Intrinsic vector potential and electromagnetic mass

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Abstract

Electric charges may have mass in part or in full because they charged. Supplying details is the electromagnetic mass problem. Here, the charge's mass is associated with intrinsic quantum mechanical quantities so that the classical problems with extended charge distributions, for example, are irrelevant. An intrinsic vector potential is defined, based on intrinsic linear momentum. The charge-electromagnetic field interaction energy is gauge-dependent and the needed mass term is placed with the interaction energy in the intrinsic gauge. Traditional electromagnetism retains its gauge invariance. The field equations make no new predictions since all dynamic dependence on intrinsic quantities can be gauged away. The field equations describe a massive, charged 4-spinor Dirac electron-like particle and an uncharged, massless neutrino-like particle, formulas that have been a part of physics for nearly a century.

1 Introduction

Opposite charges attract; like charges repel. If the interaction is mediated by an electromagnetic field and if the electromagnetic field has energy, then carrying the electromagnetic field should act as a drag on a charge's motion, an inertia. The problem is called "electromagnetic mass" and it has only controversial solutions. [3, 4, 9] Electromagnetic mass is frequently presented as an unsolved problem in introductory textbooks. [1, 2] Here we show how a charge may have mass due to its being charged, but in a way that has nothing to do the interaction being mediated by an electromagnetic field.

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Instead, we suggest that electromagnetic mass may be an intrinsic quantity like spin. Let us, for a moment, consider angular momentum. Given enough grease and quality bearings, a large flywheel rotating about a stationary axis can maintain a constant angular momentum. To keep the flywheel intact, forces are needed. Failing flywheels can do impressive damage.

Now consider quantum effects. An electron can be in a motionless state and still have angular momentum, its 'intrinsic' or 'spin' angular momentum. It is widely accepted that, with spin, "a consistent mechanical model doesn't exist", see, for example, page 374 of [7]. The physical description of intrinsic electron spin does not have a place for the forces that prevent rotating matter flying apart like a crumbling flywheel.

Intrinsic quantum quantities avoid the complications that accompany mechanical models. For example, with intrinsic electromagnetic mass, one would not need to introduce non-electromagnetic forces to counter the repulsion of like-charges in order to hold an extended charge distribution model together. There simply would not be an extended charge distribution model.

In this paper, we introduce an intrinsic matrix vector potential for a charged spin 1/2 field. Before defining an intrinsic vector potential, we first need an intrinsic momentum, a matrix that generates translations just as spin matrices generate rotations. A matrix representation (rep) of the Poincaré algebra is required. In Sec. 2, we describe the needed algebra, a representation of spacetime rotations and translations for spin 1/2.

The intrinsic matrix momentum that generates translations is needed to define the intrinsic vector potential. The intrinsic matrix translations are taken to be independent of the continuous translation rep that applies to functions. The matrix rep has its own matrix momentum generators and, we contend, its own displacements with its own intrinsic coordinate system y. Assuming that intrinsic coordinates y are independent of spacetime coordinates x avoids the mechanical model problem mentioned above.

Sec. 3 shows how to upgrade a "Lorentz" quantum field that already transforms under a matrix representation of the Lorentz group of spacetime rotations to a "Poincaré" field that transforms via Poincaré transformations so that it responds to matrix translations.

Sec. 4 begins with a Lorentz 8-spinor quantum field ψ_0 that, as is traditional, transforms under spacetime rotations but not translations. Since mass is introduced with electromagnetic interactions, we can, and do, assume that the free field ψ_0 is massless, so that ψ_0 is a linear combination of massless annihilation and creation operators. The Lorentz field ψ_0 is then upgraded in Sec. 4, by the process in Sec. 3, to a Poincaré field Φ_0 that transforms by the matrix rep of Sec. 2, including matrix translations.

The current J of the field Φ_0 is like the current j of the traditional field ψ_0 except that more complicated matrices $\alpha(y)$ in J replace the Dirac gamma matrices γ in j. It happens that the matrices $\alpha(y)$ obey an identity that looks like a Maxwell's equation in intrinsic coordinates y. This is an opportunity to identify the matrix $\alpha(y)$ as proportional to the intrinsic vector potential. The Maxwell-like equation also shows that one 4-spinor is charged and the other 4-spinor is uncharged in the 8-spinor field.

The lagrangian L, developed in Sec. 5, modifies a traditional lagrangian L_0 by including intrinsic quantities. The lagrangian L_0 combines lagrangians for a free massless 8-spinor ψ_0, a free continuous rep vector potential $A(x)$, and an interaction term for $A(x)$ with the charged 4-spinor current as source.

Adding the intrinsic vector potential $\alpha(y)$ to $A(x)$ produces the lagrangian in its final form, final aside from choosing the intrinsic gauge. The intrinsic gauge removes the intrinsic y-dependence and provides the mass term. The mass is arbitrary.

A traditional Maxwell equation is found for the continuous vector potential $A(x)$. For the 4-spinors, the field equations are the traditional Dirac equation for the charged, massive 4-spinor. The Dirac equation for the uncharged, massless 4-spinor is the same as the field equation for a free massless Dirac field.

These are familiar field equations, one for a Dirac electron and the other for a massless 4-spinor neutrino, so there are no new consequences from the field equations. However, the fact that one is massive and the other massless is a consequence of their electric charge values. Concluding remarks are presented in Sec. 6.

2 The 8-spinor Rep of the Poincar'e group

This section details the matrix representation of the Poincaré group of spacetime rotations and translations needed to transform the 8-spinor fields. There are angular momentum matrices σ that generate spacetime rotations and (linear) momentum matrices π that generate translations. The generators are 8×8 matrices that obey the Poincaré algebra.

As functions of spacetime, fields also need the differential representation of the spacetime symmetries with momentum proportional to the divergence. We call this rep the "continuous rep", while the other rep is the "matrix rep".

Suppose the points, also called events, of $3+1$ spacetime are labeled with Minkowski coordinates x^{μ}, with $\mu, \nu, ... \in \{1,2,3,4\}$ and $x^4 = x^t$ the time component. Having $\mu = 4$ as the time index conflicts with the convention $\mu = 0$ for time when one calculates the antisymmetric tensor: with time last, we have $\epsilon_{1234} = +1$, and, time first, $\epsilon_{4123} = -1$. Let the spacetime metric be the diagonal tensor $\eta_{\mu\nu}$ with diagonal $\eta_{11} = \eta_{22} = \eta_{33} = +1$ and $\eta_{44} = -1$. As shorthand, the metric combines with vectors and tensors to "raise" and "lower" indices. For example, for vector v^{μ}, we write $v_{\nu} \equiv \eta_{\mu\nu} v^{\mu}$. Repeated indices are summed, unless noted otherwise.

A Poincaré transformation applied to spacetime preserves scalar products, such as the scalar product of one primed, one unprimed interval, $\delta x^{\prime}_{\sigma} \delta x^{\sigma}$, where the interval δx^{μ} is the difference of the coordinates of points 1 and 2. A translation adds the same amount b^{μ} to each and, therefore, there is no change in the difference,

$$\delta x^{\mu} = (x_2^{\mu} + b^{\mu}) - (x_1^{\mu} + b^{\mu}) = x_2^{\mu} - x_1^{\mu} \quad . \tag{1}$$

Intervals, and hence their scalar products, are invariant. Translations are inhomogeneous, additive transformations.

Any Poincaré transformation, written (Λ, b), can be considered to be a spacetime rotation Λ followed by a translation along some displacement b. We have $(\Lambda, b) = (1, b)(\Lambda, 0)$.

Applying first transformation A followed by B gives the product transformation

$$(\Lambda_B, b_B)(\Lambda_A, b_A) = (\Lambda_B \Lambda_A, \Lambda_B b_A + b_B) \quad . \tag{2}$$

This is the rule for successive Poincaré transformations.

The 8×8 matrices in our rep of the Poincaré algebra are conveniently arranged into four 4×4 blocks,

$$\begin{pmatrix} 11 & 12 \\ 21 & 22 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 12 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 21 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 22 \end{pmatrix} \quad . \tag{3}$$

Write M_{ij} for a matrix that has nonzero components only in the ij block, $i, j \in \{1, 2\}$. It follows that $M_{ij} M_{kl} = \delta_{jk} N_{il}$, where matrix N_{il} has nonzero components confined to the block il and δ_{jk} is the Kronecker delta which is unity for equal indices $j = k$ and vanishes otherwise.

