TRIGONOMETRIC IDENTITIES

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ABSTRACT: I present the proof of Trigonometric Identities involving **Sin(x) and Cos(x)**.

1. INTRODUCTION

We are aware of the following identities;

$2SinxCosx = Sin2x$

$2Cos^2x = 1 + Cos2x$

The question is have we ever been aware of the fact that the above two identities are special cases of some other identities? Well, they actually are, and i will try to show that.

2. NEW IDENTITIES

$$2^nCos^nxSin(n)x = \sum_{k=0}^{n}\binom{n}{k}Sin2kx$$

$$2^nCos^nxCos(n)x = \sum_{k=0}^{n}\binom{n}{k}Cos2kx$$

$$Sin(n)x\sum_{k=0}^{n}\binom{n}{k}Cos2kx = Cos(n)x\sum_{k=0}^{n}\binom{n}{k}Sin2kx$$

3. PROOF OF THE NEW IDENTITIES

Using binomial expansion, we see that;

$$(1+e^{2ix})^n = \sum_{k=0}^{n}\binom{n}{k}e^{2ikx} \qquad (1)$$

Now, let's try to manipulate LHS and RHS of (1);

LHS;

$$(1+e^{2ix})^n = (e^{ix}(e^{-ix}+e^{ix}))^n$$

$$(1+e^{2ix})^n = (2e^{ix}(\frac{e^{ix}+e^{-ix}}{2}))^n$$

We know that;

$$Cosx = (\frac{e^{ix}+e^{-ix}}{2})$$

So,

$$(1+e^{2ix})^n = 2^ne^{inx}cos^nx$$

We know that;

$e^{inx} = \text{Cosnx} + i\,\text{Sinnx}$

Therefore;

$(1 + e^{2ix})^n = 2^n(\text{Cosnx} + i\text{Sin}(n)x)\text{Cos}^n x$

$$(1 + e^{2ix})^n = 2^n\text{Cos}^n x\text{Cosnx} + i(2^n\text{Cos}^n x\text{Sin}(n)x) \quad (2)$$

RHS;

$\sum_{k=0}^{n}\binom{n}{k}e^{2ikx} = \sum_{k=0}^{n}\binom{n}{k}(\text{Cos}2kx + i\,\text{Sin}2kx)$

$$\sum_{k=0}^{n}\binom{n}{k}e^{2ikx} = \sum_{k=0}^{n}\binom{n}{k}\text{Cos}2kx + i\sum_{k=0}^{n}\binom{n}{k}\text{Sin}2kx \quad (3)$$

We see from (2) and (3) that;

$$2^n\text{Cos}^n x\text{Cosnx} + i(2^n\text{Cos}^n x\text{Sin}(n)x) = (\sum_{k=0}^{n}\binom{n}{k}\text{Cos}2kx) + i(\sum_{k=0}^{n}\binom{n}{k}\text{Sin}2kx) \quad (4)$$

Equating the real and imaginary parts of (4), we see that;

$$2^n\text{Cos}^n x\text{Sin}(n)x = \sum_{k=0}^{n}\binom{n}{k}\text{Sin}2kx \quad (5)$$

$$2^n\text{Cos}^n x\text{Cos}(n)x = \sum_{k=0}^{n}\binom{n}{k}\text{Cos}2kx \quad (6)$$

Dividing (6) by (5), we see that;

$$\text{Sin}(n)x\sum_{k=0}^{n}\binom{n}{k}\text{Cos}2kx = \text{Cos}(n)x\sum_{k=0}^{n}\binom{n}{k}\text{Sin}2kx \quad (7)$$

4. GENERALIZATION OF THE NEW IDENTITIES

- $2^n\text{Cos}^n ax\text{Sin}(an + m)x = \sum_{k=0}^{n}\binom{n}{k}\text{Sin}(2ak + m)x$
- $2^n\text{Cos}^n ax\text{Cos}(an + m)x = \sum_{k=0}^{n}\binom{n}{k}\text{Cos}(2ak + m)x$
- $\text{Sin}(an + m)x\sum_{k=0}^{n}\binom{n}{k}\text{Cos}(2ak + m)x = \text{Cos}(an + m)x\sum_{k=0}^{n}\binom{n}{k}\text{Sin}(2ak + m)x$

PROOF

We can see that;

$$(e^{(a + \frac{m}{n})ix}.(e^{-iax} + e^{iax}))^n = (e^{(ai - ai + i\frac{m}{n})x} + e^{(ai + ai + i\frac{m}{n})x})^n$$

$$(e^{(a + \frac{m}{n})ix}.(e^{-iax} + e^{iax}))^n = (e^{(\frac{m}{n})ix} + e^{(2a + \frac{m}{n})ix})^n \quad (8)$$

