Phase-Transition means: Fractal invariant in Renormalizations mediates Transitions between Manifolds of different Topologies.

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1. Abstract.

Phase–transitions are normally known from physics, e.g. when a matter changes its state of aggregation while an appropriate transition–threshold changes into a decisive quality. Phenomena of such kinds are not reserved for physics only, they can also be observed – detached from any physical application – in a wide range of mathematical contexts. These scenarios – generally called as phase–transitions now – will mediate between manifolds of different topologies. Some physics–examples are known where the threshold of the appropriate transition is excelled by a fractal structure with the property of invariance in renormalizations. Distribution of magnetized micro–cells in a Ferro–magnet at a critical temperature may be mentioned as a typical example in this sense. Similar qualities can also be verified for phase–transitions in pure mathematical contexts, especially when the transitions between manifolds of different topologies are mediated by fractals that turn out to be invariant in renormalizations. Therefore it seems, this kind of phenomena will always happen in space as soon the aforementioned conditions are met.

2. Introduction.

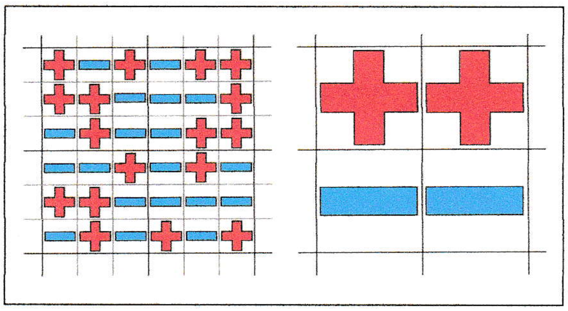
A phase–transition is normally known in physical contexts, but which content may be behind this term, shall now be described by the phenomenon of Ferro–magnetism.
A 2_dimensional Ferro–magnet may subsequently be considered as a square–lattice subdivided into a large number of small quadratic micro–magnets (sub–squares of the lattice).

- Likewise the north–poles of the micro–magnets are considered to be either up (positively) or down (negatively) oriented.
- The micro–magnets can be summarized by renormalization of the lattice into objects of increasing complexity on increasing scaling–levels.
- Magnetization of the whole lattice–system on a certain scaling–level results from the overlay of objects of the appropriate level.

This may become more obvious by the following scenario:

- Initially 3×3 micro–magnets may be summarized into a new object. This new object will be either up or negatively down depending on the orientation–majority of the included parts.
- Every new object consisting of 4 or less up and 5 or more down oriented parts will gain a negative alignment. An analogous statement can be made with regard to a positive alignment of an appropriate object if its group of micro–magnets are alternatively oriented.

This may graphically be represented in the following picture:



Renormalization of objects will be continued until an appropriate granulation of the lattice has been reached. In this way the system will become more and more coarse–grained.
This renormalisation will enable to track changes of physical properties inside the lattice, if it is assumes, that neighbouring groups will try to interact with each other to gain alignment of their orientations.

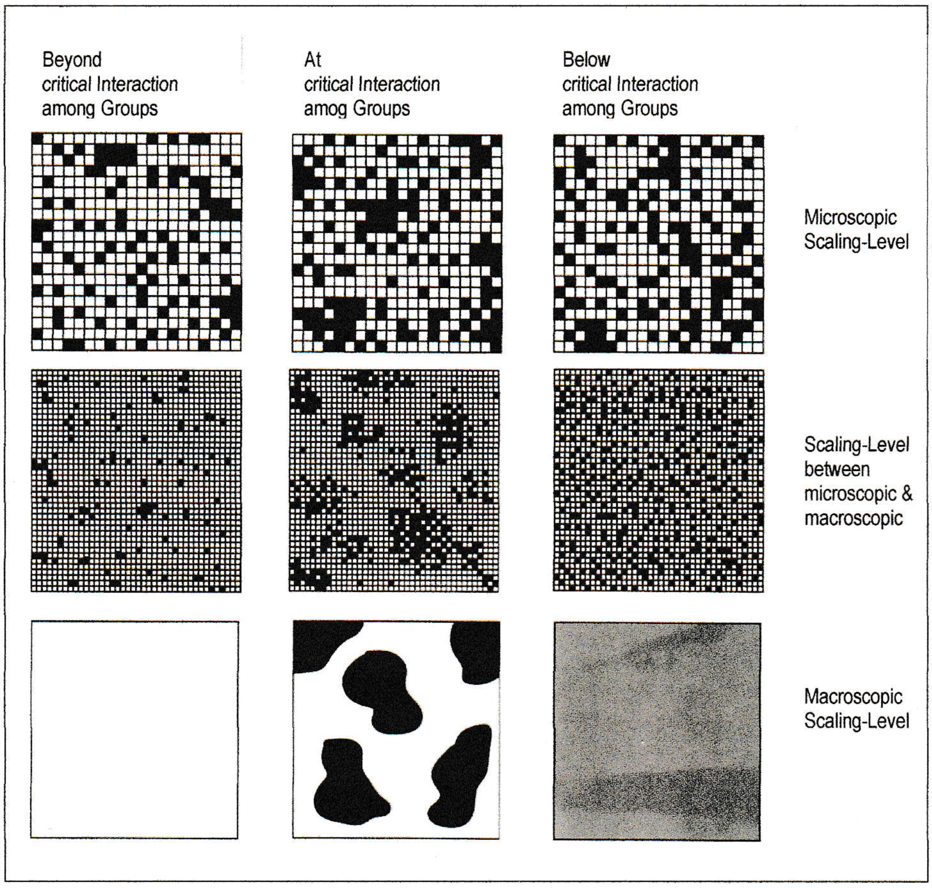
- If the interaction is weak compared to the influencing temperature from outside, thermal fluctuations will cause random orientation patterns among the renormalization–groups. No order of orientation can be found inside the lattice, whatever the scaling–level will be. The lattice appears un–magnetized macroscopically.
- As far as the interaction among the groups becomes strong compared to the influence of outside temperature, various up and/or down oriented regions inside the lattice will arise.
- Because probabilities are equal for a positive or negative orientation, small disturbances of the equilibrium will be enough to magnetize the whole lattice in one of the optional directions.

Starting without interaction and turning it on slowly, various distributions among up and down magnetized groups inside the lattice can be observed. At a critical interaction this distribution suddenly keeps similar throughout all renormalizations inside the lattice.

- The distribution of magnetized groups has become independent from renormalization–levels.
- This behaviour at the critical magnetization of the groups is typical for a phase–transition inside the lattice.
- This critical distribution of magnetization marks the transition from a state without into a state with macroscopic magnetization of the lattice.

In order to further clarify the just mentioned situations, the following picture should be observed.
On microscopic level it is hardly to decide whether the lattice is up (black) or down (white) magnetized. Therefore it's advisable to look on it more coarse–grained. But only when the critical granulation finally has been reached, the magnetization will become obvious macroscopically.