The 4×4 gamma matrices γ_D^{μ} of the Dirac formalism are taken to be the following matrices

$$\gamma_D^{\mu} = i \begin{pmatrix} 0 & -\tau^{\mu} \\ \tau_{\mu} & 0 \end{pmatrix} \tag{4}$$

with Pauli matrices τ^{μ},

$$\tau^{\mu} = \{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \} \, , \tag{5}$$

for $\mu = 1, 2, 3, 4$, respectively. One can check the defining requirement of Dirac gamma matrices, $\gamma_D^{\mu} \gamma_D^{\nu} + \gamma_D^{\nu} \gamma_D^{\mu} = 2 \eta^{\mu\nu} \mathbf{1}$. [5]

Let γ_{ij}^{μ} be an 8×8 matrix with γ_D^{μ} in the ij-block and which vanishes elsewhere. Since the γ_D^{μ} matrices satisfy the defining requirement of Dirac gammas, one has

$$\gamma_{ij}^{\mu} \gamma_{kl}^{\nu} + \gamma_{ij}^{\nu} \gamma_{kl}^{\mu} = 2 \delta_{jk} \eta^{\mu\nu} \mathbf{1}_{il} \quad , \tag{6}$$

where $\mathbf{1}_{il}$ has the 4×4 identity matrix in the il block with the other three 4×4 blocks null. We reserve the notation γ^μ for the following 8×8 matrices

$$\gamma^\mu \equiv \gamma^\mu_{11} + \gamma^\mu_{22} \quad , \tag{7}$$

which has 4×4 gamma matrices γ^μ_D in the two diagonal blocks 11 and 22.

Angular momentum matrices $\sigma^{\mu\nu}$ generate spacetime rotations. For the 8-spinor rep, choose the following 8×8 matrices

$$\sigma^{\mu\nu} = -\frac{i}{4}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) = -\frac{i}{4}\left(\gamma^\mu_{11}\gamma^\nu_{11} - \gamma^\nu_{11}\gamma^\mu_{11}\right) - \frac{i}{4}\left(\gamma^\mu_{22}\gamma^\nu_{22} - \gamma^\nu_{22}\gamma^\mu_{22}\right) \quad . \tag{8}$$

Thus the 8-spinor rep of the group of spacetime rotations reduces to two 4-spinor reps, placed diagonally in the 11- and 22-blocks.

The linear momentum matrices π^μ are defined to be

$$\pi^\mu = k\gamma^\mu_{21} \quad . \tag{9}$$

These four 8×8 matrices generate translations.

The angular and linear momentum matrices $\sigma^{\mu\nu}$ and π^μ obey the Poincaré algebra,

$$i\left[\sigma^{\mu\nu}, \sigma^{\rho\lambda}\right] = \eta^{\nu\rho}\sigma^{\mu\lambda} - \eta^{\mu\rho}\sigma^{\nu\lambda} - \eta^{\nu\lambda}\sigma^{\mu\rho} + \eta^{\mu\lambda}\sigma^{\nu\rho} \, , \tag{10}$$

$$i\left[\sigma^{\mu\nu}, \pi^\rho\right] = \eta^{\nu\rho}\pi^\mu - \eta^{\mu\rho}\pi^\nu \quad \text{and} \quad i\left[\pi^\mu, \pi^\nu\right] = 0 \, . \tag{11}$$

Equivalent reps with matrix generators differing from $\sigma^{\mu\nu}$ and π^μ by similarity transformations also satisfy the Poincaré algebra.

The generators combined with real-valued parameters make a transformation that acts on 8-spinors. Since angular momentum generators are antisymmetric, the associated parameters $\omega_{\mu\nu}$ might as well be antisymmetric, $\omega_{\nu\mu} = -\omega_{\mu\nu}$ since any symmetric part would not contribute. Let Λ be the transformation of spacetime 4-vectors in the continuous rep and $D(\Lambda, 0)$ be the 8×8 matrix transformation for 8-spinors.

For translations, the parameters are called displacements. For a displacement b^μ, spacetime coordinate transform as follows $x^\mu \rightarrow x^\mu + b^\mu$. Denote the associated 8×8 matrix transformation by $D(1, b)$.

Let the symbol $D(\Lambda, b)$ stand for the 8×8 matrix transformation resulting from a rotation followed by a translation. We consider only transformations that can be connected to the identity by successive infinitesimal transformations. Building transformations infinitesimally involves the matrix exponential and one finds that the 8×8 matrix transformation $D(\Lambda, b)$ is given by

$$D(\Lambda, b) = D(1, b)D(\Lambda, 0) = e^{-ib_\mu\pi^\mu} e^{i\omega_{\mu\nu}\sigma^{\mu\nu}/2} \quad . \tag{12}$$

It represents the Poincaré transformation (Λ, b). Since $\sigma^{\mu\nu}$ and π^{μ} are 8×8 matrices, the matrix $D(\Lambda, b)$ is an 8×8 matrix.

By definition, (9), the nonzero components of the momentum matrices π^{μ} are in the 21-block. This implies that the product of two momentum matrices vanishes and, therefore, the translation matrix $D(1, b)$ is linear in b^{μ},

$$\pi^{\mu}\pi^{\nu} = 0 \quad \text{and} \quad D(1,b) = e^{-ib_{\mu}\pi^{\mu}} = \mathbf{1} - ib_{\mu}\pi^{\mu} \quad . \tag{13}$$

When the translation matrix $D(1, b)$ is applied to an 8-spinor, $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, a combination of the first 4-spinor's components is added to the second 4-spinor. We have

$$D(1,b)\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - ib_{\mu} \begin{pmatrix} 0 & 0 \\ k\gamma^{\mu}_{D} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 - ikb_{\mu}\gamma^{\mu}_{D}\psi_1 \end{pmatrix} \quad . \tag{14}$$

The first 4-spinor ψ_1 is the "donor" and the second 4-spinor ψ_2 is the "receiver".

This additive behavior is consistent with the inhomogeneous nature of translations. Clearly there are no eigenspinors nor any eigenvalues of intrinsic translations because there is no 8-spinor ψ with $D(1, b)\psi$ proportional to ψ. No translation eigenvalues for translations means no contributions to linear momentum. Unlike spin, which does contribute to the total angular momentum, intrinsic translations do not contribute to the observed linear momentum of a quantum system. Intrinsic translations make their presence felt in other ways, as will be seen in what follows.

3 Quantum fields with translations

In this article, we need fields that transform with the full Poincaré group of spacetime symmetries, including matrix translation reps. Often, fields are defined without matrix translations. The fields are said to be "translation scalars", or, put another way, displacements vanish for matrix translations. [6, 8] Since the fields transform with matrix reps of the Lorentz group, call them "Lorentz fields." We need to take such a Lorentz field and make from it a "Poincaré" field that transforms with a matrix rep of the Poincaré group, including non-trivial reps of translations. This section shows how. While the resulting field is not as general as allowing translations earlier in the derivation of quantum fields, the result suffices for the purposes here.

A quantum field ψ is a sum of annihilation and creation operators. Under spacetime symmetries, the operators transform by a unitary representation. The unitary rep must be infinite dimensional since boosts are not compact. Infinite dimensional reps can be inconvenient for some purposes. Thus one constructs fields. [8] Fields are operators since they are sums of operators, yet they transform under spacetime rotations via nonunitary, most often finite dimensional reps.

In this article, not only do fields transform differently than operators under spacetime transformations, the fields can describe a massive particle while the operators are massless.

Given a Poincaré transformation (Λ, b), the operators transform by some infinite dimensional unitary rep denoted $U(\Lambda, b)$. One builds a Lorentz field ψ as a linear combination of these operators. The adjective "Lorentz" means that the field transforms only under the Lorentz transformation Λ and not via translations. The Lorentz transformation matrix is written $D(\Lambda, 0)$, the "0" enforcing the assumption that $b^{\mu} = 0$ for the matrix transformations of Lorentz fields.

Thus, we can begin by assuming that a Lorentz field ψ can be found that satisfies the requirement that the operators transform unitarily and field transforms non-unitarily as follows,

$$U(\Lambda, b)\psi_l(x)U^{-1}(\Lambda, b) = D^{-1}_{l\bar{l}}(\Lambda, 0)\psi_{\bar{l}}(\Lambda x + b) \quad . \tag{15}$$

Lorentz fields are standard quantum fields. See, for example, Ref. [8].

Yet one can define a field Φ, based on ψ, that responds to translation matrices. Consider applying a matrix translation $D(1, y)$ to ψ to obtain the quantity $\Phi_l(x, y)$,

$$\Phi_l(x, y) \equiv D_{ls}(1, y)\psi_{sl}(x) \quad , \tag{16}$$

where y is a 4-vector coordinate-like displacement that is independent of x. The coordinates y are called "intrinsic coordinates". Since $\psi(x)$ is a linear combination of annihilation and creation operators, it follows that Φ is such a sum as well, but with coefficients that differ from those for ψ by the matrix factor $D(1, y)$. Therefore Φ is a quantum field.

Now we need to show that Φ obeys (15) except with $D^{-1}(\Lambda, b)$ on the right. Once shown, this means that the unitary Poincaré transformation of the operators $U(\Lambda, b)$ produces a nonunitary matrix Poincaré transformation of the field Φ.

