conics

This section describes how CCGA handles the conics. All embedded CGA objects in CCGA defined in Section 3 are thus part of a more general framework.

4.1. General conics in two Euclidean dimensions

A conic in $\mathbb{R}^2$ is formulated as

$$F(x,y) = ax^2 + by^2 + cxy + dx + ey + f = 0. \tag{64}$$

A conic is constructed by wedging five points together as follows

$$\mathbf{q} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 \wedge \mathbf{x}_5. \tag{65}$$

The multivector blade $\mathbf{q}$ corresponds to the primal form of a conic in CCGA, with grade 5 and 6 components specified by the six coefficients a, b, ... , f. If we further wedge the 5-blade $\mathbf{q}$ with the simple bivector $\mathbf{I}_o^{\triangleright}$ we obtain a 7-blade and its dual vector $(\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^*$ as

$$\begin{aligned}
\mathbf{q} \wedge \mathbf{I}_o^{\triangleright} &= -\mathbf{I}_\varepsilon\big((2a\mathbf{e}_{o1} + 2b\mathbf{e}_{o2} + c\mathbf{e}_{o3}) \cdot \mathbf{I}_\infty\big) \wedge \mathbf{I}_o + \big(d\mathbf{e}_1 + e\mathbf{e}_2\big)\mathbf{I}_\varepsilon \mathbf{I}_{\infty o} \\
&\quad + f\mathbf{I}_\varepsilon \mathbf{I}_\infty \wedge (\mathbf{e}_\infty \cdot \mathbf{I}_o) \\
&= \big(-(2a\mathbf{e}_{o1} + 2b\mathbf{e}_{o2} + c\mathbf{e}_{o3}) + d\mathbf{e}_1 + e\mathbf{e}_2 - f\mathbf{e}_\infty\big)\mathbf{I} = (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^*\mathbf{I}, \tag{66}
\end{aligned}$$

where in the second equality we used the duality relationships of (42). The expression for the dual vector $(\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^*$ is therefore

$$(\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* = -(2a\mathbf{e}_{o1} + 2b\mathbf{e}_{o2} + c\mathbf{e}_{o3}) + d\mathbf{e}_1 + e\mathbf{e}_2 - f\mathbf{e}_\infty. \tag{67}$$

Proposition 4.1. *A point $\mathbf{x}$ lies on the conic curve defined by five points, i.e. by their outer product $\mathbf{q}$, if and only if $\mathbf{x} \wedge (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright}) = 0$.*

Proof.

$$\mathbf{x} \wedge (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright}) = \mathbf{x} \wedge \left((\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \mathbf{I}\right) = \mathbf{x} \cdot (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \mathbf{I}$$

$$= \mathbf{x} \cdot \left(- \left(2a\mathbf{e}_{o1} + 2b\mathbf{e}_{o2} + c\mathbf{e}_{o3}\right) + \left(d\mathbf{e}_1 + e\mathbf{e}_2\right) - f\mathbf{e}_{\infty} \right) \mathbf{I} \qquad (68)$$

$$= \left(ax^2 + by^2 + cxy + dx + ey + f\right) \mathbf{I}.$$

This indeed corresponds to the formula (64) representing a general conic.   □

Proposition 4.1 together with (67) yields the inner product condition for a point $\mathbf{x}$ to be on the conic, expressed by

Corollary 4.2. *A point $\mathbf{x}$ lies on the conic defined by $\mathbf{q}$, if and only if $\mathbf{x} \cdot (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* = 0$.*

The six coefficients $\{a, \ldots, f\}$ of the conic equation (64) can be easily extracted from the conic 5-blade $\mathbf{q}$ of (65) by computing the following scalar products with vector $(\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^*$ as

$$a = \tfrac{1}{2}(\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \cdot \mathbf{e}_{\infty 1}, \quad b = \tfrac{1}{2}(\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \cdot \mathbf{e}_{\infty 2}, \quad c = (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \cdot \mathbf{e}_{\infty 3},$$
$$d = (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \cdot \mathbf{e}_1, \quad e = (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \cdot \mathbf{e}_2, \quad f = (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \cdot \mathbf{e}_o. \qquad (69)$$