- Only if the interaction among the groups is beyond its critical value, the lattice can macroscopically be considered as a Ferro–magnet, most of the lattice's groups are either up or down oriented.
- Below the critical value of interaction, no order can be found even if the lattice get constantly less granulated; the lattice–system does not appear Ferro–magnetic.
- At the critical interaction of the groups within the lattice, the distribution of regions with positive and negative orientations becomes a fractal and keeps similar on all scaling–levels during renormalizations.



This example of phase–transition from physics is excellent by the topological properties:

- It occurs between 2 different 2–dimensional manifolds of a local Euclidean space.
- It is mediated by a fractal which keeps invariant in renormalization.

But this mathematical characterization is also valid for phenomena in a pure mathematical context, completely detached from any application in physics. Some appropriate examples will be discussed next.

3. SIERPINSKI-Gasket and its Relatives.

SIERPINSKI–gasket is:

- A 2–dimensional, symmetric fractal with strict self–similarity and invariance in renormalizations.
- It may be generated from a square by applying an appropriate Iterated Function System (IFS).

Modifying the IFS by operations from the symmetry–group of the square will enable to obtain all relatives of the gasket. All these fractals will prove to be as self–similar and therefore invariant in renormalizations; they mediate transformations between manifolds of order and chaos. Thereby chaos will be observed in a manifold of random moves, while order occurs in manifolds which contain the fractals themselves.

3.1. Deterministic iterated Function-System IFS.

A Multi–Reduction–Copy–Machine (MRCM) is a collection of contractions (similarity–transformations with angle–preservations):

- Enabled by a system of reduction–lenses.
- The lens–system of MRCM can be described by a set of affine transformations $w_{K=1\rightarrow N}$.
- For a given initial image A, small affine copies $w_{K=1\rightarrow N}$ (A) are produced, which are finally superimposed to a new image as output of MRCM by the HUTCHINSON–operator: $W(A) = w_1(A) \bigcup w_2(A) \bigcup \dots \bigcup w_N(A)$. Running the MRCM in feedback–mode corresponds to iterating the operator W(A). This is in essence the deterministic Iterated–Function–System (IFS).

In order to make IFS more obvious, the following scheme maybe helpful:

Initial picture: A_0				●		
Affine transformations: $w_1, w_2, \dots, w_N$	●	●	●			
| might be | ◆ *| might be applied to |* ◆ *| superimposed to |* ◆ *| via |*	↓	↓	↓	↓		
Transposition ∨ Contraction ∨ Shearing ∨ Rotation ∨ Reflection	●					
Picture: A		●				
| as |		↓				
$w_1(A) \wedge w_2(A) \wedge \dots \wedge w_N(A)$		●				
HUTCHINSON-operator: $W(A) = w_1(A) \bigcup w_2(A) \bigcup \dots \bigcup w_N(A)$			●	●		
| enables |				↓		
Sequence of images: $\{A_{J+1} = W(A_J), J = 0,1,2,\dots,J\}$				●	●	
| tending toward |					↓	
Final picture: A_∞					●	●
| highlighted by |					=	↓
Attractor of IFS					●	
$A_\infty = W(A_\infty)$						●

Principle of IFS

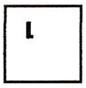
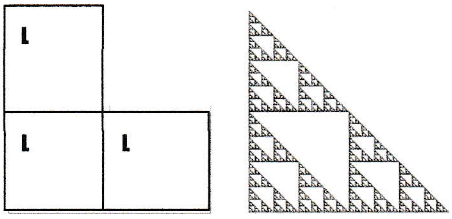
3.2. SIERPINSKI-Gasket.

The principle above will enable by the small modifications generation of an IFS appropriate to specify the topology of the SIERPINSKI–gasket:

- Affine transformations w_J are limited to number of 3.
- Any picture will become contracted by a factor 0.5 for any iteration–step
- There after:
  - w_1 keeps a picture in place from step before
  - w_2 moves a picture aside from the place of step before
  - w_3 moves a picture ahead from the place of step before

For details, please look into the scheme below:

$A[k] = W(A[k-1]) = w_1(A[k-1]) \bigcup w_2(A[k-1]) \bigcup w_3(A[k-1])$				●	●	
| with |				↓	↓	
Positioning and contraction: $(w_1 = X/2, Y/2) \wedge (w_2 = [X+1]/2, Y/2) \wedge (w_3 = X/2, [Y+1]/2)$		●	●	●		
| with |		↓	↓			
$A[1] = W(A[0]) = w_1(A[0]) \bigcup w_2(A[0]) \bigcup w_3(A[0])$ ■ $\lim\{k \rightarrow \infty\}$		●			●	
| results in |		↓		↓	↓	
IFS of SIERPINSKI-gasket	●					

| *starts with* |	↓					
$A[0]_{(X=0/Y=0)(X=1/Y=0)}^{(X=1/Y=1)(X=0/Y=1)}$ = 	●					
Blueprint: $A[1] = A[00] \bigcup A[10] \bigcup A[01]$		●	●			
| *where* | ◆ | *results in* |		↓	↓			
$A[00][_{(X=0/Y=0)(X=0.5/Y=0)}^{(X=0.5/Y=0.5)(X=0/Y=0.5)}$		●				
		∧				
$A[01]_{(X=0/Y=0.5)(X=0.5/Y=0.5)}^{(X=0.5/Y=1)(X=0/Y=1)}$		●				
		∧				
$A[10]_{(X=0.5/Y=0)(X=1/Y=0)}^{(X=1/Y=0.5)(X=0.5,Y=0.5)}$		●				
$A[2] = W(A[1]) = w_1(A[1]) \bigcup w_2(A[1]) \bigcup w_3(A[1])$			●			
$A[k+1] = W(A[k]) = w_1(A[k]) \bigcup w_2(A[k]) \bigcup w_3(A[k])$				●		
Attractor: $A[\infty] = W(\infty)$					●	●
| *represented by* |						↓
Blueprint =  **= Attractor**						●
IFS for SIERPINSKI-Gasket						

3.3. Relatives of SIERPINSKI-Gasket.

IFS for the relatives can be easily obtained by some additions to the specifications from above:

- The former affine transformations will become extended by operations from the symmetry–group of the square.
- Each optional relative will get one specific operation from the symmetric–group for every affine transformation
  $\{w_1\ w_2\ w_3\}$ individually.