To show that, first note that applying an arbitrary Poincaré transformation (Λ', b'), to spacetime gives the points x and y new coordinates, $x' = \Lambda' x + b'$ and $y' = \Lambda' y + b'$. Replacing these for the x and y in (16) and dropping the primes, we find

$$\Phi_l(\Lambda x + b, \Lambda y + b) = D_{ls}(1, \Lambda y + b)\psi_s(\Lambda x + b) \quad . \tag{17}$$

Given that, one can form the expression

$$D^{-1}_{l\bar{l}}(\Lambda, b)\Phi_{\bar{l}}(\Lambda x + b, \Lambda y + b) = D^{-1}_{l\bar{l}}(\Lambda, b)D_{\bar{l}s}(1, \Lambda y + b)\psi_s(\Lambda x + b) \quad . \tag{18}$$

We need to evaluate the product of the Poincaré transformations on the right.

The rule for successive Poincaré transformations, (2), implies $D(\Lambda^{-1}, -\Lambda^{-1}b)\ D(\Lambda, b) = D(1,0)$ and, therefore, we have

$$D^{-1}(\Lambda, b) = D(\Lambda^{-1}, -\Lambda^{-1}b) \quad , \tag{19}$$

where we have begun to hide the matrix indices. This implies, again by (2), that

$$D^{-1}(\Lambda, b)D(1, \Lambda y + b) = D(\Lambda^{-1}, -\Lambda^{-1}b)D(1, \Lambda y + b) \tag{20}$$

$$= D(\Lambda^{-1}, \Lambda^{-1}(\Lambda y + b) - \Lambda^{-1}b) = D(\Lambda^{-1}, y) = D(1, y)D^{-1}(\Lambda, 0) \quad .$$

Substituting the last expression into the right side of (18) yields

$$D^{-1}(\Lambda, b)\Phi(\Lambda x + b, \Lambda y + b) = D(1, y)D^{-1}(\Lambda, 0)\psi(\Lambda x + b) \quad . \tag{21}$$

By the Lorentz field transformation (15), we can replace $D^{-1}(\Lambda, 0)\psi(\Lambda x + b)$,

$$D^{-1}(\Lambda, b)\Phi(\Lambda x + b, \Lambda y + b) = D(1, y)U(\Lambda, b)\psi(x)U^{-1}(\Lambda, b) \quad . \tag{22}$$

Finally, the matrix $D(1, y)$ acts on $\psi(x)$ and commutes with $U(\Lambda, b)$ which acts on operators. By the definition of Φ in (16) together with switching the left and right sides of (22), we find that

$$U(\Lambda, b)\Phi(x, y)U^{-1}(\Lambda, b) = D^{-1}(\Lambda, b)\Phi(\Lambda x + b, \Lambda y + b) \quad , \tag{23}$$

which was to be shown.

Thus $\Phi(x, y) = D(1, y)\psi(x)$ is a field that transforms via the matrix Poincaré rep $D(\Lambda, b)$, which includes translations. The Poincaré field $\Phi(x, y)$ is needed to make the intrinsic vector potential.

4 Currents and the intrinsic E-M vector potential

In this section we consider the free 8-spinor field and find evidence that the first 4-spinor is charged. Terms like "the first 4-spinor" are keyed to the matrix rep in Sec. 2.

Let ψ_0 be an 8-spinor Lorentz quantum field that is constructed from the annihilation and creation operators for a massless spin 1/2 particle. This enforces the idea that mass is associated with charge. The 8-spinor can host two 4-spinors, so ψ_0 is a Lorentz free 8-spinor quantum field that is the direct sum of two free massless 4-spinor fields.

Thus, by assumption, field ψ_0 is a Lorentz quantum field. When spacetime undergoes the Poincaré transformation (Λ, b), ψ_0 transforms by the matrix $D^{-1}(\Lambda, 0)$, see (12), as if

b didn't exist, $b \rightarrow 0$. Then, by the work in the preceding section, the Poincaré field Φ_0, defined by

$$\Phi_0 \equiv D(1,y)\psi_0(x) \quad , \tag{24}$$

transforms with the matrix $D^{-1}(\Lambda,b)$ under the spacetime transformation (Λ,b). The field Φ_0, like ψ_0, is interaction free.

The 4-vector probability current density j^{μ}, for the Lorentz field ψ_0, is defined to be

$$j^{\mu} \equiv N_8 \bar{\psi}_0 \gamma^{\mu} \psi_0 \quad , \tag{25}$$

where N_8 is a normalization constant and $\bar{\psi}_0 = \psi_0^{\dagger}\gamma^t$ is the Dirac conjugate, the hermitian transpose times the time component gamma matrix. Since, by (7), $\gamma^{\mu} = \gamma^{\mu}_{11} + \gamma^{\mu}_{22}$, we have

$$j^{\mu} = N_8 \bar{\psi}_0 \gamma^{\mu} \psi_0 = \frac{N_8}{N_1} N_1 \bar{\psi}_{01} \gamma^{\mu}_{11} \psi_{01} + \frac{N_8}{N_2} N_2 \bar{\psi}_{02} \gamma^{\mu}_{22} \psi_{02} = \frac{N_8}{N_1} j^{\mu}_1 + \frac{N_8}{N_2} j^{\mu}_2 \quad , \tag{26}$$

where $j^{\mu}_i \equiv N_i \bar{\psi}_{0i} \gamma^{\mu} \psi_{0i}$ are the currents for the two 4-spinors in the 8-spinor field and the Ns are normalization constants.

The current of the Poincaré field Φ_0 in (24) is denoted J^{μ}. We find

$$J^{\mu} \equiv N_8 \bar{\Phi}_0 \gamma^{\mu} \Phi_0 = -N_8 \bar{\psi}_0(x) \gamma^t D^{\dagger}(1,y) \gamma^t \gamma^{\mu} D(1,y) \psi_0(x) = N_8 \bar{\psi}_0 \, \frac{\alpha^{\mu}}{a} \, \psi_0 \quad , \tag{27}$$

where the minus sign appears because $\gamma^{t2} = \eta^{tt} = -\mathbf{1}$ and where, for convenience, the constant a is included now to be determined later and $\alpha^{\mu}(y)$ is the matrix

$$\alpha^{\mu}(y) = -a \gamma^t D^{\dagger}(1,y) \gamma^t \gamma^{\mu} D(1,y) \quad . \tag{28}$$

Comparing (25) and (27), we see that the matrix α^{μ}/a replaces γ^{μ} in the current j^{μ} when we go from a field ψ_0 that doesn't transform with the matrix translation rep to the field Φ_0 that does translate via the matrix rep.

Aside from the scale constant a, the matrix $\alpha^{\mu}(y)$ is the given function of y in (28). Note, for reference later, that $\alpha^{\mu}(y)$ is a fixed matrix and is not free to vary when it appears in a lagrangian.

We can work with the expression for $\alpha^{\mu}(y)$. By (13), since the momentum matrix $\pi^{\mu} = k\gamma^{\mu}_{21}$ in (6) is nonzero only in the 21-block, the translation $D(1,y)$ is linear in y. With the hermitian conjugate of π^{μ} being $\pi^{\mu\,\dagger} = k\gamma_{12\,\mu}$, one can show that the matrix α^{μ}/a can be written as

$$\alpha^{\mu}(y) = a \left[\gamma^{\mu} - iky_{\rho} \left(\gamma^{\rho}_{12} \gamma^{\mu}_{22} + \gamma^{\mu}_{22} \gamma^{\rho}_{21} \right) + k^2 y^2 \gamma^{\mu}_{11} - 2k^2 y^{\mu} y_{\rho} \gamma^{\rho}_{11} \right] \quad , \tag{29}$$

which is quadratic in y and where $y^2 = y_\rho y^\rho$.

Since $\alpha^\mu(y)$ is quadratic in y, second order partial derivatives with respect to y are constant. One finds an identity,

$$\partial^{\lambda\prime}\partial^\prime_\lambda \alpha^\mu - \partial^{\mu\prime}\partial^\prime_\kappa \alpha^\kappa = 12ak^2\gamma^\mu_{11} \quad , \tag{30}$$

where the partial derivatives are with respect to the ys, meaning $\partial^\prime_\rho \alpha^\mu \equiv \partial\alpha^\mu/\partial y^\rho$. (We are saving unprimed partials for x.)

The current $J^\mu(x,y) = N_8\bar{\psi}_0\alpha^\mu\psi_0/a$ has its y-dependence confined to $\alpha^\mu(y)$. Since x and y are independent, the x-dependent functions $\bar{\psi}_0(x)$ and $\psi_0(x)$ are constants when differentiating with respect to y. Thus, immediately from (30), we have

$$\partial^{\lambda\prime}\partial^\prime_\lambda(aJ^\mu) - \partial^{\mu\prime}\partial^\prime_\kappa(aJ^\kappa) = 12ak^2N_8\bar{\psi}_0\gamma^\mu_{11}\psi_0 = 12ak^2\frac{N_8}{N_1}j^\mu_1 \quad . \tag{31}$$

where we treat x and y as independent quantities, $\partial x^\mu/\partial y^\nu = 0$.