4.2. Axis aligned conics

By wedging a point $\mathbf{x}$ with $\mathbf{e}_{\infty 3}$, the mixed coordinate term $xy\,\mathbf{e}_{\infty 3}$ in (43) is removed. This means, that an axis aligned conic can be constructed from the outer product of any four distinct points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ on the conic in the following way

$$\mathbf{q} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 \wedge \mathbf{e}_{\infty 3}. \qquad (70)$$

The dual vector of the 7-vector $\mathbf{q} \wedge \mathbf{I}_o^{\triangleright}$ is

$$(\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* = -\left(2a\mathbf{e}_{o1} + 2b\mathbf{e}_{o2}\right) + d\mathbf{e}_1 + e\mathbf{e}_2 - f\mathbf{e}_{\infty}. \qquad (71)$$

A general point $\mathbf{x}$ is on the conic represented by the 5-blade $\mathbf{q}$ in (70), if and only if,

$$\mathbf{x} \wedge (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright}) = \mathbf{x} \wedge \left((\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \mathbf{I}\right) = \mathbf{x} \cdot (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \mathbf{I}$$

$$= \mathbf{x} \cdot \left(- \left(2a\mathbf{e}_{o1} + 2b\mathbf{e}_{o2}\right) + \left(d\mathbf{e}_1 + e\mathbf{e}_2\right) - f\mathbf{e}_{\infty} \right) \mathbf{I} \qquad (72)$$

$$= \left(ax^2 + by^2 + dx + ey + f\right) \mathbf{I}.$$

which in turn means that we get from $\mathbf{x} \wedge (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright}) = 0$, or equivalently from $\mathbf{x} \cdot (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* = 0$, the implicit equation for an axis aligned conic

$$ax^2 + bx^2 + dx + ey + f = 0. \qquad (73)$$

We note that for the representation of axis aligned conics the dimensions $\mathbf{e}_{o3}$ and $\mathbf{e}_{\infty 3}$ are superfluous, and axis aligned conics could therefore be represented in the lower dimensional geometric algebra $Cl(4,2)$, which is a version for two Euclidean dimensions of the model for axis aligned quadrics in the geometric algebra $Cl(6,3)$ previously proposed for three Euclidean dimensions in [12]. But by keeping the higher dimensional embedding of axis aligned conics in $Cl(5,3)$, we preserve the freedom of performing arbitrary rotations as explained in Section 6, which is not possible in $Cl(4,2)$.
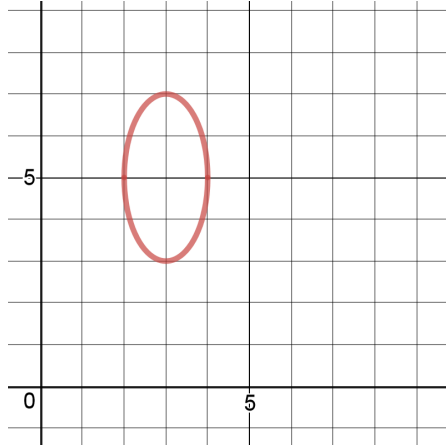
FIGURE 1. An axis aligned ellipse with half axis 1 and 2, and center coordinates $(c_x = 3, c_y = 5)$. Graph with [2].

For example, if we choose the four points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ with coordinates

$$x_1 = 4, y_1 = 5, \quad x_2 = 3, y_2 = 7, \quad x_3 = 2, y_3 = 5, \quad x_4 = 3, y_4 = 3, \tag{74}$$

of an axis aligned ellipse with half axis 1 and 2, and center coordinates $(c_x = 3, c_y = 5)$, we can use e.g. the Clifford Multivector Toolbox (for MATLAB) [11] to obtain from $\mathbf{q}$ the ellipse parameters by applying (69) as

$$a = -16, \quad b = -4, \quad d = 96, \quad e = 40, \quad f = -228, \tag{75}$$

where as expected $c = (\mathbf{q} \wedge \mathbf{I}_o^{\triangleright})^* \cdot \mathbf{e}_{\infty 3} = 0$, for an axis aligned ellipse. Figure 1 shows the resulting ellipse.