Details can be found again in the scheme below:

<table>
<tr><td>IFS for Relatives of SIERPINSKI-gasket</td><td>●</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>| <i>modified with regard to</i> |</td><td>↓</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>IFS of SIERPINSKI-gasket</td><td>●</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>| <i>by</i> |</td><td>↓</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>$w_{J\in[1\sim2\sim3]}(A[0])$</td><td>●</td><td>●</td><td></td><td></td><td></td><td></td></tr>
<tr><td>| <i>superimposed to</i> |</td><td></td><td>↓</td><td></td><td></td><td></td><td></td></tr>
<tr><td>$w_{P\in[1\sim2\sim3]} = v_{P\in[1\sim2\sim3]} \bullet d_{Q\in[0\sim1\sim2\sim3\sim4\sim5\sim6\sim7]}$</td><td></td><td>●</td><td>●</td><td>●</td><td></td><td></td></tr>
<tr><td>| <i>where</i> |</td><td></td><td></td><td>↓</td><td>↓</td><td></td><td></td></tr>
<tr><td>$[v_1 = (X/2,Y/2)] \wedge [v_2 = ([X+1]/2,Y/2)] \wedge [v_3 = (X/2,[Y+1]/2)]$</td><td></td><td></td><td>●</td><td></td><td></td><td></td></tr>
<tr><td>$d_{Q\in[0\sim1\sim2\sim3\sim4\sim5\sim6\sim7]}$</td><td></td><td></td><td></td><td>●</td><td></td><td></td></tr>
<tr><td>| <i>obtained from</i> |</td><td></td><td></td><td></td><td>↓</td><td></td><td></td></tr>
<tr><td>Symmetry-group of $A[0]$ =
<table>
<tr><th>•</th><th>d_0</th><th>d_1</th><th>d_2</th><th>d_3</th><th>d_4</th><th>d_5</th><th>d_6</th><th>d_7</th></tr>
<tr><td>d_0</td><td>d_0</td><td>d_1</td><td>d_2</td><td>d_3</td><td>d_4</td><td>d_5</td><td>d_6</td><td>d_7</td></tr>
<tr><td>d_1</td><td>d_1</td><td>d_2</td><td>d_3</td><td>d_0</td><td>d_7</td><td>d_6</td><td>d_4</td><td>d_5</td></tr>
<tr><td>d_2</td><td>d_2</td><td>d_3</td><td>d_0</td><td>d_1</td><td>d_5</td><td>d_4</td><td>d_7</td><td>d_6</td></tr>
<tr><td>d_3</td><td>d_3</td><td>d_0</td><td>d_1</td><td>d_2</td><td>d_6</td><td>d_7</td><td>d_5</td><td>d_4</td></tr>
<tr><td>d_4</td><td>d_4</td><td>d_6</td><td>d_5</td><td>d_7</td><td>d_0</td><td>d_2</td><td>d_1</td><td>d_3</td></tr>
<tr><td>d_5</td><td>d_5</td><td>d_7</td><td>d_4</td><td>d_6</td><td>d_2</td><td>d_0</td><td>d_3</td><td>d_1</td></tr>
<tr><td>d_6</td><td>d_6</td><td>d_5</td><td>d_7</td><td>d_4</td><td>d_3</td><td>d_1</td><td>d_0</td><td>d_2</td></tr>
<tr><td>d_7</td><td>d_7</td><td>d_4</td><td>d_6</td><td>d_5</td><td>d_1</td><td>d_3</td><td>d_2</td><td>d_0</td></tr>
</table></td><td></td><td></td><td></td><td>●</td><td>●</td><td>●</td></tr>
<tr><td>| <i>where</i> | ◆ | <i>included by</i> |</td><td></td><td></td><td></td><td></td><td>↓</td><td>↓</td></tr>
<tr><td>d_0 = [identity-transformation = e = 0 ⤻ 1 ⤻ 2 ⤻ 3]</td><td></td><td></td><td></td><td></td><td>●</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td>∧</td><td></td></tr>
<tr><td>⟨d_1 = [rotation = 3 ⤻ 0 ⤻ 1 ⤻ 2]⟩ ∧ ⟨d_2 = [rotation = 2 ⤻ 3 ⤻ 0 ⤻ 1]⟩</td><td></td><td></td><td></td><td></td><td>●</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td>∧</td><td></td></tr>
</table>

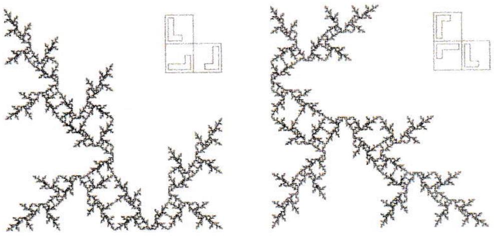
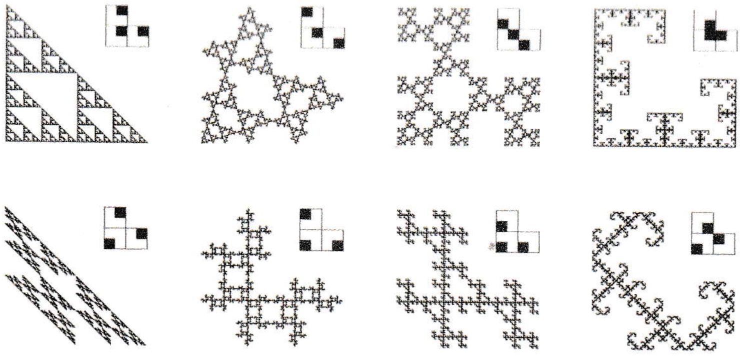
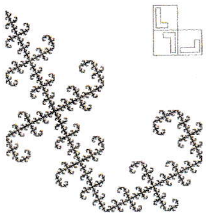
d_3 = [rotation = 1 ↝ 2 ↝ 3 ↝ 0]					●	
					∧	
⟨d_4 = [reflection = 3 ↝ 2 ↝ 1 ↝ 0]⟩ ∧ ⟨d_5 = [reflection = 1 ↝ 0 ↝ 3 ↝ 2]⟩					●	
					∧	
⟨d_6 = [reflection = 2 ↝ 1 ↝ 0 ↝ 3] ⟩ ∧ ⟨d_7 = [reflection = 0 ↝ 3 ↝ 2 ↝ 1]⟩					●	
Cyclic group of the rotations (yellow marked)						●