Compare (31) with one of Maxwell's equations,

$$\partial^\lambda\partial_\lambda A^\mu_q - \partial^\mu\partial_\kappa A^\kappa_q = \rho^\mu \quad , \tag{32}$$

where $\partial_\lambda A^\mu_q \equiv \partial A^\mu_q/\partial x^\lambda$ and A^μ_q is the vector potential due to a charged current density ρ^μ. Clearly, the two equations (31) and (32) have the same form.

We are lead to define the quantity A^μ_i, related to α^μ,

$$A^\mu_i \equiv N_8\bar{\psi}_0\alpha^\mu\psi_0 = aJ^\mu = \frac{qN_1}{12k^2N_8}J^\mu \quad , \tag{33}$$

so that the right side of (31) is the current $\rho^\mu = qj^\mu_1$ of a charge q. This has determined the constant

$$a = \frac{qN_1}{12k^2N_8} \tag{34}$$

which was introduced with J^μ in (27).

By (31) and (33), at first glance, it would seem that the quantity A^μ_i, satisfies Maxwell's equation (32). But wait, this is obviously false because the partials in (32) are with respect to x, not y. And, unlike the matrix quantity $\alpha^\mu(y)$, the vector potential $A_q(x)$ is a function of the same coordinates x as the current density $\rho^\mu = qj^\mu_1(x)$.

However, that is all right because no intrinsic quantity can be a function of x. The intrinsic quantity would then have "mechanical" properties, which is like proposing a rotating mass with a density function of x as the source of the electron's spin. No mechanical model accounts for electron spin and no mechanical model should exist for the intrinsic vector potential. Hence, spacetime coordinates x and intrinsic coordinates y should be independent.

Based on these remarks, we identify $A^{\mu}_{i} = aJ^{\mu}$ as the "intrinsic vector potential" and α^{μ} as the "intrinsic vector potential matrix" or "matrix vector potential." Thus (31) becomes the Maxwell-like equation

$$\partial^{\rho\prime}\partial^{\prime}_{\rho}A^{\mu}_{i} - \partial^{\mu\prime}\partial^{\prime}_{\kappa}A^{\kappa}_{i} = qj^{\mu}_{1} \quad , \tag{35}$$

so that the intrinsic vector potential A^{μ}_{i} satisfies Maxwell's equation (32) aside from the issue of coordinates x versus y.

The Maxwell-like equation (35) is our justification for characterizing the first 4-spinor as charged and the second 4-spinor as uncharged. The reason for the difference originates with the intrinsic momentum matrix π^{μ}. By definition (9), the matrix $\pi^{\mu} = k\gamma^{\mu}_{21}$ is off-diagonal with nonzero components in the 21-block. Thus the translation matrix $D(1, y)$ has all of its y-dependence in the 21-block and the hermitian conjugate $D^{\dagger}(1, y)$ has y-dependence in the 12-block. Since the gamma matrices γ^{t} and γ^{μ} are block-diagonal, the quadratic y-dependence follows from $D^{\dagger}(1, y)$ times $D(1, y)$ which is effectively a 12-block times a 21-block. But 12 times 21 yields a 11-block, and all the quadratic y-components in α^{μ} are in the 11-block as seen in (29). Applying two y-derivatives to α^{μ} can, therefore, only yield nonzero results in the 11-block, as in (30). Hence the second 4-spinor ψ_2 is not charged while the first 4-spinor ψ_1 is charged and that is due to the off-diagonal donor-receiver nature of the momentum matrices.

Gauge invariance of the Maxwell-like equation (35) follows when the matrix vector potential α^{μ} undergoes a gauge transformation

$$\tilde{\alpha}^{\mu} = \alpha^{\mu} + \partial^{\mu}\chi \quad , \tag{36}$$

where the gauge function $\chi(y)$ is an 8×8 matrix whose components have symmetric second partials, $\partial^{\nu\prime}\partial^{\mu\prime}\chi = \partial^{\mu\prime}\partial^{\nu\prime}\chi$.

One can find a gauge χ_0 that makes the gauge-transformed intrinsic vector potential divergence-free. We don't do that. Instead we need a different gauge, a special gauge to make a mass term.

5 Gauge, mass term and field equations

The previous section dealt with free quantum fields. In this section a lagrangian is chosen for interacting fields and field equations are found.

The initial lagrangian L_0 combines the free-field lagrangians L_{ψ} and L_{A} for electromagnetism with the lagrangian L_{int} for the electromagnetic interaction. One has

$$L_0 \equiv L_{\psi} + L_{\mathrm{A}} + L_{\mathrm{int}} = \bar{\psi}p_{\lambda}\gamma^{\lambda}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - qA_{\mu}j^{\mu}_{1} \quad , \tag{37}$$

where $p_\lambda = i\partial_\lambda$ is the momentum, the partial derivative being with respect to spacetime coordinates x^λ, $F_{\mu\nu}$ is the electromagnetic field of the vector potential A^μ, $F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$. The quantity A^μ is the vector potential and the charge current is qj_1^μ. The functions in L_0 depend on x. There are no intrinsic quantities in L_0.

The interaction lagrangian, i.e. $L_{\text{int}} = -qA_\mu j_1^\mu$, does not include j_2^μ because we interpret the Maxwell-like equation (35) as showing that j_1^μ, not j_2^μ, carries electromagnetic current.

Now the initial lagrangian L_0 is modified by including intrinsic quantities. Add the intrinsic momentum π^μ to the momentum p^μ; the replacement is $p^\mu \rightarrow p^\mu + \pi^\mu$. Also, the intrinsic vector potential $\tilde{A}_{i\mu}$ is combined with the continuous rep vector potential A_μ. The general matrix vector potential, i.e. $\tilde{\alpha}^\mu = \alpha^\mu + \partial^{\mu\prime}\chi$, is the fixed matrix vector potential α^μ in (28) and (29) with arbitrary gauge χ. The gauge χ is a function of intrinsic coordinates y and has yet to be determined.

Including the intrinsic quantities produces the lagrangian L,

$$L \equiv \bar{\psi}\left[\left(i\partial_\lambda + \pi_\lambda\right)\gamma^\lambda - \left(qA_\lambda + q\alpha_\lambda + q\partial_\lambda^\prime\chi\right)\gamma_{11}^\lambda\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad . \tag{38}$$

The lagrangian L has both functions of spacetime coordinates x and functions of intrinsic coordinates y.

The intrinsic gauge χ is a function of the ys constrained only by having symmetric second partial derivatives. We choose the special gauge to be

$$\chi = -a\left[\left(1 - \frac{m}{4aq}\right)y_\lambda\gamma^\lambda + i\frac{k}{2}y^2\left(\mathbf{1}_{12} + \mathbf{1}_{21}\right) + \frac{k^2}{3}y^2 y_\lambda\gamma_{11}^\lambda\right] + \frac{1}{q}y_\lambda\pi^\lambda \quad , \tag{39}$$

where the constant a is given by (34). The gauge-transformed matrix vector potential is then

$$\tilde{\alpha}^\mu = \alpha^\mu + \partial^{\mu\prime}\chi = \tag{40}$$

$$= -a\left[-\frac{m}{4aq}\gamma^\mu + iky_\lambda(\gamma_{12}^\lambda\gamma_{22}^\mu + \gamma_{22}^\mu\gamma_{21}^\lambda) + 2iky^\mu\left(\mathbf{1}_{12} + \mathbf{1}_{21}\right) - \frac{2}{3}k^2y^2\gamma_{11}^\mu + \frac{8}{3}k^2y^\mu y_\lambda\gamma_{11}^\lambda\right] + \frac{1}{q}\pi^\mu \ .$$

While $\tilde{\alpha}^\mu$ is quite a mess, the gauge is chosen to produce a simplified lagrangian with a mass term.

One finds that, with the gauge χ in (39), the lagrangian L in (38) becomes

$$L(\phi,\partial\phi) = \bar{\psi}\left[\left(i\partial_\lambda - qA_\lambda\right)\gamma_{11}^\lambda - m\mathbf{1}_{11} + i\partial_\lambda\gamma_{22}^\lambda\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \tag{41}$$

where the placeholder ϕ indicates the fields $\bar{\psi}, \psi, A^\mu$. All these functions depend on spacetime coordinates x.

The special gauge χ removed the functions of intrinsic coordinates y in α_λ from the Lagrangian L. By (38), this means that the y-dependence of the expression $\alpha_\lambda \gamma^\lambda_{11}$ is in the form $\partial^\prime_\lambda \chi \, \gamma^\lambda_{11}$. In one sense, α^λ and γ^λ_{11} are perpendicular, if by "perpendicular" one means that their scalar product $\alpha_\lambda \gamma^\lambda_{11}$ is effectively zero because the y-dependence can be gauged away.