Note, that the circle (50) is a special case of an axis aligned conic, because

$$Circle = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{I}_\infty^{\triangleright} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{e}_{\infty 12}^{\triangleright} \wedge \mathbf{e}_{\infty 3}, \tag{76}$$

obtained from (70) by replacing $\mathbf{x}_4$ with the vector $\mathbf{e}_{\infty 12}^{\triangleright}$.

5. Intersections

Any number of linearly independent round or flat embedded CGA objects in CCGA and any number of conics $\{\mathbf{A}, \mathbf{B}, \dots, \mathbf{Z}\}$, after wedging with the simple bivector $\mathbf{I}_o^{\triangleright}$, can be intersected by computing the dual of the outer product of their duals

$$(\text{intersect} \wedge \mathbf{I}_o^{\triangleright})^* = (\mathbf{A} \wedge \mathbf{I}_o^{\triangleright})^* \wedge (\mathbf{B} \wedge \mathbf{I}_o^{\triangleright})^* \wedge \dots \wedge (\mathbf{Z} \wedge \mathbf{I}_o^{\triangleright})^*. \tag{77}$$

The criterion for a general point $\mathbf{x}$ to be on the intersection is

$$\mathbf{x} \cdot (\text{intersect} \wedge \mathbf{I}_o^{\triangleright})^* = 0. \tag{78}$$

For cases that one object is wholly included in another object (like a line in a plane), the proper meet operation has to be defined, taking into account the subspace spanned by the join of the two objects [8].

As an example, we intersect the two conics $\mathbf{q}_1$ of (65) and its axis aligned version $\mathbf{q}_2$ of (70) (formed from the same first four points) yielding the quadvector

$$\text{intersect}_4 = -\tfrac{1}{2}\Big(\big((\mathbf{q}_1 \wedge \mathbf{I}_o^{\triangleright})^* \wedge (\mathbf{q}_2 \wedge \mathbf{I}_o^{\triangleright})^*\big)\mathbf{I}\Big)\lfloor \mathbf{I}_\infty^{\triangleright} \propto \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4, \qquad (79)$$

that is exactly proportional to the outer product of the four intersection points.

6. Versors for rotation, translation and scaling

We acknowledge inspiration from the related formalism in [9]. Note that the conic conformal point definition (2) of [9] differs from our definition (43), which partly accounts for the differences and simplifications in the versor expressions for rotations and especially translations below.

For the successful implementation of rotations together with a simplification of the translation versors, we found it essential to define the null vector pair $\{\mathbf{e}_{\infty 3}, \mathbf{e}_{o3}\}$ in the *symmetric* fashion of (2)

$$\mathbf{e}_{\infty 3} = \tfrac{1}{\sqrt{2}}(\mathbf{e}_{+3} + \mathbf{e}_{-3}), \qquad \mathbf{e}_{o3} = \tfrac{1}{\sqrt{2}}(\mathbf{e}_{-3} - \mathbf{e}_{+3}).$$

Only with this definition[6] for $\{\mathbf{e}_{\infty 3}, \mathbf{e}_{o3}\}$, were we able to reduce the number of versor factors for achieving translations from the six needed in [9], to the four featured below in (89). Moreover, as seen below, our approach has the additional advantage of completely eliminating the quadratic terms in the translation vector coordinates from the exponents of the factors in the translation operators, compared to [9].

Rotations are generated by the following three bivectors

$$e_{12}, \qquad B_2 = \tfrac{1}{2}\mathbf{e}_{o3}\mathbf{e}_{\infty 12}^{\triangleright}, \qquad B_3 = \mathbf{e}_{\infty 3}\mathbf{e}_{o12}^{\triangleright}, \qquad B_2^2 = B_3^2 = 0, \qquad (80)$$

in the rotor form of $R = R_1(R_2 \wedge R_3)$, where

$$R_1 = e^{\frac{\varphi}{2}e_{12}}, \quad R_2 = \cos\varphi + \sin\varphi B_2, \quad R_3 = \cos\varphi + \sin\varphi B_3. \qquad (81)$$

Applying the rotor R to a conic conformal point $\mathbf{x}$ leads to

$$\mathbf{x}' = \widetilde{R}\mathbf{x}R, \; \mathbf{x}'_\varepsilon = x'\mathbf{e}_1 + y'\mathbf{e}_2, \; x' = \cos\varphi x - \sin\varphi y, \; y' = \sin\varphi x + \cos\varphi y, \qquad (82)$$

an anticlockwise (mathematically positive) rotation in the $\mathbf{I}_\varepsilon$-plane. Our formulation of the rotors above has some differences to Proposition 5.1 in [9], but is of the same level of algebraic complexity.