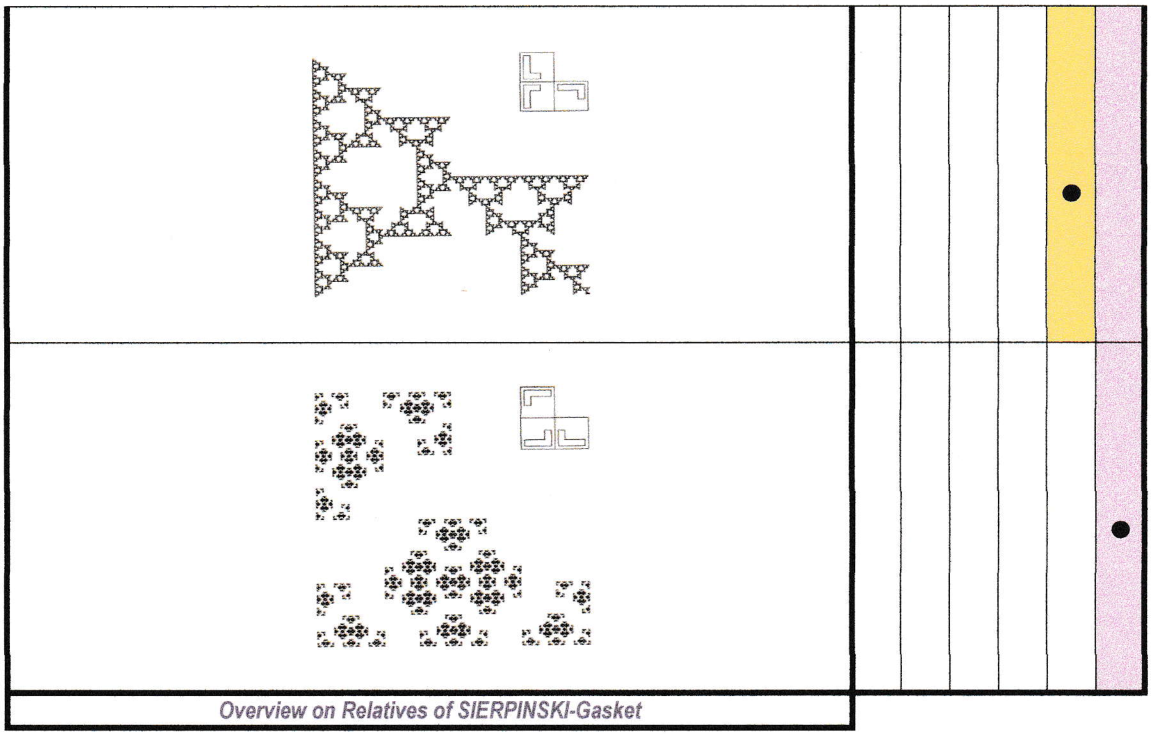
IFS from Relatives of SIERPINSKI-Gasket

For the members of the family it is valid:

- Each one is specified by $w_J = v_J \cdot d_K$ (where J ∈ [1,2,3] and K ∈ [0,1,2,3,4,5,6,7])
- A number of $8^3 = 512$ different collages can be obtained.
- These may be divided into several sub–classes.

A comprehensive description of this is contained in the scheme below.

Number of relatives from SIERPINSKI-gasket: 8^3 = 512	●					
consisting of	↓					
Relatives non symmetric with respect to diagonal: 2·224 = 448	●	●				
symmetric together with		↓				
Counterparts:		●				
	∧					
Relatives symmetric with respect to diagonal: 64	●		●			
	∧					
Subset of simply-connected fractals	●			●		
	∧					
Subset of not-simply-connected fractals	●				●	
	∧					
Subset of disconnected fractals	●					●
specified by example		↓	↓	↓	↓	↓
		●				
			●			
				●		



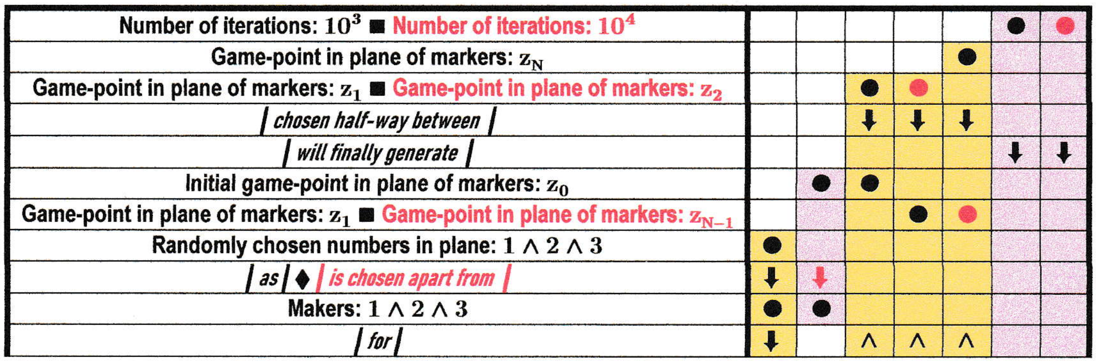
Overview on Relatives of SIERPINSKI-Gasket

3.4. Members of SIERPINSKI-Family can be produced by a Chaos-Game.

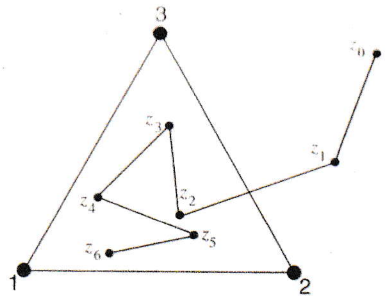
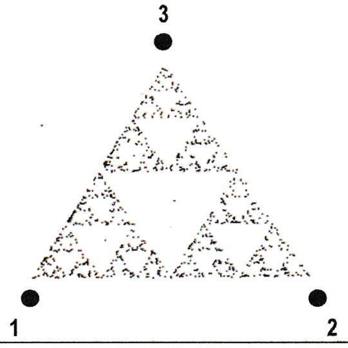
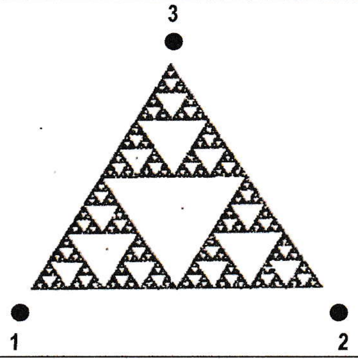
Small particles of solid matter suspended in a liquid and observed in a microscope will show particle–movements in an irregular, erratic way. This is the called BROWNIAN–motion due to the random molecular impacts of neighbouring particles. It may be an appropriate picture for a randomly steered motion described next.

- Beginning at a point of the plane.
  - A walk is started in a direction chosen randomly, it moves for some distant and stops.
  - Another random direction is chosen, it is walked along for some distant and again comes to rest.
  - This procedure is repeated again and again.
- After hundreds or thousands of steps more or less the same pattern of the random move will become evolved but each time a bit more dense.
- In any event there doesn't seem to be much to expect from randomness in conjunction with the images in such generations.
- One may try a variant, which – on a first glance – could well belong to that category. Following M. F. BARNSLEY a family of games is introduced, which can potentially change the intuitive idea of randomness drastically.