Also, we made the intrinsic momentum term π^μ disappear. This is possible because π^μ is the gradient $\pi^\mu = \partial^{\mu\prime} \left(k y_\lambda \gamma^\lambda_{21} \right)$ and, therefore, can be absorbed by the gauge.

The field equations are the Euler-Lagrange equations,

$$\frac{\partial L}{\partial \phi} - \partial_\lambda \left[\frac{\partial L}{\partial(\partial_\lambda \phi)} \right] = 0 \quad , \tag{42}$$

with suitable boundary conditions, i.e. the fields vanish properly at infinity.

The field equations for the continuous rep vector potential $\phi \rightarrow A^\mu(x)$ are Maxwell's equations for a current source $q j^\mu_1$,

$$\partial^\lambda \partial_\lambda A^\mu - \partial^\mu \partial_\lambda A^\lambda = q j^\mu_1 \quad . \tag{43}$$

Thus, we have set up the lagrangian so that the source of the vector potential $A^\mu(x)$ is the current $q j^\mu_1$. The assumption that $q j^\mu_1$ is the electromagnetic current is based on the identity (30) which became the Maxwell-like equation (35).

Finally, varying L with $\phi \rightarrow \bar{\psi}(x)$ and $\phi \rightarrow \psi(x)$ gives Euler-Lagrange equations for the 8-spinor. The 8-spinor fields $\psi(x)$ and $\bar{\psi}(x)$ obey field equations for the first 4-spinor ψ_1 and a distinct set of field equations for the second 4-spinor ψ_2.

For the first 4-spinor ψ_1 we get Dirac equations for a charged, massive particle,

$$(i\partial_\lambda - qA_\lambda) \gamma^\lambda_D \psi_1 = m_1 \psi_1 \quad \text{and} \quad \left(-i\partial_\lambda \bar{\psi}_1 - qA_\lambda \bar{\psi}_1 \right) \gamma^\lambda_D = m_1 \bar{\psi}_1 \quad . \tag{44}$$

For the second 4-spinor ψ_2, one finds Dirac equations for a massless particle,

$$\gamma^\lambda_D \partial_\lambda \psi_2 = 0 \quad \text{and} \quad \partial_\lambda \bar{\psi}_2 \gamma^\lambda_D = 0 \quad . \tag{45}$$

The first 4-spinor ψ_1 obeys the Dirac equation for a massive fermion with charge q in an electromagnetic field with vector potential A^μ. See, for example, Ref. [5], Chapter XX.9 for a discussion. The second fermion ψ_2 has neither electromagnetic charge nor mass and obeys free-particle field equations.

6 Concluding remarks

The mass m of the 4-spinor ψ_1 arises because it is charged. The current j_1^μ is assumed to be charged and not j_2^μ because of a matrix identity, (30), which looks like one of Maxwell's equations with the first 4-spinor ψ_1 supplying the current density. That places ψ_1, and not ψ_2, in the interaction energy with the intrinsic vector potential where it can get mass from the intrinsic gauge χ.

Electromagnetic field energy does not enter into the process. Intrinsic quantities offer an alternative way of including a mass term that has nothing to do with any inertia caused by carrying along an electromagnetic field as the charge moves. It might, therefore, be more consistent to associate electromagnetism with one of the non-traditional theories that discards the need for an electromagnetic field. Here, we have followed the conventional treatment of electromagnetism by incorporating a vector potential A^μ.

The field equation (43) for the vector potential A^μ is gauge invariant. The special gauge χ in (39) that gives the mass term is an intrinsic 8×8 matrix function associated with the intrinsic counterpart to A^μ. The continuous rep vector potential A^μ is not involved in the origin of the mass term. Thus the vector potential A^μ can be gauge-transformed and the consequences of gauge invariance continue here unchanged from conventional electromagnetism.

The intrinsic gauge χ, Eq. (39), removes all interactions between the first 4-spinor ψ_1 and the second 4-spinor ψ_2. These are the off-diagonal blocks in χ, the terms proportional to $\mathbf{1}_{12}$, $\mathbf{1}_{21}$, and π^μ which is nonzero in the 21-block. In this way, the charged massive electron-like particle does not interact with the massless chargeless neutrino-like particle. This is an assumption. In general, an interaction between the two 4-spinors ψ_1 and ψ_2 could be implemented by introducing suitable off-diagonal terms χ_{12} and χ_{21} with nonzero components confined to the 12- and 21-blocks. Such speculation is well beyond the horizon of the present paper. Our focus is to provide a mass for the otherwise massless charged 4-spinor ψ_1. Interactions between the first and second 4-spinors originating with an intrinsic gauge may be considered in some future work.

The field equations (44) and (45) for the fermion fields ψ_1 and ψ_2 are those of a charged massive particle and a massless particle, respectively. These are long-studied equations and so there is nothing new. We keep four components for the massless particle because, if it represents a neutrino, nonzero neutrino mass is indicated by experimental results.

In summary, the consideration of new intrinsic quantities has allowed a mass to be assigned to the charged fermion without becoming entrapped by the quandaries of charge distribution models.

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(*Content-type:application/mathematica*)(***Wolfram Notebook File***)(*http://www.wolfram.com/nb*)(*CreatedBy='Mathematica 12.0'*)

An 8x8 matrix rep of the Poincare group and an intrinsic vector potential, by Richard Shurtleff, Wentworth Institute of Technology, Boston, MA, USA

20191114IntrinsicVectorPotential.nb

This notebook contains an 8x8 matrix representation of the Poincare group of spacetime transformations rotations, boosts and translations that preserve the scalar product of two 4-vectors.

The representation is set up to be applied to an 8-spinor wave function made of two 4-spinor Dirac fermions.

The intrinsic vector potential is a quantity that obeys a Maxwell-like equation whose source is the first 4-spinor which is, therefore, charged.

The intrinsic gauge is chosen in part to give the first 4-spinor a mass. Thus the charged 4-spinor acquires mass due to its being charge.

The calculations are discussed in my 2019 paper[1] `Intrinsic vector potential and electromagnetic mass'. Many equations from the article are verified in this notebook.

A ready-to-run version of this notebook should be available on the Wolfram User Notebook Archive, current URL: https://notebookarchive.org/ (as of Nov. 21, 2019)

Symbols

The table has the symbol from the paper, its Mathematica notation, and its definition

$\eta^{\mu\nu}$, $\eta_{\mu\nu}$	$\eta\mu\nu$[[μ,ν]]	the spacetime metric, flat, a 4x4 diagonal matrix with diagonal = {+1,+1,+1,-1}
δ_{ij}	δij[[i,j]]	Identity matrix, one along the diagonal i = j and zero otherwise
	ZeroMatrix[n]	an nxn matrix of zeros
1	IdentityMatrix[n]	the unit nxn matrix
x^{μ}	x[[μ]]	the four Minkowski coordinates of a point (event)
δx^{μ}	δxμ[μ]	the μ^{th} component of a coordinate interval

b^μ	b[[μ]]	4-vector displacement parameters for a translation
y^μ	y[[μ]]	intrinsic Minkowski coordinates, independent of x^μ
x^2	xSQUARED[x]	$x^2 = x_\mu x^\mu = {x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2$; scalar product of x with itself
$\omega^{\mu\nu}$	ω[[μ,ν]]	antisymmetric tensor of parameters for a spacetime rotation (Lorentz transformation)
ψ	ψ8[[i]]	8-spinor with component index i = 1,2,...,8
ψ_1	ψ1[[i]]	the first 4-spinor in ψ, i = 1,2,3,4 with $\psi_1 = 0$ for i = 5,6,7,8
ψ_2	ψ2[[i]]	the second 4-spinor in ψ, i = 5,6,7,8 and $\psi_2 = 0$ for i = 1,2,3,4
$\overline{\psi}$	ψ8bar[[i]]	the Dirac conjugate $\psi^\dagger \gamma^4$
τ^μ	$\tau\mu$[[μ]]	the four Pauli 2x2 spin matrices with μ = 1,2,3,4 and time is μ = 4.
${\gamma_D}^\mu$	γDμ[[μ]]	the four Dirac 4x4 gamma matrices
${\gamma_D}^5$	γD5	the product ${\gamma_D}^5 = i {\gamma_D}^4 {\gamma_D}^1 {\gamma_D}^2 {\gamma_D}^3$, note that time ${\gamma_D}^4$ is first.
M_{ij}		an 8x8 matrix with nonzero components only in the *ij*-block with 4x4 blocks *ij* = 1,2.
${\gamma_{ij}}^\mu$	$\gamma\mu$ij[[μ]]	four 8x8 matrices with the 4x4 matrix ${\gamma_D}^\mu$ in the ij block, where i,j = 1,2
1_{ij}	oneij	an 8x8 matrix with the unit 4x4 matrix in the ij block, where i,j = 1,2
	γALL[μ]	the four 8x8 matrices ${\gamma_{ij}}^\mu$ arranged in a 2x2 matrix so that γALL[μ][[i,j]] is ${\gamma_{ij}}^\mu$
	oneALL	the four 8x8 matrices 1_{ij} arranged in a 2x2 matrix so that oneALL[[i,j]] is 1_{ij}
γ^μ	$\gamma\mu$[[μ]]	four 8x8 matrices with ${\gamma_D}^\mu$ in the 11 and 22 diagonal blocks
$\sigma^{\mu\nu}$	$\sigma\mu\nu$[[μ,ν]]	the $\mu\nu^{th}$ angular momentum 8x8 matrix
π^μ	$\pi\mu$[[μ]]	the μ^{th} linear momentum 8x8 matrix
k	*kc*	scale factor for the π^μ matrices
(Λ,b)		a Poincare transformation, the rotation $\Lambda(\omega)$ followed by the translation along b^μ.
D(Λ,0)	DΛ0[ω]	the 8x8 matrix representing a pure spacetime rotation (no translation)
D(**1**,b)	D1b[b]	the 8x8 matrix representing a pure translation (no spacetime rotation)
$D^\dagger$(**1**,b)	D1bDagger[b]	hermitian conjugate of the translation D(1,b); transpose of the complex conjugate
D(Λ,b)	DΛb[ω,b]	the 8x8 matrix representing the transformation (Λ,b)