We note the useful invariance relationships, that

$$\widetilde{R}\mathbf{a}R = \mathbf{a} \quad \forall \mathbf{a} \in \{1, \mathbf{e}_o, \mathbf{e}_\infty, E, \mathbf{I}, \mathbf{I}_\varepsilon, \mathbf{I}_C, \mathbf{I}_{\infty o}, \mathbf{I}_\infty, \mathbf{I}_o, \mathbf{I}_\infty^{\triangleright}, \mathbf{I}_o^{\triangleright}\}. \qquad (83)$$

Because embedded flat and round CGA objects and all conics are constructed from outer products of between one and five points and $\mathbf{I}_\infty^{\triangleright}$, these objects are naturally covariant under rotations. Another consequence of this invariance is, that equations like (77), involving outer products with $\mathbf{I}_o^{\triangleright}$, are also covariant under rotations. Rotation covariance will certainly be of great value, when CCGA is employed, e.g.,

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[6] For $\{\mathbf{e}_{\infty 1}, \mathbf{e}_{o1}\}$ and $\{\mathbf{e}_{\infty 2}, \mathbf{e}_{o2}\}$, we could also have chosen $\lambda_1 = \lambda_2 = \sqrt{2}$ as in [9], without altering our form of the transformation versors given below. But for aesthetic reasons, we decided in (2) to simply set all three λ coefficients to 1.

for the construction of feature multivectors. We further note, that in general the vector $\mathbf{e}_{\infty 3}$ is not rotation invariant, which is natural for an outer product factor used to yield axis aligned conics, which being axis aligned, can not be expected to be rotation covariant.

Different from [9], *translation* by distance $a \in \mathbb{R}$ in the direction of $\mathbf{e}_1$ is achieved by the versor $T_x = T_1 T_2 = T_2 T_1$, with

$$T_1 = e^{\frac{1}{2}a\mathbf{e}_1\mathbf{e}_{\infty 1}} = 1 + \tfrac{1}{2}a\mathbf{e}_1\mathbf{e}_{\infty 1}, \quad T_2 = e^{\frac{1}{2}a\mathbf{e}_2\mathbf{e}_{\infty 3}} = 1 + \tfrac{1}{2}a\mathbf{e}_2\mathbf{e}_{\infty 3}, \qquad (84)$$

which leads to

$$\mathbf{x}' = \widetilde{T}_x\mathbf{x}T_x, \qquad \mathbf{x}'_\varepsilon = (x+a)\mathbf{e}_1 + y\mathbf{e}_2. \qquad (85)$$

Note, that the product of the two null bivector generators $(\mathbf{e}_1\mathbf{e}_{\infty 1})(\mathbf{e}_2\mathbf{e}_{\infty 3})$, commutes and is a null quadvector (squaring to zero).

Translation by distance $b \in \mathbb{R}$ in the direction of $\mathbf{e}_2$ is similarly achieved by the versor $T_y = T_3 T_4 = T_4 T_3$, with

$$T_3 = e^{\frac{1}{2}b\mathbf{e}_2\mathbf{e}_{\infty 2}} = 1 + \tfrac{1}{2}b\mathbf{e}_2\mathbf{e}_{\infty 2}, \quad T_4 = e^{\frac{1}{2}b\mathbf{e}_1\mathbf{e}_{\infty 3}} = 1 + \tfrac{1}{2}b\mathbf{e}_1\mathbf{e}_{\infty 3}, \qquad (86)$$

which leads to

$$\mathbf{x}' = \widetilde{T}_y\mathbf{x}T_y, \qquad \mathbf{x}'_\varepsilon = x\mathbf{e}_1 + (y+b)\mathbf{e}_2. \qquad (87)$$