One of these games considered next is applied on SIERPINSKI–gasket and – with small additional modifications – later on its relatives as well:



Number of iterations: 10^3 ■ Number of iterations: 10^4						●	●
Game-point in plane of markers: z_N					●		
Game-point in plane of markers: z_1 ■ Game-point in plane of markers: z_2			●	●			
| chosen half-way between |			↓	↓	↓		
| will finally generate |						↓	↓
Initial game-point in plane of markers: z_0		●	●				
Game-point in plane of markers: z_1 ■ Game-point in plane of markers: z_{N-1}				●	●		
Randomly chosen numbers in plane: 1 ∧ 2 ∧ 3	●						
| as | ♦ *| is chosen apart from |*	↓	↓					
Makers: 1 ∧ 2 ∧ 3	●	●					
| for |	↓		∧	∧	∧		

Chaos-game = 	●						
Randomly chosen marker: 1 ∨ 2 ∨ 3			●	●	●		
						●	
							●
/ neglected /						↓	↓
Spurious dots outside maker-connecting lines						●	●

Chaos-Game generates SIERPINSKI-gasket

- SIERPINSKI–gasket has been generated in a completely random process as deterministic structure of order inform of a self–similar fractal.
  - Following the time–process step by step one cannot predict where the next game–point will settle down.
  - Nevertheless the pattern which all game–points together leave behind is absolutely predictable.
  - The SIERPINSKI–gasket discloses itself as structure of order in a phase–transition from random moves.
- Relatives will be generated principally in a similar way as the gasket in chaos–game from above.
  - IFSs for the relatives will be principally the same as for the IFS in case of SIERPINSKI–gasket.
  - All game–points therefore will land on pictures A_K as in game before although the $w_{P\in[1,2,3]}$ have been modified by the $d_{Q\in[0,1,2,3,4,5,6,7]}$.
  - This is guaranteed by the fact that the $w_{P\in[1,2,3]}$ despite of their modifications remain qualitatively the same with respect to the IFS–procedure.

4. JULIA-Set.

A JULIA–set can be characterized:

- As a fractal in complex plane with the property of self–similarity and therefore of invariance in renormalizations.
- Whether it is a connected or not, it encloses one or more sets of orbits converging to fix–points and it is enclosed by another set of orbits escaping to infinity.

- It acts as repeller for the enclosed and for the enclosing sets as well.

For the subsequent discussions a JULIA–set is considered as a connected one.

4.1. Determination of JULIA-Set.

Orbits of kind $h \rightarrow h^2+\ell$ totally contained in the complex plane are divided into h's which:

- Escape to infinity and thus belong to the escape–set or
- Are influenced by fix–points from a limited area, called fix–points–set.
- The fix–points–set itself consists of the prisoner–set and the JULIA–set as disjunctive subsets.

A summary about the relations and individual properties of the sets is presented next:

Flow: $(h_{(N=0,1,2...)} \in \mathbb{C}) \rightarrow h_N^2+(\ell \in \mathbb{C})$	●			
/*enters*/	↓			
Escape-set: $E_\ell = \{h_{(N=0,1,2...)} \mid (h_N \rightarrow h_N^2+\ell) \rightarrow \text{infinity}\}$	●			●
	∨			∧
Fix-points-set: $L_\ell = \{h_{(N=0,1,2...)} \mid (h_N \rightarrow h_N^2+\ell) \rightarrow \text{fix-points}\}$	●	●		
/*separated into*/		↓		
Prisoner-set: $P_\ell = \{h_{(N=0,1,2...)} \mid (h_N \rightarrow h_N^2+\ell) \rightarrow \text{(attracting fix-point)}\}$		●		●
		∧		
JULIA-set: $J_\ell = h_{(N=0,1,2...)} \mid (h_N \rightarrow h_N^2+\ell) \rightarrow \text{(repelling fix-point)}\}$		●	●	
/*pushed away by*/			=	↓
Repeller				●
JULIA-set : $J_\ell = \{P_\ell \bigcap \{L_\ell \setminus P_\ell\} = \{0\}\}$			●	
Specification of {Escape-set} ∧ {Limit-set = {Prisoner-set} ⋃ {JULIA-set}}				

This shall be more clarified by the example $h \rightarrow h^2+ 0.12+0.74i$. It shows:

- How JULIA–set and prisoner–set as subsets of an appropriate fix–point–set can be specified.
- How they can be separated from each other by the properties of their fix–points only.

Fix-points of iteration: $h \rightarrow h^2+(\ell = 0.12+0.74i)$	●											
/*obtained by*/	↓											
Quadratic equation: $h^2-h+0.12+0.74i = 0$	●	●										
/*solved by*/		↓										
$h_{1,2} = [1\pm(1-4[0.12+0.74i])^{1/2}]/2 = [1\pm(0.52-2.96i)^{1/2}]/2$		●								●		
/*where*/		↓										
$(0.52-2.96i)^{1/2} = p+qi$		●							●			
		∧							∧	∧		
$0.52-2.96i = (p+qi)(p+qi) = (p^2-q^2)+(2pq)i$		●	●									
/*leads to*/			↓									
$(0.52 = p^2-q^2) \wedge ([-2.96 = 2pq] = [-1.48 = pq])$			●					●				
			∨					∧				
$0.52 = p^2-2.19/p^2$			●	●								
/*leads to*/				↓								
$p^4-0.52p^2-2.19 = 0$				●	●	●						
/*is*/ ♦ /*leads to*/					↓	↓						
Quadratic equation for: p^2					●							
$p^2 = [0.52+(0.27+8.76)^{1/2}]/2 = [0.52+(9.03)^{1/2}]/2 \approx 1.762$						●	●					
/*because*/ ♦ /*leads to*/						↓	↓					
$(p^2 > 0) \wedge (0.52 < (9.03)^{1/2})$						●						
$p = (1.762)^{1/2} \approx 1.33$							●	●	●			
/*leads to*/								↓	∧			
$q = -1.48/p \approx -1.11$								●	●			
/*leads to*/									↓			
$(0.52-2.96i)^{1/2} \approx 1.33-1.11i$									●	●		
/*leads to*/										↓		
$h_1 = [1+1.33-1.11i]/2 = [2.33-1.11i]/2 = 1.165-0.555i$										●	●	
										∧		
$h_2 = [1-1.33+1.11i]/2 = [-0.33-1.11i]/ 2 = -0.165+0.555i$										●		●
/*leads to*/											↓	↓

$\lvert 2h_1\rvert = \lvert 2.33-1.11i\rvert = \lvert([2.33]^2+[1.11]^2)^{1/2}\rvert > 1$											●	
$\lvert 2h_2\rvert = \lvert -0.33+1.11i\rvert = \lvert([0.33]^2-[1.11]^2)^{1/2}\rvert < 1$												●
| thus identified as |											↓	↓
Repelling fix-point											●	
Attractive fix-point												●
| has to be |											↓	↓
Element of JULIA-set											●	
Element of prisoner-set												●
Fixed-Points of Sink-Set												

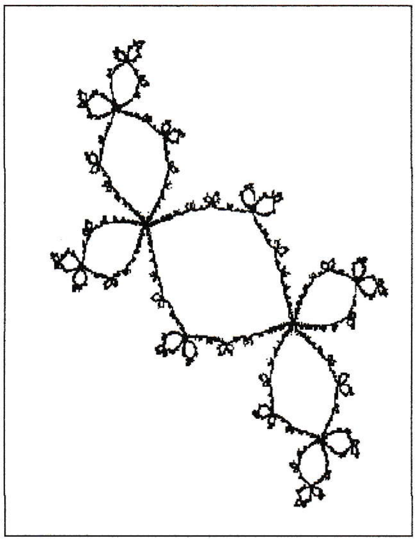
4.2. JULIA-Set may be obtained by a Chaos-Game.