α^{μ}	$\alpha\mu$[[μ]]	the 8x8 matrix intrinsic vector potential
a	ca	a constant associated with the intrinsic vector potential α^{μ}
χ	χ	intrinsic gauge, an 8x8 matrix function of intrinsic coordinates y
m	m	mass
q	q	charge

Definitions

In[∘]:=
```
(*metric, Kronecker delta, null matrix*)

ημν = { {+1, 0, 0, 0}, {0, +1, 0, 0}, {0, 0, +1, 0}, {0, 0, 0, -1} };
δij = IdentityMatrix[50]; (* 50 = ∞ *)
ZeroMatrix[n_] := ZeroMatrix[n] = IdentityMatrix[n] - IdentityMatrix[n]


(*coordinate 4-vectors*)

x = {x1, x2, x3, x4}; (*Spacetime*)
b = {b1, b2, b3, b4}; (*displacement*)
y = {y1, y2, y3, y4}; (*Intrinsic*)
xSQUARED[x_] := Sum[ημν[[μ1, μ2]] × x[[μ1]] × x[[μ2]], {μ1, 4}, {μ2, 4}]
(*Use this for any coordinates*)
```

In[∘]:=
```
(*rotation parameters, spinor wave functions*)
ω = {{0, ω12, ω13, ω14}, {-ω12, 0, ω23, ω24}, {-ω13, -ω23, 0, ω34}, {-ω14, -ω24, -ω34, 0}};
ψ8 = {ψ81, ψ82, ψ83, ψ84, ψ85, ψ86, ψ87, ψ88};
ψ1 = {ψ81, ψ82, ψ83, ψ84, 0, 0, 0, 0};
ψ2 = {0, 0, 0, 0, ψ85, ψ86, ψ87, ψ88};
ψ8bar = {ψ8b1, ψ8b2, ψ8b3, ψ8b4, ψ8b5, ψ8b6, ψ8b7, ψ8b8};
```

In[∘]:=
```
(*2x2 Pauli spin matrices, 4x4 Dirac gamma matrices*)
τμ = { {{0, 1}, {1, 0}}, {{0, -i}, {i, 0}}, {{1, 0}, {0, -1}}, {{1, 0}, {0, 1}}  };
γDμ = +i Table[ArrayFlatten[{{0, -τμ[[μ]]}, {Sum[(+ημν[[μ, ν]]) τμ[[ν]], {ν, 4}], 0}}], {μ, 4}];
γD5 = i γDμ[[4]].γDμ[[1]].γDμ[[2]].γDμ[[3]];
```

In[ ]:=
```
(*There are four 4x4 blocks in an 8x8 matrix. *)
(*These are 8x8 matrices with nonzero components in just one block.*)
γμ11 = Table[ArrayFlatten[{{γDμ[[μ]] , ZeroMatrix[4]}, {0, ZeroMatrix[4]}}], {μ, 4}];
γμ22 = Table[ArrayFlatten[{{ZeroMatrix[4] , 0}, {0, γDμ[[μ]]}}], {μ, 4}];
γμ21 = Table[ArrayFlatten[{{0, ZeroMatrix[4] }, {γDμ[[μ]], 0}}], {μ, 4}];
γμ12 = Table[ArrayFlatten[{{0, γDμ[[μ]]}, {ZeroMatrix[4] , 0}}], {μ, 4}];
one11 = ArrayFlatten[{{IdentityMatrix[4], 0}, {0, ZeroMatrix[4]}}];
one22 = ArrayFlatten[{{ZeroMatrix[4], 0}, {0, IdentityMatrix[4]}}];
one12 = ArrayFlatten[{{0, IdentityMatrix[4]}, {ZeroMatrix[4] , 0}}];
one21 = ArrayFlatten[{{0, ZeroMatrix[4]}, {IdentityMatrix[4] , 0}}];
```

In[ ]:=
```
(*The four 8x8 matrices γij^μ arranged in a 2x2 matrix so that γALL[μ][[i,j]] = γij^μ.*)
γALL[μ_] := {{γμ11[[μ]], γμ12[[μ]]}, {γμ21[[μ]], γμ22[[μ]]}}
(*The four 8x8 matrices 1ij arranged in a 2x2 matrix so that oneALL[[i,j]] = 1ij.*)
oneALL = {{one11, one12}, {one21, one22}};
```

In[ ]:=
```
(*8x8 gamma matrices*)
γμ = Table[γμ11[[μ]] + γμ22[[μ]], {μ, 4}];
```

In[ ]:=
```
(*Poincare generators*)
(* Angular momentum matrices,
i.e. Lorentz generators for rotations and boosts ( together = spacetime rotations)*)
σμν = Table[(-ⅈ/4) (γμ[[μ]].γμ[[ν]] - γμ[[ν]].γμ[[μ]]), {μ, 4}, {ν, 4}];
(*Momentum matrices, i.e. generators for translations.*)
πμ = Table[kc γμ21[[μ]], {μ, 4}];
```

In[ ]:=
```
(*Poincare Group transformations*)
(*Lorentz group of spacetime rotations*)
DΛ0[ω_] :=
 Simplify[MatrixExp[Sum[+ⅈ ημν[[μ1, μ2]] × ημν[[ν1, ν2]] × ω[[μ1, ν1]] × σμν[[μ2, ν2]] / 2,
    {μ1, 4}, {μ2, 4}, {ν1, 4}, {ν2, 4}]], ω ∈ Reals]
(*Group of translations and hermitian conjugate (transpose plus complex conjugate)*)
D1b[b_] :=
  Simplify[MatrixExp[Sum[-ⅈ ημν[[μ1, μ2]] × b[[μ1]] × πμ[[μ2]], {μ1, 4}, {μ2, 4}]]];
D1bDagger[b_] := FullSimplify[Transpose[Conjugate[
    (IdentityMatrix[8] - ⅈ Sum[ημν[[μ1, μ2]] × b[[μ1]] × πμ[[μ2]], {μ1, 4}, {μ2, 4}])]],
  {kc, b1, b2, b3, b4} ∈ Reals]
(*general spacetime transformation, spacetime rotation followed by a translation*)
DΛb[ω_, b_] := D1b[b].DΛ0[ω] (* Eqn. 12, definition of D(Λ,b). *)
```

In[ ]:=
```
(*Intrinsic vector potential and intrinsic gauge*)
αμ = Table[FullSimplify[
    -ca γμ[[4]].D1bDagger[y].γμ[[4]].γμ[[μ]].D1b[y], {kc, y} ∈ Reals], {μ, 4}];
χ[y_] := -ca Sum[ ημν[[λ1, λ2]] ((1 - m/(4 ca q)) y[[λ1]] × γμ[[λ2]] +
      kc^2 (xSQUARED[y]/3) y[[λ1]] × γμ11[[λ2]] + ⅈ (kc/2) xSQUARED[y] (one21 + one12) -
      (1/(q ca)) y[[λ1]] × πμ[[λ2]]), {λ1, 4}, {λ2, 4}]
```

