A Theory of second order linear variable coefficient equations

Ayéna Vignon Régis Yehossou[1], Marcellin Nonti[1], Jean Akande[1], Marc Delphin Monsia[1*]

[1]Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01. BP.526, Cotonou, BENIN

Abstract

A class of second order linear variable differential equations which may exhibit elementary function solutions is developed in this paper. It is shown that this class includes as special case the equation record in Kamke's book as equation 2. 80.

Theory

Consider the second order linear harmonic oscillator equation

$$\frac{d^2y}{d\tau^2}+cy(\tau)=0 \tag{1}$$

By using the nonlocal transformation [1]

$$y(\tau)=u(x)e^{l\,f(x)},\ d\tau=e^{\gamma g(x)}dx \tag{2}$$

The following theorem may be proved.

Theorem 1

Consider equation (1). Then by applying the nonlocal transformation (2), to equation may be mapped into

$$u''(x)+[2l\,f'(x)-\gamma g'(x)]u'(x)+\left[l\,f''(x)+l^2 f'(x)^2-l\gamma\, f'(x)g'(x)+ce^{2\gamma g(x)}\right]u(x)=0 \tag{3}$$

Proof. Using (2) one may compute the first derivate $y'(\tau)=\frac{dy(\tau)}{d\tau}$ as

$$\frac{dy(\tau)}{d\tau}=\left[u'(x)e^{l\,f(x)}+l\,f'(x)e^{l\,f(x)}u(x)\right]e^{-\gamma g(x)} \tag{4}$$

In this way the second derivative

$$\frac{d^2y(\tau)}{d\tau^2}=\left[u''(x)+\left(2lf'(x)-\gamma g'(x)\right)u'(x)+\left(lf''(x)+l^2 f'(x)^2-l\gamma f'(x)g'(x)\right)u(x)\right]e^{l\,f(x)}e^{-2\gamma g(x)} \tag{5}$$

Substituting (5) into (1) and using (2) leads to obtain immediately (3).

---

* Corresponding author. Email: monsiadelphin@yahoo.fr

Theorem 2

If $f(x) = x$ *, and* $g(x) = \ell n\, h(x)$*, where* $h(x) \succ 0$ *, then equation (3) reduce to*

$$u''(x) + u'(x)\left(2l - \gamma \frac{h'(x)}{h(x)}\right) + \left(l^2 - l\gamma \frac{h'(x)}{h(x)} + c\, h(x)^{2\gamma}\right) u(x) = 0 \qquad (6)$$

Equation (6) is recorded in Kamke's book [2] as equation (2. 80).

References

[1] J. Akande, D. K. K Adjaï, M. D. Monsia, On Schrödinger equations equivalent to constant coefficient equations, (2018), Math. Phys., viXra.org/1804.0222v2.pdf.

[2] E. Kamke, Differentialgleichungen lösungsmethoden und lösungen, Springer Fachmedien Weisbaden GMBH, 10th edition, 1977.