A JULIA–set can be obtained in a chaos–game if a certain kind of orbits is considered; the appropriate procedure works in the following way:

- One may think about the inverted transformation, i.e., the transformation that takes a point g to the point $\hbar$ where $g = \hbar^2+\ell$; effectively one is iterating backwards.
- With respect to this iteration the character of the JULIA–set changes temporarily from a repeller to an attractor.
- The procedure to be applied demands for solving $\hbar^2-g+\ell = 0$ and writing the appropriate solutions in the form of $\hbar_1 = +(g-\ell)^{1/2}$ and $\hbar_2 = -(g-\ell)^{1/2}$, as pre–images $\hbar_{1,2}$ of the point g.

This may be shown by the following example:

- The process starts at fix–point $g = 1.165-0.555i$ from JULIA–set.
- One of the two pre–images $\hbar_{1,2}$ is selected at random and replaces g.
- g is replaced by the $\hbar$ and the process is repeated this way again and again until enough points have been collected.
- The calculated points drawn in series will show (depending on extent of the point–set) the following picture of the JULIA–set appropriate for $\hbar \to \hbar^2+0.12+0.74i$.



There may be some region in the JULIA–set that are hard to go to. In these situations certain modifications are appropriate to improve the method just mentioned (please look into [3]).

JULIA–set is to be seen as the attractor in the chaos–game and this reveals an important fact on its self–similarity.

- The situation is similar to that in a Multiple–Reduction–Copy–Machine (MRCM) where the whole attractor becomes covered by small copies of itself.
- This makes obvious, any copy is nothing else but an image of the whole attractor under the transformations of the IFS.
- Similar applies to the JULIA–set in the chaos–game above.

One can deduce from that:

- By applying one of the two transformations $\hbar_1 = +(g-\ell)^{1/2}$ or $\hbar_2 = -(g-\ell)^{1/2}$ to any point g of the JULIA–set, one will obtain another point of the JULIA–set.
- Therefore the JULIA–set is invariant with respect to the inverse transformation of $\hbar \to \hbar^2+\ell$.
- Moreover, if $\hbar$ is a point from the JULIA–set, $\hbar^2+\ell$ cannot be part of the escape–set otherwise the initial point g would have to be a point of the escape–set too, but g was initially chosen from JULIA–set.
- On the other hand $\hbar^2+\ell$ cannot be in the prisoner–set. Due to the continuity of the quadratic transformation, it must be on JULIA–set (the boundary of sink–set).

And thus it follows: A JULIA–set is invariant with respect to:

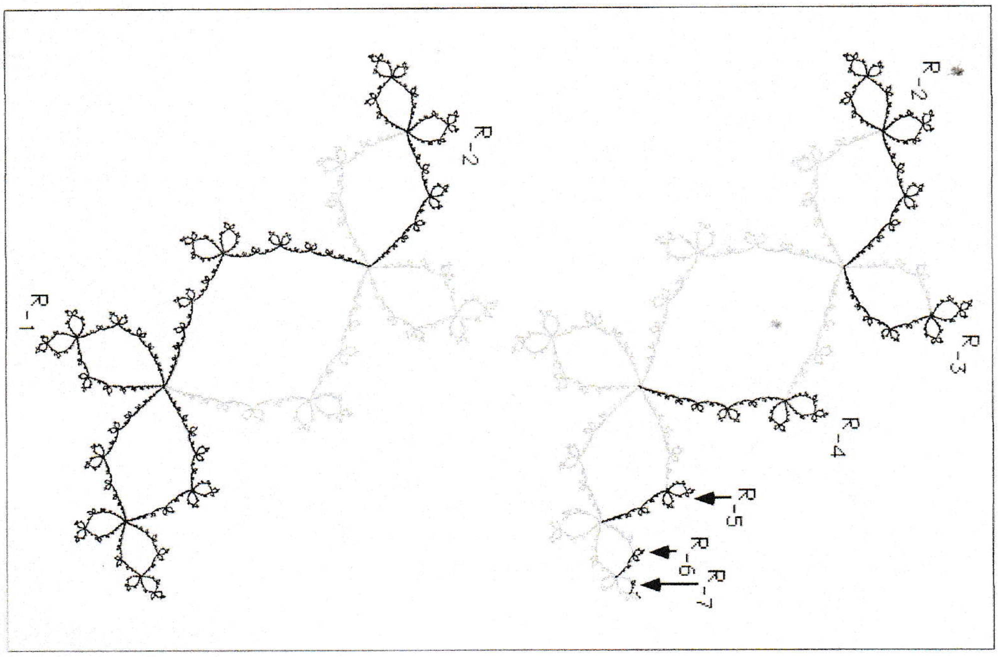
- transformation $\hbar \to \hbar^2+\ell$ and
- $\hbar = \pm(g-\ell)^{1/2}$ as well.

In other words, the JULIA–set remains invariant under forward– and backward–iterations as well. This property is called complete invariance.

The global structure of the JULIA–set must appear in its:

- Image and
- Pre–images as well.

This explains the appearance of self–similarity associated with $\hbar \to \hbar^2+0.12+0.74i$ shown in the next picture:



The similarity is based on a non–linear transformation, thus the smaller copies contained in itself are not exact copies but distorted in a way, that they are folded back on themselves.

- One may take any small section of the JULIA–set (i.e. the intersection of a small disk with the JULIA–set which is not empty) and apply the iteration $h \to h^2+\ell$ to every point of the section.
- The result will be in a new typically larger subset of the JULIA–set. Iterating further in this way a finite number of times will reinstall the complete JULIA–set again.

This can be expressed by:

- The complicated global structure of the JULIA–set is already contained in any arbitrarily small section of it, thus it is self–similar.
- The JULIA–set thus turns out to remain invariant under renormalizations.
- It can be considered as a phase–transition between orbits of the prisoner–set, converging to a finite fix–point, and those of the escape–set, tending towards infiniteness.