Poincare algebra

In[∘]:=

```
Print["check Angular momentum commutators (Eqn. 10)"]
Print["  i[σ^μν,σ^ρσ] = η^νρσ^μσ-η^μρσ^νσ-η^νσσ^μρ+η^μσσ^νρ : ",
 {0} == Union[Flatten[Table[i (σμν[[μ, ν]].σμν[[ρ, σ]] - σμν[[ρ, σ]].σμν[[μ, ν]]) -
     (ημν[[ν, ρ]] × σμν[[μ, σ]] - ημν[[μ, ρ]] × σμν[[ν, σ]] - ημν[[ν, σ]] × σμν[[μ, ρ]] +
       ημν[[μ, σ]] × σμν[[ν, ρ]]), {μ, 4}, {ν, 4}, {ρ, 4}, {σ, 4}]]]]
```

```
check Angular momentum commutators (Eqn. 10)
 i[σ^μν,σ^ρσ] = η^νρσ^μσ-η^μρσ^νσ-η^νσσ^μρ+η^μσσ^νρ : True
```

In[∘]:=

```
Print["check that (linear) momentum generators form a vector (Eqn. 11)"]
Print["  i[σ^μν,π^ρ] = η^νρπ^μ-η^μρπ^ν : ",
 {0} == Union[Flatten[Table[i (σμν[[μ, ν]].πμ[[ρ]] - πμ[[ρ]].σμν[[μ, ν]]) -
     (ημν[[ν, ρ]] × πμ[[μ]] - ημν[[μ, ρ]] × πμ[[ν]]), {μ, 4}, {ν, 4}, {ρ, 4}]]]]
```

```
check that (linear) momentum generators form a vector (Eqn. 11)
 i[σ^μν,π^ρ] = η^νρπ^μ-η^μρπ^ν : True
```

In[∘]:=

```
Print[
 "check translations form an Abelian group because the generators commute (Eqn. 11)"]
Print["  i[π^μ,π^ν] = 0 : ",
 {0} == Union[Flatten[Table[i (πμ[[μ]].πμ[[ν]] - πμ[[ν]].πμ[[μ]]), {μ, 4}, {ν, 4}]]]]
```

```
check translations form an Abelian group because the generators commute (Eqn. 11)
 i[π^μ,π^ν] = 0 : True
```

Translations are linear functions of displacements b^{μ}

In[∘]:=

```
Print["(Eqn. 13) check the product of two momenta vanishes;   π^μπ^ν = 0 : ",
 {0} == Union[Flatten[Table[πμ[[μ]].πμ[[ν]], {μ, 4}, {ν, 4}]]]]
```

```
(Eqn. 13) check the product of two momenta vanishes;   π^μπ^ν = 0 : True
```

In[∘]:=

```
Print[
 "(Eqn. 13) check  translations are linear in displacements b^μ;  D(1,b) = 1 - ib_μπ^μ:  ",
 {0} == Union[Flatten[Simplify[D1b[b] - (IdentityMatrix[8] -
       i Sum[ημν[[μ1, μ2]] × b[[μ1]] × πμ[[μ2]], {μ1, 4}, {μ2, 4}]), b ∈ Reals]]]]
```

```
(Eqn. 13) check  translations are linear in displacements b^μ;  D(1,b) = 1 - ib_μπ^μ:  True
```

Fields

It is a long story. The essentials:

1. Quantum Field Theory discounts translations of fields. Fields transform via spacetime ROTATIONS, but not translations. (Confusing fact: operators do transform by the full Poincare group including translations. That is where waves, $e^{i p^{\sigma} x_{\sigma}}$, come from.)
2. Call a field that transforms just by spacetime rotations a Lorentz field ψ. One can make a Poincare field Φ from ψ by multiplying by a translation:

$$\Phi(x,y) = D(1,y)\psi(x),$$

where x are spacetime coordinates and y are "intrinsic coordinates" that can be, and are assumed to be, independent of the x coordinates.
3. The probability current j^{μ} for the Lorentz field ψ is given by $j^{\mu} = \overline{\psi}\gamma^{\mu}\psi$. For the Poincare field we have $J^{\mu} = \overline{\Phi}\gamma^{\mu}\Phi$.
4. One can show that the Poincare current can be written like the Lorentz current, i.e.

$$aJ^{\mu} = \overline{\psi}\alpha^{\mu}\psi,$$

except that α^{μ}/a replaces γ^{μ}. The constant "a" is placed there for convenience and determined in the article[1].
5. The four matrices are functions of intrinsic coordinates y, $\alpha^{\mu}(y)$. They are called the Intrinsic Vector Potential because they satisfy a Maxwell-like equation.

The intrinsic vector potential $\alpha^{\mu}(y)$ is defined above. It satisfies a Maxwell-like equation:

In[ ]:=

```
Print["(Eqn. 29) check the expression for α^μ ;
  α^μ/a = +γ^μ - ⅈky_ρ(γ12^ρ γ22^μ + γ22^μ γ21^ρ) + k²y²γ11^μ - 2k²y^μ y_ρ γ11^ρ :    ",
 {0} == Union[Flatten[Table[FullSimplify[( αμ[[μ]]/ca - (+γμ[[μ]] - ⅈ kc Sum[ημν[[ρ1, ρ2]] ×
        y[[ρ1]] (γμ12[[ρ2]].γμ22[[μ]] + γμ22[[μ]].γμ21[[ρ2]]), {ρ1, 4}, {ρ2, 4}] +
      kc² xSQUARED[y] × γμ11[[μ]] - 2 kc² y[[μ]] × Sum[ημν[[ρ1, ρ2]] × y[[ρ1]] ×
        γμ11[[ρ2]], {ρ1, 4}, {ρ2, 4}]) ), {kc, b} ∈ Reals], {μ, 4}]]]]
```

(Eqn. 29) check the expression for α^{μ} ;

$$\frac{\alpha^{\mu}}{a} = +\gamma^{\mu} - i k y_{\rho}(\gamma_{12}{}^{\rho}\gamma_{22}{}^{\mu} + \gamma_{22}{}^{\mu}\gamma_{21}{}^{\rho}) + k^2 y^2 \gamma_{11}{}^{\mu} - 2k^2 y^{\mu} y_{\rho}\gamma_{11}{}^{\rho} : \quad \text{True}$$

An equation that looks like one of Maxwell's equations

In[ ]:=

```
Print["(Eqn. 30) check the Maxwell-like equation ;  ∂ρ′∂ρ′αμ - ∂μ′∂ρ′αρ = 12ak²γ11μ : ",
 {0} == Union[
   Flatten[Table[(Sum[ημν[[ρ1, ρ2]] × D[αμ[[μ]], y[[ρ1]], y[[ρ2]]], {ρ1, 4}, {ρ2, 4}] -
        Sum[ημν[[μ, ρ2]] × D[αμ[[ρ1]], y[[ρ1]], y[[ρ2]]], {ρ1, 4}, {ρ2, 4}]) -
      (12 ca kc² γμ11[[μ]]), {μ, 4}]]]]
Print["The appearence of γ11μ means the current of the first
   4-spinor ψ1 carries charge."]
```

(Eqn. 30) check the Maxwell-like equation ; $\partial^{\rho\prime}\partial_{\rho}{}'\alpha^{\mu} - \partial^{\mu\prime}\partial_{\rho}{}'\alpha^{\rho} = 12ak^2\gamma_{11}{}^{\mu}$: True

The appearence of $\gamma_{11}{}^{\mu}$ means the current of the first 4-spinor ψ_1 carries charge.

The intrinsic gauge χ(y) must have symmetric second partials to preserve the Maxwell-like equation above, but is otherwise arbitrary. For a certain Lagrangian, see the article[1], the following gauge removes all dependence on intrinsic coordinates *y*, removes the intrinsic momentum π^{μ}, and provides a mass term.

In[ ]:=

```
Print["(Eqn. 39) The chosen intrinsic gauge ;  χ =
   -a[(1 - m/(4 aq))yλγλ + i k/2 y²(1_12 + 1_21) + k²/3 y²yλγ11λ ] + 1/q yλπλ  : ",
 {0} == Union[Flatten[χ[y] - (-ca Sum[ ημν[[λ1, λ2]] ((1 - m/(4 ca q)) y[[λ1]] × γμ[[λ2]] +
          i kc/2 xSQUARED[y] (one12 + one21) + kc²/3 xSQUARED[y] × y[[λ1]] × γμ11[[λ2]] -
          1/(q ca) y[[λ1]] × πμ[[λ2]] ), {λ1, 4}, {λ2, 4}])]]]
```

(Eqn. 39) The chosen intrinsic gauge ; χ =

$$-a[(1 - \frac{m}{4\,aq})y_{\lambda}\gamma^{\lambda} + i\frac{k}{2}y^2(1_{12} + 1_{21}) + \frac{k^2}{3}y^2y_{\lambda}\gamma_{11}{}^{\lambda}] + \frac{1}{q}y_{\lambda}\pi^{\lambda}$$: True