We note the commutation relations

$$T_1 T_3 = T_3 T_1, \qquad T_2 T_4 = T_4 T_2, \quad \text{but} \quad T_x T_y \neq T_y T_x. \qquad (88)$$

In total we have for the application to conic conformal points

$$\mathbf{x}' = \widetilde{T}\mathbf{x}T, \quad T = T_x T_y = T_1 T_2 T_3 T_4, \quad \mathbf{x}'_\varepsilon = (x+a)\mathbf{e}_1 + (y+b)\mathbf{e}_2. \qquad (89)$$

It is interesting to note, that in spite of (88), we do have applied to any point $\mathbf{x}$ that

$$\mathbf{x}' = \widetilde{T}\mathbf{x}T = \widetilde{T_x T_y}\,\mathbf{x}\,T_x T_y = \widetilde{T_y T_x}\,\mathbf{x}\,T_y T_x. \qquad (90)$$

We further note, that Proposition 5.2 [9] needs *six* factors for a general translation, whereas we only need *four* factors. Equations (25) and (28) in Proposition 5.2 [9] are versors with quadratic terms in displacement distances a, b, which we do not need in (84) and (86). We have only bivector terms linear in displacement distances a, b, in each of the four versors $T_1, \ldots T_4$. We expect this to be advantageous in application to optimization problems.

We note the useful invariance relationships, that

$$\widetilde{T}\mathbf{a}T = \mathbf{a} \quad \forall \mathbf{a} \in \{1, \mathbf{e}_{\infty 1}, \mathbf{e}_{\infty 2}, \mathbf{e}_{\infty 3}, \mathbf{e}_\infty, \mathbf{I}, \mathbf{I}_\infty, \mathbf{I}_\infty^{\triangleright}\}. \qquad (91)$$

By construction therefore all flat and round CGA objects and all conics are translation covariant, this includes, because of $T\mathbf{e}_{\infty 3}\widetilde{T} = \mathbf{e}_{\infty 3}$, axis aligned conics as well, since translation has no effect on axis alignment.

As in [9], *scaling* by positive scalar $\alpha \in \mathbb{R}$ is achieved with the help of the scaling operator (scalor) $S = S_1 S_2 S_3$, where

$$S_k = \tfrac{1}{2}\left(\tfrac{\alpha+1}{\sqrt{\alpha}} + \tfrac{\alpha-1}{\sqrt{\alpha}}E_k\right), \qquad \widetilde{S}_k S_k = S_k\widetilde{S}_k = 1, \qquad 1 \leq k \leq 3. \qquad (92)$$

Note that the three factors S_k, $1 \leq k \leq 3$, mutually commute. This leads to (isotropic) scaling of points

$$\mathbf{x}' = \alpha\,\widetilde{S}\mathbf{x}S, \qquad \mathbf{x}'_\varepsilon = \alpha\,\mathbf{x}_\varepsilon. \qquad (93)$$

Note that the overall factor α could be omitted in $\mathbf{x}' = \alpha\, S\mathbf{x}\widetilde{S}$, due to the homogeneity of the conic point representation $\mathbf{x}$, but we include it for convenience, such that $\mathbf{x}' \cdot \mathbf{e}_\infty = \mathbf{x} \cdot \mathbf{e}_\infty = -1$.

The following multivector elements are invariant under scaling (93)

$$\mathbf{e}_o, \mathbf{e}_{o1}, \mathbf{e}_{o2}, \mathbf{e}_{o3}, \mathbf{I}_o, \mathbf{I}_{o12}, \mathbf{I}_o^{\triangleright}, \mathbf{e}_{o12}^{\triangleright}. \tag{94}$$

For bivectors this means

$$\{\mathbf{I}_{o12}, \mathbf{I}_o^{\triangleright}\} = \alpha^2 \widetilde{S}\{\mathbf{I}_{o12}, \mathbf{I}_o^{\triangleright}\}S, \tag{95}$$

and for the trivector

$$\mathbf{I}_o = \alpha^3 \widetilde{S}\mathbf{I}_o\, S. \tag{96}$$

Because conics (65) are constructed from outer products of points, the scaling operators S_k, $k = 1, 2, 3$, of (92) act (even individually) via outermorphisms covariantly on conics. And the expressions for intersections of conics (77), also remain covariant under scaling (93), because scaling (93) maps $\mathbf{I} \to \alpha^8 \mathbf{I}$ to a scalar multiple of itself, and because the representation of points and conics is homogeneous.

