RedShift and an Exact Solution of the Einstein Equations with Cosmological term for Cylindrically Symmetric Empty Space

Xuan Zhong Ni, Campbell, CA, USA

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Abstract

In this article, we derived an exact solution of the Einstein equations for cylindrically symmetric empty space . we discussed the red shift can be a result of cylindrically symmetric of Einstein equation. We found that the red shift is larger when the distance of the light source is larger.

Here we derive an exact solution of the Einstein equations for cylindrically symmetric empty space. The static *condition*[1] means that, with a static coordinate system, the fundamental tensors, the $g_{\mu\nu}$ are independent of the time x^0 or t and also $g_{0m} = 0$. The spatial coordinates may be taken to be cylindrical coordinates $x^1 = r, x^2 = \phi\ , x^3 = z$. The general form for the square of invariant distance, the ds^2 compatible with cylindrical symmetry is

$$ds^2 = e^{2v}dt^2 - e^{2h}dr^2 - r^2d\phi^2 - e^{2u}dz^2 \tag{1}$$

where v, h, and u are functions of r and z only,

We can read off the values of $g_{\mu\nu}$ from equation (1), namely,

$g_{00} = e^{2v}, g_{11} = -e^{2h}, g_{22} = -r^2, g_{33} = -e^{2u}$, and $g_{\mu\nu} = 0, for \mu \neq \nu$

We can find Christoffel symbols and the calculate the Ricci tensors $R_{\mu\nu}$,

The non vanishing components of $R_{\mu\nu}$ are following,

$$R_{00}=e^{2(v-h)}\{\frac{\partial v}{\partial r}[-\frac{\partial(v+u-h)}{\partial r}-\frac{1}{r}]-\frac{\partial^2 v}{\partial r^2}\}+e^{2(v-u)}\{\frac{\partial v}{\partial z}[-\frac{(v-u+h)}{\partial z}]-\frac{\partial^2 v}{\partial z^2}\} \quad (2)$$

$$R_{11}=e^{2(h-u)}\{\frac{\partial h}{\partial z}[\frac{\partial(v-u+h)}{\partial z}]+\frac{\partial^2 h}{\partial z^2}\}-\frac{\partial h}{\partial r}[\frac{\partial(v+u)}{\partial r}+\frac{1}{r}]+(\frac{\partial v}{\partial r})^2+(\frac{\partial u}{\partial r})^2+\frac{\partial^2(v+u)}{\partial r^2} \quad (3)$$

$$R_{22}=e^{-2h}r\frac{\partial(v+u-h)}{\partial r} \quad (4)$$

$$R_{33}=e^{2(u-h)}\{\frac{\partial u}{\partial r}[\frac{\partial(v+u-h)}{\partial r}+\frac{1}{r}]+\frac{\partial^2 u}{\partial r^2}\}+(\frac{\partial v}{\partial z}^2)+\frac{\partial^2(v+h)}{\partial z^2}+(\frac{\partial h}{\partial z^2})^2-\frac{\partial u}{\partial z}\frac{\partial(v+h)}{\partial z} \quad (5)$$

Einstein's law requires all $R_{\mu\nu}$ satisfy the following relation. Thus,

$$R_{\mu\nu}=\Lambda g_{\mu\nu} \quad (6)$$

for all μ,ν.

The equations $R_{00}=\Lambda g_{00}, R_{11}=\Lambda g_{11}, R_{22}=\Lambda g_{22}, and R_{33}=\Lambda g_{33}$ will be,

$$e^{2(v-h)}\{\frac{\partial v}{\partial r}[-\frac{\partial(v+u-h)}{\partial r}-\frac{1}{r}]-\frac{\partial^2 v}{\partial r^2}\}+e^{2(v-u)}\{\frac{\partial v}{\partial z}[-\frac{(v-u+h)}{\partial z}]-\frac{\partial^2 v}{\partial z^2}\}=\Lambda e^{2v} \quad (7)$$

$$e^{2(h-u)}\{\frac{\partial h}{\partial z}[\frac{\partial(v-u+h)}{\partial z}]+\frac{\partial^2 h}{\partial z^2}\}-\frac{\partial h}{\partial r}[\frac{\partial(v+u)}{\partial r}+\frac{1}{r}]+(\frac{\partial v}{\partial r})^2+(\frac{\partial u}{\partial r})^2+\frac{\partial^2(v+u)}{\partial r^2}=-\Lambda e^{2h} \quad (8)$$

$$e^{-2h}r\frac{\partial(v+u-h)}{\partial r}=-\Lambda r^2 \quad (9)$$

$$e^{2(u-h)}\{\frac{\partial u}{\partial r}[\frac{\partial(v+u-h)}{\partial r}+\frac{1}{r}]+\frac{\partial^2 u}{\partial r^2}\}+(\frac{\partial v}{\partial z}^2)+\frac{\partial^2(v+h)}{\partial z^2}+(\frac{\partial h}{\partial z^2})^2-\frac{\partial u}{\partial z}\frac{\partial(v+h)}{\partial z}=-\Lambda e^{2u}, \quad (10)$$

We are looking for the solution with function u is a function of z only and with the following conditions,

$$e^{2v}\to 1, e^{2u}\to 1, e^{2h}\to 1 \quad (11)$$

when $r\to 0$, and $z\to 0$,

We can get the exact solution of the following,

$$e^{2v} = cos^2(\sqrt{\Lambda}z) \tag{12}$$

$$e^{2u} = 1 \tag{13}$$

$$e^{2h} = \frac{1}{1 - \Lambda r^2} \tag{14}$$

Let us consider an event that a light source at rest in a remote point on the z-axis emitting monochromatic radiation, and an observer at the original point receiving the light. The physics with this event is cylindrically symmetric around z-axiz. The wavelenght of the light is Δs.

$$\Delta s^2 = g_{00}(\Delta x^0)^2 \tag{15}$$

here Δx^0 is the period of the light.

This period is thus dependent on g_{00} at the place where the light was emitted:

$$\Delta x^0 :: g_{00}^{-\frac{1}{2}} = \frac{1}{cos(\sqrt{\Lambda}z)} \tag{16}$$

when $z \ll \frac{1}{\sqrt{\Lambda}}$, The redshift will be larger when the distance z is larger. The BIG BAN theory is based on this redshift phenomena. But we think it is only a result of Einstein equations.

References

[1] P.A.M. Dirac,General Theory of Relativity, Wiley, 1975.