5. Final-States-Characteristics of the Quadratic-Iterator within different Parameter-Regions.

The final–state–history of iterations $\langle 0 \leq x_{(N=0\rightarrow\infty)} \leq 1\rangle \rightarrow p\cdot x_N(1-x_N)$ are considered subsequently for the parameter–ranges:

- $p \leq 4$
- $p > 4$

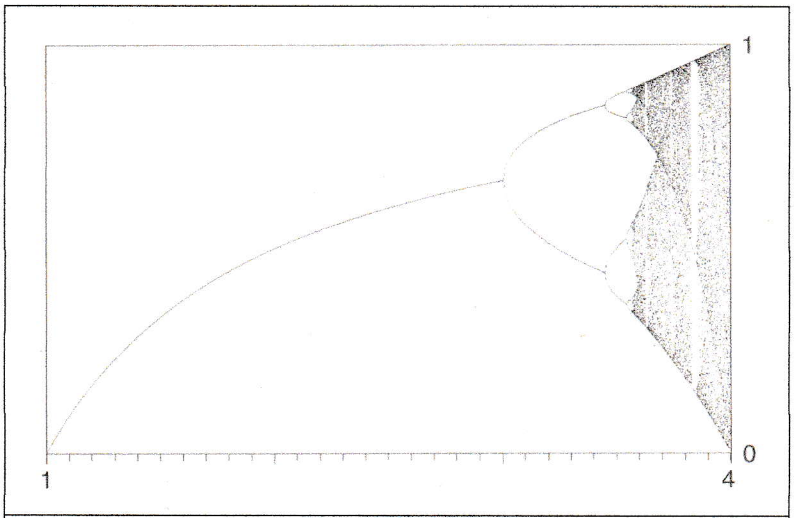
It will follow that, whether iterations are approaching either $p \rightarrow 4$ or $p \leftarrow 4$, the behaviour of the appropriate histories will become qualitatively different. The transition between both regimes is mediated by a threshold structured as a CANTOR–set, which shows the property of invariance in renormalization–transformations.

The final–states from iteration with a specific parameter value can be obtained in the following way:

- An initial value $x_0 \in [0,1]$ is chosen randomly and iterated for ≥ 200 times.
- The appropriate iterations will settle dawn at one or more final–states.

5.1. Final-States Distribution for $p \leq 4$.

The resulting plot for histories of final–states in the range $1 \leq p \leq 4$ is made obvious in the next picture (the famous FEIGENBAUM–diagram):



Final–State–Diagram for $1 \leq p \leq 4$

(for supplement, please look into [4]).

- For interval $1 \leq p \leq 3$ a final–state for each individual p–value can found.
- Whereas for increasing p–values in the range $3 < p < p_\infty \approx 3.5699456...$ a cascade of 2, 4, ..., $2^{J=3\rightarrow\infty\ J=3\rightarrow\infty}$ final–states is created; $p_\infty \approx 3.5699456...$ is called FEIGENBAUM–point.
- For $p_\infty \leq p \leq 4$ the final–states–distribution for p–value will become chaotic and fills–up finally the whole unit–interval perpendicular to the parameter–scale.
- The unit–interval shall be called for future–use as the prisoner–set for iterations for p–value of the appropriate interval.

5.2. Final-States Distribution for $4 < p$.

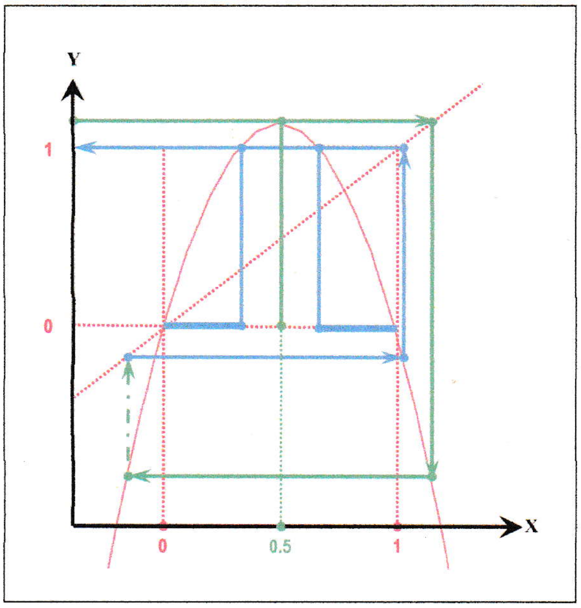
Prisoner–set will change qualitatively in $(p > 4)$–situations if compared to $(p < 4)$–cases from before; details shall be discussed next.

In situations where the parameter p exceeds a value of 4, only a subset of orbits will start inside [0,1] and thus finally not end in the prisoner–set [0,1]. This alternate portion of orbits will escape to infinity and therefore belong to the escape–set of the appropriate iterations. The question pops–up, what is now the structure of the prisoner–set under these changed conditions? A method to answer the question is to follow iterations:

- Not in forward–directions: $p \cdot x_N(1-x_N) \rightarrow x_{N+1}$,
- But in backward–direction: $x_N \leftarrow 0.5 \cdot (p \pm [p^2+4 \cdot p \cdot x_{N+1}]^{1/2})$

By backward–iteration orbits will be generated, which can be described by tree–structures. Given x_{N+1} one will obtain 2 or 1 or 0 pre–images x_N. In cases where no further pre–images x_N will exist, trees are pruned at the corresponding branches.

For the following discussion the specific example for $p = 4.5$ is selected, the appropriate picture is shown below. The backward–iteration will be started at $Y_0 = 1.125$, because all iterations beyond this value will escape to infinity and therefore will definitely not have pre–images in the prisoner–set.



The following scheme is provided in order to let become the method of construction in above picture more obvious:

Horizontal line					●		●	●		●	
Vertical line				●		●			●		
Horizontal line from: $Y_0 = 1.125$	●		●								
/ tanging with / ◆ */ intersects with /*	↓		↓								
/ starting from /				↓	↓	↓	↓	↓	↓	↓	
Top of graph g(X) = 4.5•X(1-X) at: P_1	●	●									
/ defines /		↓									
Image at: $X_0 = 0.5$		●									
Main-diagonal of unit-square: P_2			●	●							
/ interconnects with /				↓							
Graph-point: P_3				●	●						
/ interconnects with /					↓						
Opposite graph-point: P_4					●	●					
/ interconnects with /						↓					
Main-diagonal of unit-square: P_5						●	●				
/ interconnects with /							↓				
Graph-point: P_6							●	●			
/ interconnects with /								↓			
Opposite graph-point: P_7								●	●		
/ interconnects with /									↓		
Main-diagonal of unit-square: P_8										●	
/ intersects with /										↓	
Graph-points at: $Y_1 \wedge Y_2$										●	●
/ define /											↓
Pre-images of $X_0 = 0.5$: $X_1 \wedge X_2$											●
Image and Pre–Images of $X_{1\wedge 2} \leftarrow 0.5 \cdot (4.5 \pm [20.25+4 \cdot 4.5 \cdot X_0]^{1/2})$											

To step forward this way, a few more backward–iterations will have to be considered, starting from the pre–images obtained above.