Furthermore, due to the relationships (8) and (9) one can show that the scalor (93) maps the three infinity vectors $\{\mathbf{e}_{\infty 1}, \mathbf{e}_{\infty 2}, \mathbf{e}_{\infty 3}\} \to \alpha^2\{\mathbf{e}_{\infty 1}, \mathbf{e}_{\infty 2}, \mathbf{e}_{\infty 3}\}$ to scalar multiples of themselves. This together with (94), that means that scaling can be covariantly applied to axis aligned conics, due to the homogeneity of the CCGA representation. This also applies to intersection operations involving axis aligned conics.

Isotropic scaling does apply to embedded CGA objects, because it simply maps $\mathbf{e}_\infty \to \alpha^2 \mathbf{e}_\infty$, $\mathbf{I}_\infty^{\triangleright} \to \alpha^4 \mathbf{I}_\infty^{\triangleright}$, $\mathbf{I}_{\infty 12} \to \alpha^4 \mathbf{I}_{\infty 12}$, and $\mathbf{I}_\infty \to \alpha^6 \mathbf{I}_\infty$, to scalar multiples of themselves, respectively, which is no problem due to the homogeneity of the representation.

Translators permit rotations and scaling relative to arbitrary centers of rotation and scaling, by first translating the respective center position to the origin, rotating or scaling with the above versors R, S, followed by back translation to the center.

Alternatively, it is possible to combine conic conformal geometric algebra with double conformal geometric algebra (DCGA) [5]. For that the coefficients of the conic a, b, c, d, e, f, can be extracted with (69) from $\mathbf{q}$ (see (75) for an example), and then used to define the same conic in DCGA. The rotors for reflection, rotation, translation and scaling are substantially simpler in DCGA. So if preferred, versor operations can be executed within DCGA, and the coefficients a', b', c', d', e', f', of the resulting new conic can be extracted from its DCGA bivector representation and transferred back to CCGA to define the same conic in CCGA.

7. Conclusion

This work on conic CGA (CCGA) $Cl(5,3)$ provides a detailed algebraic study and refinement of C. Perwass' *ansatz* [10]. An alternative recent approach we refer to is by J. Hrdina et al [9]. The complete CGA of two-dimensional Euclidean geometry appears as algebraic subset of CCGA. We showed how in CCGA conics can be defined from outer products of five on-curve points. This enables the computation of general intersections by outer products of duals of conics. We showed that CCGA

has versor operations, and compared with the versors given in [9]. A modification of basic algebraic definitions allowed us to simplify the formulation of translators to the form they had in [10], while preserving the full consistent set of versors for the operations of rotation, translation and scaling.

Nevertheless, it can be said, that versors in double CGA (DCGA) [1] are structurally simpler and a hybrid approach where the corresponding conics are transformed by reflection, rotation, translation, and dilation with versors in DCGA may be beneficial (e.g. in terms of numerical cost and accuracy of computation). Furthermore, DCGA has more general objects (cyclides) than conics, and differentiation is possible via simple commutator computations. But DCGA has no means to compute general intersections of conics, hence the use of CCGA becomes inevitable. Finally, extensions to three dimensions exist for quadric CGA [1], and to any finite dimension for DCGA [5].

Acknowledgments

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[7]J.C. Maxwell had inscribed at the gate of his Cavendish laboratory: *The works of the Lord are great, Studied by all who have pleasure in them.* [Psalm 111:2, New King James Version]. D. Capkova writes in Colloquium Comenius and Descartes, Foundation Comenius Museum, Naarden, 1997, p. 15, that: "Comenius' warning against onesided rationalism and against universal application of rationalism speaks to our time very urgently."

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