Expanding the right hand side of (8) using binomial expansion, we can see that;

$$(e^{\left(\frac{m}{n}\right)ix} + e^{\left(a+\frac{m}{n}\right)ix})^n = \binom{n}{0}.(e^{n\left(\frac{m}{n}\right)ix}) + \binom{n}{1}.(e^{(n-1)\left(\frac{m}{n}\right)ix}).e^{\left(2a+\frac{m}{n}\right)ix} + \binom{n}{2}.(e^{(n-2)\left(\frac{m}{n}\right)ix}).e^{2\left(2a+\frac{m}{n}\right)ix} + \ldots + \binom{n}{n}.e^{n\left(2a+\frac{m}{n}\right)ix}$$

$$(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix})^n = \binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{\left(2ai+\left(\frac{im}{n}\right)+\left(\frac{inm}{n}\right)-\left(\frac{im}{n}\right)\right)x}) + \binom{n}{2}.(e^{\left(4ai+\left(\frac{2im}{n}\right)+\left(\frac{inm}{n}\right)-\left(\frac{2im}{n}\right)\right)x}) + \ldots + \binom{n}{n}.e^{(2an+m)ix}$$

$$(e^{\left(\frac{m}{n}\right)ix} + e^{\left(2a+\frac{m}{n}\right)ix})^n = \binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{(2a+m)ix}) + \binom{n}{2}.(e^{(4a+m)ix}) + \ldots + \binom{n}{n}.e^{(2an+m)ix} \quad (9)$$

We can see clearly that;

$$\binom{n}{0}.(e^{imx}) + \binom{n}{1}.(e^{(2a+m)ix}) + \binom{n}{2}.(e^{(4a+m)ix}) + \ldots + \binom{n}{n}.e^{(2an+m)ix} = \sum_{k=0}^{n}\binom{n}{k}e^{(2ak+m)ix}$$

Now,

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax} + e^{iax}))^n = \sum_{k=0}^{n}\binom{n}{k}e^{(2ak+m)ix} \quad (10)$$

Now, let's try to manipulate LHS and RHS of (10);

LHS;

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax} + e^{iax}))^n = (2e^{\left(a+\frac{m}{n}\right)ix}(\frac{e^{iax} + e^{-iax}}{2}))^n$$

We know that;

$$\mathrm{Cos}(a)x = (\frac{e^{iax} + e^{-iax}}{2})$$

So,

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax} + e^{iax}))^n = 2^n e^{n\left(a+\frac{m}{n}\right)ix}\cos^n ax$$

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax} + e^{iax}))^n = 2^n e^{(an+m)ix}\cos^n ax$$

We know that;

$$e^{(an+m)ix} = \mathrm{Cos}(an+m)x + i\,\mathrm{Sin}(an+m)x$$

Therefore;

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax} + e^{iax}))^n = 2^n(\mathrm{Cos}(an+m)x + i\mathrm{Sin}(an+m)x)\mathrm{Cos}^n ax$$

$$(e^{\left(a+\frac{m}{n}\right)ix}.(e^{-iax} + e^{iax}))^n = 2^n\mathrm{Cos}^n ax\mathrm{Cos}(an+m)x + i(2^n\mathrm{Cos}^n ax\mathrm{Sin}(an+m)x) \quad (11)$$

RHS;

$$\sum_{k=0}^{n}\binom{n}{k}e^{(2ak+m)ix} = \sum_{k=0}^{n}\binom{n}{k}(\mathrm{Cos}(2ak+m)x + i\,\mathrm{Sin}(2ak+m)x)$$

$$\sum_{k=0}^{n}\binom{n}{k}e^{(2ak+m)ix} = \sum_{k=0}^{n}\binom{n}{k}\mathrm{Cos}(2ak+m)x + i\sum_{k=0}^{n}\binom{n}{k}\mathrm{Sin}(2ak+m)x \quad (12)$$

Since (11) equals (12), then;

$$2^n Cos^n ax Cos(an+m)x + i(2^n Cos^n ax Sin(an+m)x) = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x + i\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x \quad (13)$$

Equating the real and imaginary parts of (13), we see that;

$$2^n Cos^n ax Sin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x \quad (14)$$

$$2^n Cos^n ax Cos(an+m))x = \sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x \quad (15)$$

Dividing (14) by (15), we see that;

$$Sin(an+m)x\sum_{k=0}^{n}\binom{n}{k}Cos(2ak+m)x = Cos(an+m)x\sum_{k=0}^{n}\binom{n}{k}Sin(2ak+m)x \quad (16)$$

5. SOME OTHER NEW IDENTITIES

- $2^n Sin^n ax Sin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^k Sin(2ak+m)x$ (n is even)
- $2^n Sin^n ax Sin(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^k Cos(2ak+m)x$ (n is odd)
- $2^n Sin^n ax Cos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^k Sin(2ak+m)x$ (n is odd)
- $2^n Sin^n ax Cos(an+m)x = \sum_{k=0}^{n}\binom{n}{k}(-1)^k Cos(2ak+m)x$ (n is even)