In[ ]:=

```
Print["(Eqn. 40) the gauge-transformed intrinsic vector
  potential ;   α^μ + η^μλ∂_λ′χ  = -a[- m/(4 aq) γ^μ  + ikyλ(γ12^λγ22^μ + γ22^μγ21^λ)
  + 2iky^μ(1_12 + 1_21) - 2/3k^2y^2γ11^μ + 8/3k^2y^μy_λγ11^λ] + 1/q π^μ  : ", {0} == Union[
  Flatten[Table[FullSimplify[( + αμ[[μ]] + Sum[ημν[[μ, λ]] × D[χ[y], y[[λ]]], {λ, 4}] ) -
     (-ca (- m/(4 ca q) γμ[[μ]] + i kc Sum[ημν[[ρ1, ρ2]] × y[[ρ1]]
          (γμ12[[ρ2]].γμ22[[μ]] + γμ22[[μ]].γμ21[[ρ2]]), {ρ1, 4}, {ρ2, 4}] +
        2 i kc y[[μ]] (one12 + one21) - 2/3 kc^2 xSQUARED[y] × γμ11[[μ]] +
        8/3 kc^2 y[[μ]] × Sum[ημν[[λ1, λ2]] × y[[λ1]] × γμ11[[λ2]], {λ1, 4}, {λ2, 4}]) +
      1/q πμ[[μ]]), {ca, kc, m, y} ∈ Reals], {μ, 4} ]]]]
```

(Eqn. 40) the gauge-transformed intrinsic vector potential ; $\alpha^{\mu} + \eta^{\mu\lambda}\partial_{\lambda}'\chi = -a[-\frac{m}{4\,aq}\gamma^{\mu} + i k y_{\lambda}(\gamma_{12}{}^{\lambda}\gamma_{22}{}^{\mu} + \gamma_{22}{}^{\mu}\gamma_{21}{}^{\lambda}) + 2 i k y^{\mu}(1_{12} + 1_{21}) - \frac{2}{3}k^2y^2\gamma_{11}{}^{\mu} + \frac{8}{3}k^2y^{\mu}y_{\lambda}\gamma_{11}{}^{\lambda}] + \frac{1}{q}\pi^{\mu}$: True

In[ ]:=

```
Print["(Eqn. 41, matrix part) Matrix terms in the
  lagrangian combine to make a mass term for the charged 4-spinor ψ1."]
Print["   π_λγ^λ - qα_λγ11^λ -  q∂_λ′χ γ11^λ = -m 1_11 : ",
 {0} == Union[Flatten[FullSimplify[Sum[ημν[[κ1, κ2]]
       ( + πμ[[κ1]].γμ[[κ2]] - q αμ[[κ1]] .γμ11[[κ2]]), {κ1, 4}, {κ2, 4} ] -
     Sum[q D[χ[y], y[[κ]]].γμ11[[κ]], {κ, 4}] - (-m one11), {ca, kc, m, y} ∈ Reals]]]]
```

(Eqn. 41, matrix part) Matrix terms in the
lagrangian combine to make a mass term for the charged 4-spinor ψ_1.

$\pi_{\lambda}\gamma^{\lambda} - q\alpha_{\lambda}\gamma_{11}{}^{\lambda} - q\partial_{\lambda}'\chi\,\gamma_{11}{}^{\lambda} = -m\,1_{11}$: True

Checks of various other equations in the article[1]

In[ ]:=

```
(*Dirac gammas must obey the following*)
Print["check  that γD^μγD^ν + γD^νγD^μ = 2 η^μν1 : ",
 {0} == Union[Flatten[Table[(γDμ[[μ]].γDμ[[ν]] + γDμ[[ν]].γDμ[[μ]]) -
     2 ημν[[μ, ν]] × IdentityMatrix[4], {μ, 4}, {ν, 4}]]]]
```

check that $\gamma_D{}^{\mu}\gamma_D{}^{\nu} + \gamma_D{}^{\nu}\gamma_D{}^{\mu} = 2\,\eta^{\mu\nu}1$: True

```
In[◦]:= (*Eqn. 6*)
Print["check Eqn. 6:  γij^μ γkl^ν + γij^ν γkl^μ = 2 δjk η^μν 1il : ",
 {0} == Union[Flatten[Table[Table[γALL[μ][[i, j]].γALL[ν][[k, l]] + γALL[ν][[i, j]].
      γALL[μ][[k, l]] - (2 IdentityMatrix[8][[j, k]] × ημν[[μ, ν]] × oneALL[[i, l]]),
    {i, 2}, {j, 2}, {k, 2}, {l, 2}], {μ, 4}, {ν, 4}]]]]

check Eqn. 6:  γij^μ γkl^ν + γij^ν γkl^μ = 2 δjk η^μν 1il : True

In[◦]:= (*Eqn. 7*)
Print["check Eqn. 7:  γ^μ = γ11^μ + γ22^μ : ",
 {0} == Union[Flatten[Table[γμ[[μ]] - (γμ11[[μ]] + γμ22[[μ]]), {μ, 4}]]]]

check Eqn. 7:  γ^μ = γ11^μ + γ22^μ : True

In[◦]:= (*Eqn. 8*)
Print["check Eqn. 8; σ^μν = -(ⅈ/4)(γ11^μ γ11^ν - γ11^ν γ11^μ) - (ⅈ/4)(γ22^μ γ22^ν - γ22^ν γ22^μ): ", {0} ==
  Union[Flatten[Table[σμν[[μ, ν]] - (-(ⅈ/4) (γμ11[[μ]].γμ11[[ν]] - γμ11[[ν]].γμ11[[μ]]) -
       (ⅈ/4) (γμ22[[μ]].γμ22[[ν]] - γμ22[[ν]].γμ22[[μ]])), {μ, 4}, {ν, 4}]]]]

check Eqn. 8; σ^μν = -(ⅈ/4)(γ11^μ γ11^ν - γ11^ν γ11^μ) - (ⅈ/4)(γ22^μ γ22^ν - γ22^ν γ22^μ): True

In[◦]:= (*Eqn. 9*)
Print["check Eqn. 9:  π^μ = kγ21^μ : ",
 {0} == Union[Flatten[Table[πμ[[μ]] - (kc γμ21[[μ]]), {μ, 4}]]]]

check Eqn. 9:  π^μ = kγ21^μ : True

In[◦]:= (*Eqn. 12*)
Print["check Eqn. 12:  D(Λ,b) = ⅇ^(-ⅈbμ π^μ) ⅇ^(+ (ⅈ/2) ωμν σ^μν) : ",
 {0} == Union[Flatten[DΛb[ω, b] - (Simplify[
      MatrixExp[Sum[-ⅈ ημν[[μ1, μ2]] × b[[μ1]] × πμ[[μ2]], {μ1, 4}, {μ2, 4}]]].Simplify[
      MatrixExp[Sum[+ⅈ ημν[[μ1, μ2]] × ημν[[ν1, ν2]] × ω[[μ1, ν1]] × σμν[[μ2, ν2]] / 2,
        {μ1, 4}, {μ2, 4}, {ν1, 4}, {ν2, 4}]], ω ∈ Reals])]]]

check Eqn. 12:  D(Λ,b) = ⅇ^(-ⅈbμ π^μ) ⅇ^(+ (ⅈ/2) ωμν σ^μν) : True

In[◦]:= (*Eqn. 14*)
Print["check Eqn. 14;   D(1,b)ψ8 = {{ψ1},{ψ2 - ⅈkbμ γD^μ ψ1}} :  ",
 {0} == Union[Flatten[Simplify[D1b[b].ψ8 -
     (ψ8 - ⅈ Sum[ημν[[μ1, μ2]] × b[[μ1]] × πμ[[μ2]].ψ8, {μ1, 4}, {μ2, 4}]), b ∈ Reals]]]]

check Eqn. 14;   D(1,b)ψ8 = {{ψ1},{ψ2 - ⅈkbμ γD^μ ψ1}} :  True
```

Bibliography

(refer to the article for a more detailed list of references. For general items, I would try Wikipedia first.)

[1]Intrinsic vector potential and electromagnetic mass

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Abstract

Electric charges may have mass in part or in full because they charged. Supplying details is the electromagnetic mass problem. Here, the charge's mass is associated with intrinsic quantum mechanical quantities so that the classical problems with extended charge distributions, for example, are irrelevant. An intrinsic vector potential is defined, based on intrinsic linear momentum. The charge-electromagnetic field interaction energy is gauge-dependent and the needed mass term is placed with the interaction energy in the intrinsic gauge. Traditional electromagnetism retains its gauge invariance. The field equations make no new predictions since all dynamic dependence on intrinsic quantities can be gauged away. The field equations describe a massive, charged 4-spinor Dirac electron-like particle and an uncharged, massless neutrino-like particle, formulas that have been a part of physics for nearly a century.

[2]Many equations appearing in the article are verified by the calculations in this notebook. The equations checked are equations numbered (6), (7), (8), (9), (10), (11), (12), (13), (14), (29), (30), (39), (40), and (41).

[3]The article is expected to be uploaded to Arxiv or Vixra by the end of November 2019. It has been submitted to *The Foundations of Physics*, and so it may appear there eventually.