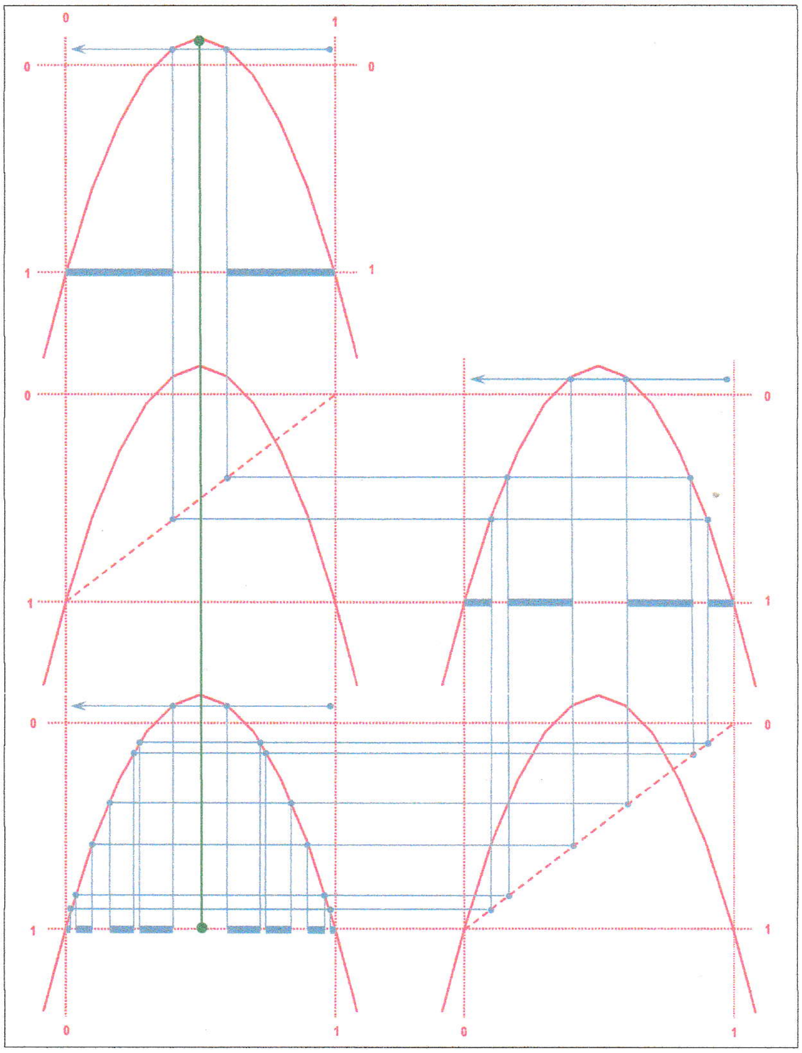
- Several graphs $g(X) = 4.5 \cdot X \cdot (1-X)$ may be positioned in a way, that the first step can feed into the second a.s.o.. Please consider the next picture below.
- One will observe, the resulting prisoner–set of the graph $g(x)$ will tend to a kind of CANTOR–set.

Usually a CANTOR–set is referred to as an interval from which the (open) middle thirds are removed recursively:

- Therefore, all pieces in a certain stage of construction will have same lengths.
- The resulting limit–object is strictly self–similar (invariant under renormalization–transformations).

By the construction below one obtains something very similar,

- But the here pieces of a given step in construction have different sizes and therefore the limit–object in the picture below (the prisoner–set of $g(X) = 4.5 \cdot X \cdot (1-X)$) is not as symmetrical as the usual CANTOR–set.
- It is a CANTOR–set slightly disordered but still invariant with respect to any renormalization of the prisoner–set.



Similar results will be obtained for:

- All 2–point intersections of images $0 < Y < Y_0$ with quantitative, not qualitative modifications of appropriate CANTOR–sets and appropriate situations.
- Appropriate situations with $4 < p < 4.5$.

Thus, one may summarize:

- For $p \leq 4$ the prisoner–set is a connected interval.

- While for $p > 4$ it becomes a fractal.

In other words, in going from $p \leq 4$ to $p > 4$ a phase–transition takes place and the threshold for this event changes from a connected to the disconnected line of a CANTOR–set. This phenomenon is to be observed independent from renormalizations of (X,Y)–measures.

6. Conclusion.

From previous discussion of Ferro–magnetism a phase–transition had to been characterized by two fundamental properties:

- It mediates between two topologically different manifolds in space.
- Threshold for a change from one domain into the alternate one will have a fractal structure, which keeps invariant at renormalizations.

This example from physics shows a direct parallelism with a few events departed from any physical applications in pure mathematical contexts:

- Random moves attracted by the SIERPINSKI–gasket and/or its relatives:
    - The history of chaos formed by random–moves is topologically different from the history of order according to the generations of appropriate fractals.
    - The fractals themselves mediate transformations between order and chaos.
    - Each fractal is self–similar (overlay of small pieces of the attractor) and thus keeps invariant with regard to renormalization–transformations.
- Orbit–history in $\mathbb{C}$–plane inside and outside of a connected JULIA–set:
    - The histories are topologically different from each other; inside the JULIA– set history is tending towards a fix–point, outside the JULIA–set history escapes to infinity.
    - The JULIA–set itself acts as repeller for and object of mediation between both history–sets.
    - The JULIA–set it is self–similar (can be reproduced from any small part of itself) and therefore is invariant with regard to renormalizations.
- The iteration $x_{P+1} = a \cdot x_{[P\rightarrow\infty]} \cdot (1 - x_{[p\rightarrow\infty]})$ in $\mathbb{R}$–plane for Parameter–value $(a<4) \rightarrow 4$ and $4 \leftarrow (a>4)$:
    - The histories are topologically different from each other, for $(a<4) \rightarrow 4$ history tends chaotically towards [0,1], for $4 \leftarrow (a>4)$ history partially escapes to infinity and partially tends to a CANTOR–set on [0,1].
    - The CANTOR–set is distorted with regard to the classical one but it is invariant with regard to renormalizations. It mediates between history–sets from parameter–regions $(a<4) \rightarrow 4$ and $4 \leftarrow (a>4)$.

Due to the fact that there exist strong similarities among these examples in such a way that the critical transitions between the topological manifolds obeys the same fundamental qualities mentioned in the physical example above, it seems to be justified to classify all these transitions as phase–transitions; their properties are:

- Mediations between topologically different manifolds.
- Fractals with invariance under renormalizations as mediation–thresholds.

7. References.

[1] S. Erne Quantenwelt im Nichtgleichgewicht, Spektrum der Wissenschaft, 10.2019.

[2] H. O. Pleitgen, H. Jürgens, D. Saupe Chaos and Fractals, New Frontiers of Science, Springer 1992.

[3] H. O. Pleitgen, D. Saupe Modified Inverse Iteration Method, in ″The Science of Fractal Images°, Springer 1988.

[4] U. E. Steinemann Final–States–Diagram of quadratic Iterator topologically equivalent with an Eddy′s Decay–Cascade in turbulent Fluids, 2018, http://vixra.org/abs/1809.0202.